Equations: Check your Understanding

Section 1: Solving Quadratic Equation

1. **CJC Prelim 8865/2018/Q2**

Given that $2x^4 + x^2 - 1 = 0$, use the substitution $u = x^2$ to find the exact values of x. [4]

Answer: $x = \pm\dfrac{1}{\sqrt{2}}$

Soln:

1	Substituting $u = x^2$ into $2x^4 + x^2 - 1 = 0$, $2u^2 + u - 1 = 0$ $(u+1)(2u-1) = 0$ $u = -1$ or $u = \dfrac{1}{2}$ $\Rightarrow x^2 = -1$ $\left(\text{reject} \because x^2 \geq 0\right)$ or $x^2 = \dfrac{1}{2}$ $x^2 = \dfrac{1}{2}$ $x = \pm\dfrac{1}{\sqrt{2}}$

2. **TJC Prelim 8865/2018/Q2(a)**

By means of the substitution $u = \sqrt{x}$, and without the use of a graphing calculator, find the value of x which satisfies the equation $5\sqrt{x} - \dfrac{8}{\sqrt{x}} = 6.$ [3]

Answer: $x = 4$

Soln:

2	$5\sqrt{x} - \dfrac{8}{\sqrt{x}} = 6 \Rightarrow 5u - \dfrac{8}{u} = 6$ $5u^2 - 6u - 8 = 0$ $(5u+4)(u-2) = 0$ $u = -\dfrac{4}{5}$ or $u = 2$ $\sqrt{x} = -\dfrac{4}{5}$ (Reject) or $\sqrt{x} = 2$ $x = 4$

Section 2: Nature of Roots of a Quadratic Equation

1. **RI Prelim 8865/2018/Q3(a)**

Find the set of values of k for which the equation $3x^2 - (k-1)x + 3 = 0$ has real roots. [4]

Answer: $\{k \in \mathbb{R}: k \le -5 \text{ or } k \ge 7\}$

Soln:

	Since $3x^2 - (k-1)x + 3 = 0$ has real roots, Discriminant ≥ 0 $\Rightarrow (k-1)^2 - 4(3)(3) \ge 0$ $\Rightarrow (k-1)^2 - (6)^2 \ge 0$ $\Rightarrow (k+5)(k-7) \ge 0$ Hence the set of values of k is $\{k \in \mathbb{R}: k \le -5 \text{ or } k \ge 7\}$

2. **MJC Prelim 8865/2018/Q2(a)**

Find the range of values of p for which the equation $x^2 + px + 2 = 0$ has no real roots. [2]

Answer: $-2\sqrt{2} < p < 2\sqrt{2}$

Soln:

2	Since $x^2 + px + 2 = 0$ has no real roots, Discriminant < 0 $p^2 - 4(1)(2) < 0$ $\left(p-\sqrt{8}\right)\left(p+\sqrt{8}\right) < 0$ $-\sqrt{8} < p < \sqrt{8}$ $-2\sqrt{2} < p < 2\sqrt{2}$

3. **PJC Prelim 8865/2018/Q2**

Given that p and q are real numbers, show that the equation, $(x-p)(x-2) = q^2$ has real roots. State the conditions for the roots to be equal. [3]

Answer: $p = 2$ and $q = 0$

Soln:

	$(x-p)(x-2) = q^2 \cdots (1)$ $x^2 - (p+2)x + 2p - q^2 = 0$ Discriminant $= (p+2)^2 - 4(1)(2p - q^2) = p^2 + 4p + 4 - 8p + 4q^2 = (p-2)^2 + 4q^2$ ≥ 0 since p and q are real numbers Therefore the roots of the equation are real. For (1) to have equal roots, Discriminant = 0. Therefore, $p = 2$ and $q = 0$

Section 3: Quadratic expression to be always positive ot always negative

1. SRJC Prelim 8865/2018/Q2(a)

Find the exact range of values of *k* for which $y = x^2 + (k-1)x + 3$ is always positive. [3]

Answer: $1-2\sqrt{3} < k < 1+2\sqrt{3}$

Soln:

Since $y = x^2 + (k-1)x + 3$ is always positive,

Discriminant < 0 and coefficient of x^2 = 1 > 0,

$(k-1)^2 - 4(1)(3) < 0$

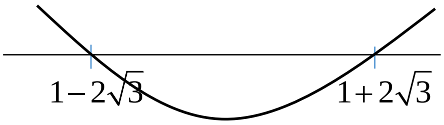
Method 1

$k^2 - 2k - 11 < 0$

Consider $k^2 - 2k - 11 = 0$.

$$k = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2(1)}$$

$k = 1+2\sqrt{3}$ or $k = 1-2\sqrt{3}$



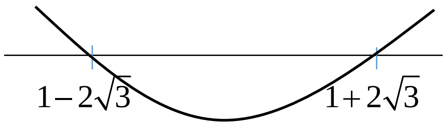
Hence, $1-2\sqrt{3} < k < 1+2\sqrt{3}$

Method 2

$(k-1)^2 - \left(\sqrt{12}\right)^2 < 0$

$\left(k-1-\sqrt{12}\right)\left(k-1+\sqrt{12}\right) < 0$

$\left[k-\left(1+2\sqrt{3}\right)\right]\left[k-\left(1-2\sqrt{3}\right)\right] < 0$



Hence, $1-2\sqrt{3} < k < 1+2\sqrt{3}$

2. CJC Prelim 8865/2018/Q1

Find, algebraically, the range of values of *k* for which $kx^2 - 4x + k < 0$ for all real values of *x*. [4]

Answer: $k < -2$

Soln:

Since $kx^2 - 4x + k < 0$,

Discriminant <0 and coefficient of $x^2 = k < 0 \cdots (1)$

$(-4)^2 - 4(k)(k) < 0$

$16 - 4k^2 < 0$

$4 - k^2 < 0$

$(2+k)(2-k) < 0$

$k < -2$ or $k > 2 \quad \cdots (2)$

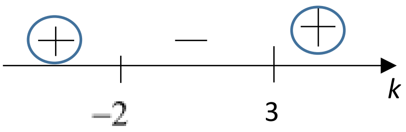
Combining (1) and (2), $k < -2$.

3. IJC Prelim 8865/2018/Q1

Find algebraically the set of values of k for which $(k-2)x^2 - 2kx + (2k+3) < 0$ for all real values of x. [4]

Answer: $\{k : k \in \mathbb{R}, k < -2\}$

Soln:

	Since $(k-2)x^2 - 2kx + (2k+3) < 0$, Discriminant < 0 and coefficient of $x^2 = k-2 < 0 \Rightarrow k < 2 \cdots (1)$ $(-2k)^2 - 4(k-2)(2k+3) < 0$ $4k^2 - (8k^2 - 4k - 24) < 0$ $-4k^2 + 4k + 24 < 0$ $k^2 - k - 6 > 0$ $(k-3)(k+2) > 0$  $\therefore\ k < -2$ or $k > 3$ Combining (1) and (2), $k < -2$ Therefore, the set of solution is $\{k \in \mathbb{R} : k < -2\}$

4. ACJC Prelim 8865/2018/Q1

Find, algebraically, the set of exact values of m for which $3mx^2 - 24x + 7m > 0$ for all real values of x. [4]

Answer: $m > 4\sqrt{\frac{3}{7}}$

Soln:

1	Since $3mx^2 - 24x + 7m > 0$, Discriminant < 0 and coefficient of $x^2 = 3m > 0 \Rightarrow m > 0 \cdots (1)$ $(-24)^2 - 4(3m)(7m) < 0$ $84m^2 > 576$ $84m^2 - 576 > 0$ $m^2 - \frac{48}{7} > 0$ $\left(m - \sqrt{\frac{48}{7}}\right)\left(m + \sqrt{\frac{48}{7}}\right) > 0$ $m < -4\sqrt{\frac{3}{7}}$ or $m > 4\sqrt{\frac{3}{7}} \quad \cdots (2)$

	Combining (1) and (2), $m > 4\sqrt{\frac{3}{7}}$ Hence the set of values of *m* is $\left\{m \in \mathbb{R} : m > 4\sqrt{\frac{3}{7}}\right\}$

5. **EJC Prelim 8865/2018/Q1**

Show that there are no real values of *k* for which $(k-8)x+2k-2x^2$ is always negative. [4]

Soln:

	$\text{Discriminant} = (k-8)^2 - 4(-2)(2k)$ $= k^2 - 16k + 64 + 16k$ $= k^2 + 64 > 0$ for all real values of k Therefore discriminant will never be less than 0 and there are not real values of *k* for $(k-8)x+2k-2x^2$ is always negative

Section 4: Intersection Problems leading to Quadratic Equation

1. **JJC Prelim 8865/2018/Q3**

Curve *C* has equation $(x+1)^2 + y^2 = 2$.

(i) Find the range of values of *p* if the line $y = x + p$ does not intersect *C*. [4]

(ii) Deduce the values of *p* if the line $y = x + p$ is a tangent to *C*. [1]

Answer: (i) $p < -1$ or $p > 3$, (ii) $p = -1$ or $p = 3$

Soln:

(i)	Sub $y = x + p$ into $(x+1)^2 + y^2 = 2$: $(x+1)^2 + (x+p)^2 = 2$ $x^2 + 2x + 1 + x^2 + 2px + p^2 = 2$ $2x^2 + (2+2p)x + (p^2 - 1) = 0 \cdots (1)$ Since the line does not intersect C, (1) has no real roots. Discriminant < 0 $(2+2p)^2 - 4(2)(p^2-1) < 0$ $4 + 4p^2 + 8p - 8p^2 + 8 < 0$ $-4p^2 + 8p + 12 < 0$ $p^2 - 2p - 3 > 0$ $(p-3)(p+1) > 0$ $p < -1$ or $p > 3$
(ii)	$p = -1$ or $p = 3$

2. **MJC Prelim 8865/2018/Q2(b)**

Find the range of values of k for the line $y = 2x + 3$ to intersect the curve $y = kx^2 + (2k+1)x - 3$ at least once. [4]

Answer: $k \le -4.95$ or $k \ge -0.0505$

Soln:

$kx^2 + (2k+1)x - 3 = 2x + 3$

$kx^2 + (2k+1)x - 3 - 2x - 3 = 0$

$kx^2 + (2k-1)x - 6 = 0 \cdots (1)$

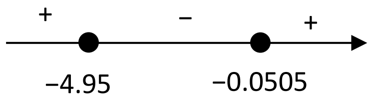
Since the line intersect the curve at least once, (1) has real solutions.

$\therefore$ Discriminant ≥ 0

$(2k-1)^2 - 4k(-6) \ge 0$

$4k^2 - 4k + 1 + 24k \ge 0$

$4k^2 + 20k + 1 \ge 0$



Using GC,

$k \le -4.95$ or $k \ge -0.0505$ (3 s.f.)

3. **DHS Prelim 8865/2018/Q1**

Show algebraically that for all real non-zero k, the line $y - kx = 1$ intersects the curve $y^2 - 3x - 2 = 0$ at two distinct points. [4]

Soln:

1 Substitute $y = kx + 1$ into $y^2 - 3x - 2 = 0$,

$(kx+1)^2 - 3x - 2 = 0$

$k^2x^2 + (2k-3)x - 1 = 0$

Discriminant = $(2k-3)^2 - 4(k^2)(-1)$

$= 8k^2 - 12k + 9$

$= 8\left(k^2 - \frac{3}{2}k\right) + 9$

$= 8\left(\left(k - \frac{3}{4}\right)^2 - \frac{9}{16}\right) + 9$

$= 8\left(k - \frac{3}{4}\right)^2 + \frac{9}{2} > 0$ for all $k \in \mathbb{R}$

Alternative:

Discriminant

$= (2k-3)^2 - 4(k^2)(-1)$

$= (2k-3)^2 + 4k^2 > 0$ for all $k \in \mathbb{R}$

Therefore, $k^2x^2 + (2k-3)x - 1 = 0$ has 2 distinct real roots for all $k \in \mathbb{R}$. Hence, the line $y - kx = 1$ will always intersect the curve $y^2 - 3x - 2 = 0$ at two distinct points.

4. **SAJC Prelim 8865/2018/Q2**

Two curves are given by the equation $\mathrm{f}(x) = ax^2 + 2x - 3$ and $\mathrm{g}(x) = -ax - 4$ respectively where $a \in \mathbb{R}$. Show algebraically that these two curves will intersect each other at 2 distinct points for all real values of *a*. [3]

Soln:

4	At intersection point(s): Equate the 2 equations: $ax^2 + 2x - 3 = -ax - 4$ Rearrange: $ax^2 + (a+2)x + 1 = 0 \ldots(1)$ Discriminant $= (a+2)^2 - 4(a)(1)$ $= a^2 + 4a + 4 - 4a$ $= a^2 + 4 > 0$ for all real values of a The 2 curves will intersect each other at 2 distinct points.

5. **VJC Prelim 8865/2018/Q1**

The curve $y = (k-6)x^2 - 5x$ has a minimum point. Find algebraically the set of values of *k* for which the curve intersects the line $y = 3x - k$ at two distinct points for all real values of *x*. [4]

Answer: $\{k \in \mathbb{R} : 6 < k < 8\}$

Soln:

	Since the curve has a minimum turning point, $k - 6 > 0 \Rightarrow k > 6$ $(k-6)x^2 - 5x = 3x - k$ $(k-6)x^2 - 8x + k = 0$ Since the line intersect the curve at two distinct points, (1) has real and distinct solutions. $\therefore$ Discriminant > 0 $(-8)^2 - 4(k-6)k > 0$ $64 - 4k^2 + 24k > 0$ $k^2 - 6k - 16 < 0$ $(k-8)(k+2) < 0$ $-2 < k < 8$ Since $k > 6$ and $-2 < k < 8 \Rightarrow 6 < k < 8$ The set of values of *k* is $\{k \in \mathbb{R} : 6 < k < 8\}$

Section 5: SOLE

1. **SAJC Prelim 8865/2018/Q1**

A company manufactures three different types of candy. Each candy is filled with either fruit, cream or nut. The candies are packed into three types of boxes and sold as follows:

Filling / Type of box	Fruit	Cream	Nut	Selling Price
Square	4	4	12	\$9.40
Heart	12	4	4	\$7.60
Round	8	8	8	\$11

The company manufactures 4800 fruit-filled candies, 4000 cream-filled candies and 5600 nut-filled candies weekly.

(i) Find the number of boxes of each type of candies the company manufactures each week. **[3]**

The manufacturing cost per candy is \$0.20 for fruit-filled candy, \$0.25 for cream-filled candy and \$0.30 for nut-filled candy.

(ii) Assuming that the cost of boxes are negligible, find the profit the company makes if all the boxes are sold each week. **[2]**

Answer: (i) 200 square boxes, 100 heart boxes and 350 round boxes (ii) \$2850

Soln:

(i)	Let x, y, z be the number of square, heart and round type boxes manufactured weekly respectively. $4x+12y+8z=4800$.....(1) $4x+4y+8z=4000$.....(2) $12x+4y+8z=5600$.....(3) Using GC, $x=200$, $y=100$, $z=350$ The company manufactures 200 square boxes, 100 heart boxes and 350 round boxes.
(ii)	Total Cost = $(4800\times0.20)+(4000\times0.25)+(5600\times0.30)=3640$ Total revenue $=(200\times9.4)+(100\times7.6)+(350\times11)=6490$ Profit = \$6490 – \$3640 = \$2850

2. EJC Prelim 8865/2018/Q2

Chia has a total of 100 blue, red and yellow matchsticks. All the matchsticks are indistinguishable apart from their colour. He uses all the blue matchsticks to form triangles, all the red matchsticks to form squares and all the yellow matchsticks to form pentagons as shown in the diagram.

The total number of triangles and pentagons formed is four times the number of squares formed. If Chia were to exchange $\frac{1}{5}$ of his yellow matchsticks for red matchsticks, he will have the same number of blue and red matchsticks. Assume that each matchstick can only be part of one shape. Find the number of matchsticks that Chia has for each colour and deduce the number of triangles, squares and pentagons formed. [5]

Answer: $r = 20$, $b = 30$ and $y = 50$; 5 squares, 10 triangles and 10 pentagons

Soln:

2	Let r, b and y be the number of red, blue and yellow matchsticks respectively. $r + b + y = 100 ---(1)$ $\frac{y}{5} + r = b \Rightarrow 5r - 5b + y = 0 ---(2)$ $\frac{b}{3} + \frac{y}{5} = 4\left(\frac{r}{4}\right) \Rightarrow 15r - 5b - 3y = 0 ---(3)$ Using GC, $r = 20$, $b = 30$ and $y = 50$ 5 squares, 10 triangles and 10 pentagons formed

3. VJC Prelim 8865/2018/Q5

A stall in a food bazaar sells three types of cupcakes: vanilla, red velvet and white chocolate. The price of a vanilla cupcake, red velvet cupcake and white chocolate cupcake is \$1.50, \$3.50 and \$3.00 respectively. On Monday, 400 cupcakes were baked. At the end of the day, $\frac{5}{9}$ of the vanilla cupcakes baked were sold and $\frac{3}{4}$ of the red velvet cupcakes baked were sold. The number of vanilla cupcakes sold was 70 more than the number of white chocolate cupcakes sold. The amount collected from selling these cupcakes was \$450.

If all the cupcakes were sold at the end of Monday, the stall would have collected \$970.

(i) By writing down three linear equations, find the number of each type of cupcake baked on Monday. [4]

(ii) Given that the production cost of a vanilla cupcake, red velvet cupcake and white chocolate cupcake is \$0.60, \$ 2.00 and \$1.80 respectively, find the profit earned by the stall on Monday and interpret the numerical value obtained in the context of the question. [2]

In order to attract more customers, the stall is trialling a new product – cookies. The cookies are sold by weight. It is predicted that the total profit \$$P$ will be related to the weight of cookies produced (x kg) by the equation

$$P=-10x^2+140x-400.$$

You may assume that all the cookies produced are sold.

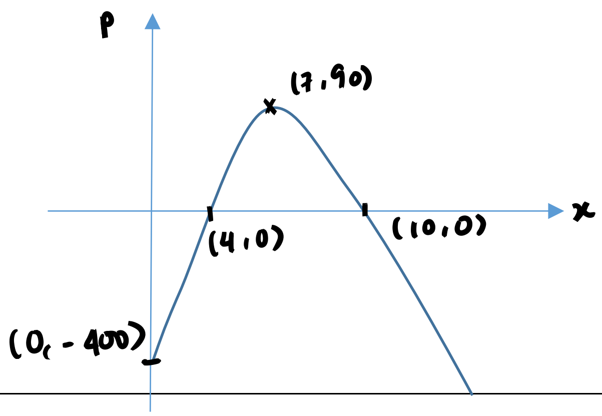
(iii) Sketch the graph of P against x, stating the coordinates of the intersections with the axes. [2]

(iv) State the weight of cookies produced when the profit is a maximum. Give this value of P. [2]

(v) Give an interpretation, in context, of the value of P when $x=0$. [1]

Answer: (i) $v=180, r=80, w=140$, (ii) -70, (iv) 7kg ; \$90

Soln:

(i)	Let v, r and w represent the number of vanilla cupcakes, red velvet cupcakes and white chocolate cupcakes **baked** respectively. $v+r+w=400-(1)$ $1.50v+3.50r+3.00w=970-(2)$ $\frac{5}{9}v(1.50)+\frac{3}{4}r(3.50)+\left(\frac{5}{9}v-70\right)(3.00)=450$ $\frac{5}{2}v+2.625r=660-(3)$ By GC, $v=180, r=80, w=140$
(ii)	Profit = $450-180(0.60)-80(2)-140(1.80)=-70$ The store made a loss of \$70 on Monday.
(iii)	

(iv) (v)	The weight of cookies produced is 7 kg. The maximum value of P is \$90. It is the fixed cost of the stall.

4. NYJC Prelim 8865/2018/Q1

David went to a seafood restaurant on three different days to eat lobster, fish and crab. He observed that the price per kilogram of lobster and fish remained constant for all his three visits and the price per kilogram of crab was the same for his first two visits but increased by 20% on his third visit. In addition, the restaurant gave a fifty dollars discount for any bill exceeding \$360. The mass of lobster, fish and crab that he ordered as well as the bill before discount for each visit are shown in the table below.

	First visit	Second visit	Third visit
Lobster (kg)	3.20	4.50	5.60
Fish (kg)	1.50	1.20	2.00
Crab (kg)	6.00	5.20	4.80
Bill after discount (\$), where applicable	289.39	309.43	322.76

Find the price per kilogram of lobster, fish and crab during his first visit to the restaurant. [4]

Answer: $x = 34.7, y = 12.9, z = 26.5$

Soln:

	Let x, y and z be the price per kilogram of the lobsters, fish and crabs respectively during his first visit. $3.20x + 1.50y + 6.00z = 289.39$ $4.50x + 1.20y + 5.20z = 309.43$ $5.60x + 2.00y + 4.80(1.2z) = 322.76 + 50$ Using GC, $x = 34.70, y = 12.90, z = 26.50$