Candidate Name: ______________________ Class: ____________

PIONEER JUNIOR COLLEGE
While I Live, I Learn

JC2 PRELIMINARY EXAM
Higher 2

MATHEMATICS **9740/01**

Paper 1

14 Sept 2015

3 hours

Additional Materials: Cover page
Answer papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your full name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 The volume of water V in a filtration tank at time t satisfies the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t}=5-kV\ ,$$

where k is a positive constant. Find V in terms of k and t, given that the tank is initially empty. [5]

State what happens to V for large values of t. [1]

2 **(i)** Given that $y=\mathrm{e}^{\sin^{-1}(2x)}$, show that $\left(1-4x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y^2$. [2]

(ii) By further differentiation of this result, find the first three terms of the Maclaurin series for y in ascending powers of x. [3]

(iii) Deduce the first three terms of the Maclaurin series for $\dfrac{\mathrm{e}^{\sin^{-1}(2x)}}{\cos x}$ in ascending powers of x. [3]

3 The curve C has equation

$$y=\frac{4x^2}{x-q}\ ,$$

where q is a non-zero constant.

It is given that C has a stationary point at $x=4$ and an asymptote $y=4x+r$, where r is a non-zero constant.

(i) Find the values of q and r. [3]

(ii) Sketch C, stating clearly the equations of its asymptotes, stationary points and the coordinates of any point(s) of intersection with the axes. [3]

(iii) State the set of values that y can take. [1]

(iv) Using the graph in part **(ii)**, find the range of values of a such that the equation

$$\left(x-2\right)^2+\left(\frac{4x^2}{x-q}-16\right)^2=a$$

has a negative real root. [2]

4 **(a)** State a sequence of transformations which transform the graph of $y = \ln x$ to the graph of $y = \ln(1-2x)$. [3]

(b) It is given that $\mathrm{f}(x) = 2a - x$, where $a > 0$.

By considering the graphs of $y = \left|\mathrm{f}(x) + a\right|$ and $y = \mathrm{f}(|x|) + a$, find the value of the constant k for which $\int_0^{4a} \left|\mathrm{f}(x) + a\right| \mathrm{d}x = k\int_{-2a}^{2a} \left(\mathrm{f}(|x|) + a\right) \mathrm{d}x$. [4]

5 Relative to the origin *O*, the position vectors of two points *A* and *B* are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $|\mathbf{a}| = 2$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{3\pi}{4}$ radians. Given that $\mathbf{a}$ and $2\mathbf{a} + \mathbf{b}$ are perpendicular, find

(i) the exact length of **b**, [3]

(ii) the exact length of projection of **a** onto **b**. [1]

The point *P* lies on *OB* such that the ratio $OP : OB = 3 : 5$.

(iii) Find the exact area of triangle *APB*. [4]

6 An art exhibition features sculptures of Singa and Nila, which are the mascots for the SEA Games held in Singapore in 1993 and 2015 respectively. The number of Singa and Nila sculptures corresponds to the number of sports contested at the two editions of the games respectively. The sculptures are of varying heights.

The Singa sculptures are displayed in a line such that the tallest Singa sculpture is in the middle. Starting from both ends of the line, the height of each subsequent Singa sculpture is 10 cm more than the preceding Singa sculpture, up to the middle Singa sculpture.

(i) Given that 29 sports were contested at the 1993 SEA Games and the total height of all the Singa sculptures is 3120 cm, find the height of the tallest and shortest sculptures. [3]

The Nila sculptures are displayed in order of descending height. The height of the tallest Nila sculpture is 210 cm. The height of each subsequent Nila sculpture is 5% shorter than the height of the preceding Nila sculpture.

(ii) Given that the shortest Nila sculpture is the only Nila sculpture to have a height of less than 35 cm, find the number of sports contested at the 2015 SEA Games. [2]

(iii) Find, to 2 decimal places, the height of the shortest Nila sculpture and the total height of all the Nila sculptures. [3]

7 The polynomial P(z) has real coefficients. The equation P(z) = 0 has a root $re^{i\theta}$, where $r > 0$ and $0 < \theta \le \pi$.

(i) Write down a second root in terms of r and θ, and hence show that a quadratic factor of P(z) is $z^2 - 2rz\cos\theta + r^2$. [3]

(ii) Solve the equation $z^4 = -625$, expressing the solutions in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(iii) Use your answers in parts **(i)** and **(ii)** to express $z^4 + 625$ as the product of two quadratic factors with real coefficients, giving each factor in non-trigonometrical form. [3]

8

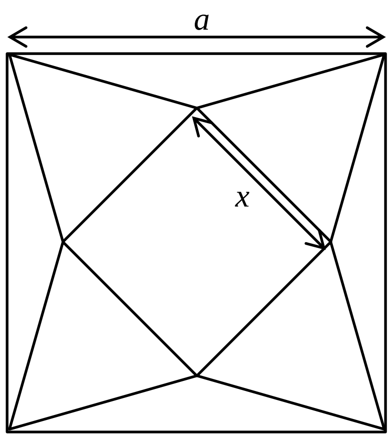


Fig.1

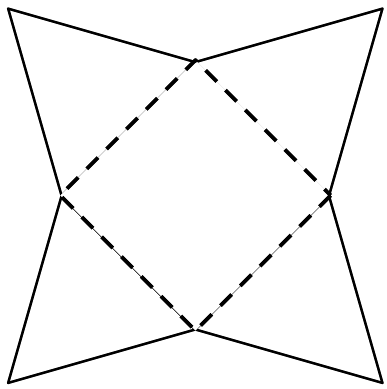


Fig. 2

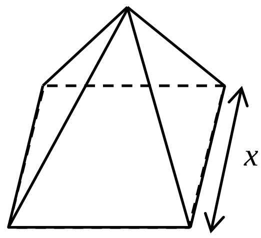


Fig.3

Fig. 1 shows a card in the form of a square of fixed side a. A triangle is cut from each side, to give the shape shown in Fig. 2. The remaining card shown in Fig. 2 is folded along the dotted lines, to form the right pyramid with square base of side x as shown in Fig. 3.

(i) Show that the volume V of the pyramid is given by $V = \dfrac{x^2}{3}\sqrt{\dfrac{a^2 - \sqrt{2}ax}{2}}$. [4]

(ii) Use differentiation to find in surd form the value of $\dfrac{a}{x}$ that gives a stationary value of V. [4]

9 **(a)** **(i)** By considering the derivative of e^{x^2}, find $\int x e^{x^2} \, dx$. [2]

(ii) Hence, find $\int x^3 e^{x^2} \, dx$. [3]

(b) Use the substitution $u = \sin^2 x$ to find

$$\int \sqrt{\frac{1-u}{u}} \, du .$$ [5]

10 Prove by mathematical induction that $\sum_{r=1}^{n} \sin(2r\theta) = \frac{\cos\theta - \cos(2n+1)\theta}{2\sin\theta}$ for all positive integers n. [6]

Hence, find an expression for $\sin\theta\cos\theta + \sin 2\theta\cos 2\theta + \ldots + \sin n\theta\cos n\theta$ in terms of θ and n. [2]

11 A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 1$ and $u_{r+1} = u_r + \frac{1}{(\ln k)^r}$, where k is a positive constant and $r \geq 1$.

(i) Using the method of differences, show that

$$u_n = \frac{1 - \left(\frac{1}{\ln k}\right)^n}{1 - \frac{1}{\ln k}} .$$ [4]

(ii) Given that u_n converges,

(a) state the limit of the sequence in terms of k, [1]

(b) find the range of values of k. [2]

12 **Do not use a graphing calculator in answering this question.**

The planes p_1 and p_2 have equations $2x - z = 3$ and $x - 3y = -3$ respectively. The point A with position vector $\lambda\mathbf{i} + \mathbf{j} + \mu\mathbf{k}$, where λ and μ are constants, is in both p_1 and p_2.

(i) Find the values of λ and μ. [2]

(ii) The planes p_1 and p_2 intersect in a line l. Find a vector equation of l. [2]

(iii) A third plane p_3 has equation $\mathbf{r}\cdot(\alpha\mathbf{i} + \mathbf{j}) = \beta$, where α and β are constants. Given that the three planes have no point in common, find the value of α. What can be said about the value of β? [3]

(iv) The point B has position vector $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Find the position vector of N, the foot of the perpendicular from B to p_1. Hence find the vector equation of the line of reflection of AB in p_1. [5]

Pioneer Junior College
H2 Mathematics **JC2-2015**
Prelim Exam Paper 1(Solution)

Q1

$\frac{\mathrm{d}V}{\mathrm{d}t}=5-kV,\ \ k>0$

$\int \frac{1}{5-kV}\,\mathrm{d}v=\int 1\ \mathrm{d}t$

$-\frac{1}{k}\ln\left|5-kV\right|=t+c$

$\ln\left|5-kV\right|=-kt-kc$

$5-kV=\pm e^{-kc}\mathrm{e}^{-kt}$

$5-kV=A\mathrm{e}^{-kt},\qquad A=\pm e^{-kc}$

$V=\frac{1}{k}\left(5-A\mathrm{e}^{-kt}\right)$

t = 0, V = 0.

Hence, $0=\frac{1}{k}(5-A)\Rightarrow A=5$.

$V=\frac{1}{k}\left(5-5\mathrm{e}^{-kt}\right)$

$V=\frac{5}{k}\left(1-\mathrm{e}^{-kt}\right)$

As $t\to\infty$, $V\to\frac{5}{k}$

Q2
(i)

$y=\mathrm{e}^{\sin^{-1}(2x)}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{\sin^{-1}(2x)}\left(\frac{2}{\sqrt{1-(2x)^2}}\right)$

$\sqrt{1-4x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x}=2y$

$\left(1-4x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y^2$ [Shown]

Alternative solution

$\ln y=\sin^{-1}(2x)$

differentiate wrt x:

$\frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{2}{\sqrt{1-(2x)^2}}$

$\sqrt{1-4x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x}=2y$

$\left(1-4x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y^2$ [Shown]

(ii)

Differentiate implicitly,

$$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\left(1-4x^2\right)-8x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=8y\frac{\mathrm{d}y}{\mathrm{d}x}$$

When $x=0,\ y=1,\ \frac{\mathrm{d}y}{\mathrm{d}x}=2\ ,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=4$

$$\therefore\ y=1+2x+\frac{x^2}{2}\left(4\right)+\ldots$$

$$=1+2x+2x^2+\ldots$$

$$\frac{\mathrm{e}^{\sin^{-1}(2x)}}{\cos x}=(1+2x+2x^2+\ldots)(1-\frac{x^2}{2}+\ldots)^{-1}$$

$$=(1+2x+2x^2+\ldots)\left[1+\frac{x^2}{2}\ldots\right]$$

$$=1+\frac{x^2}{2}+2x+2x^2+\ldots$$

$$=1+2x+\frac{5}{2}x^2+..$$

Q3

(i)

$$y=\frac{4x^2}{x-q}=4x+4q+\frac{4q^2}{x-q}$$

The oblique asymptote is $y=4x+4q$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=4-\frac{4q^2}{\left(x-q\right)^2}$$

When $x=4,\quad \frac{\mathrm{d}y}{\mathrm{d}x}=0$

Therefore,

$$4-\frac{4q^2}{\left(x-q\right)^2}=0$$

$$\left(\frac{2q}{4-q}\right)^2=4$$

$$\frac{2q}{4-q}=\pm2$$

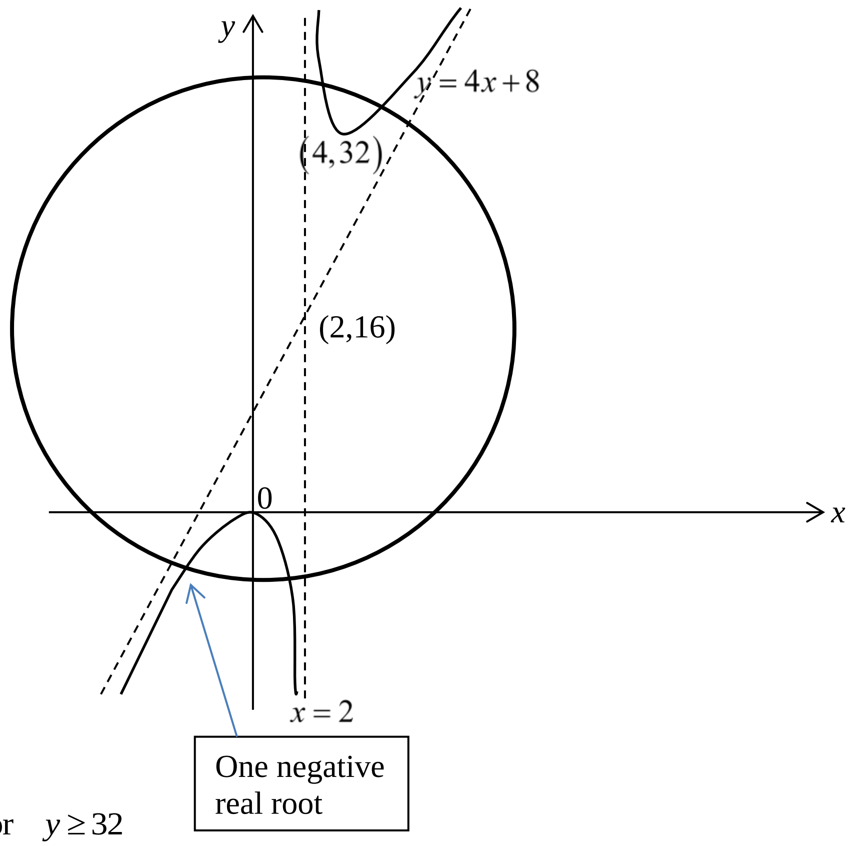
$\frac{2q}{4-q}=2$ *or* $\frac{2q}{4-q}=-2$ (no solution)

$q=2$

$r=4q=8$

$\therefore q=2, r=8$

(ii)



(iii)
$y \le 0$ or $y \ge 32$

(iv)

$$\left(x-2\right)^2 + \left(\frac{4x^2}{x-q} - 16\right)^2 = a$$

The equation given is actually $\left(x-2\right)^2 + \left(y-16\right)^2 = \left(\sqrt{a}\right)^2$

Therefore, the solution to the equation is given by the intersection between the $y = \dfrac{4x^2}{x-q}$ and circle with radius $\sqrt{a}$ centred at $\left(2,16\right)$ which is the point of intersection between the 2 asymptotes.

Distance between origin and $\left(2,16\right)$ = Distance between $\left(2,16\right)$ and $\left(4,32\right)$

$= \sqrt{2^2 + 16^2} = \sqrt{260}$

For the equation to have 1 negative real root the graphs must intersect at least once for $x < 0$.
Hence, $a > 260$

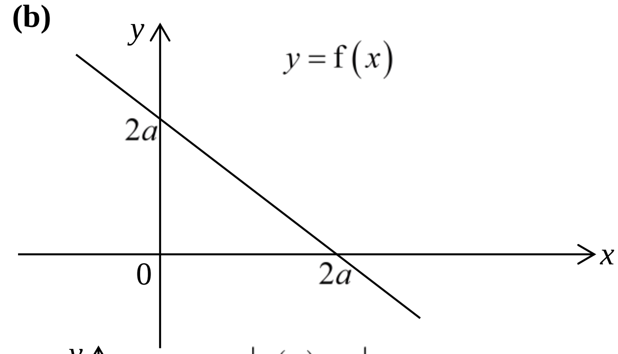
Q4

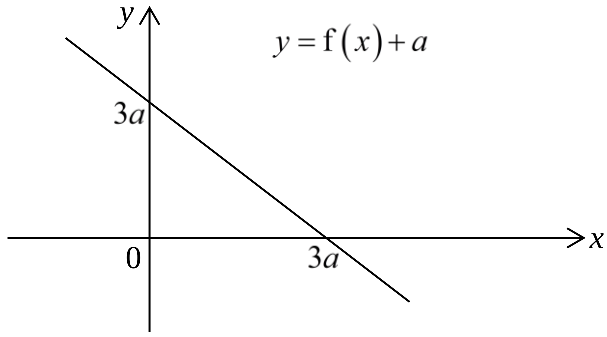
(a)

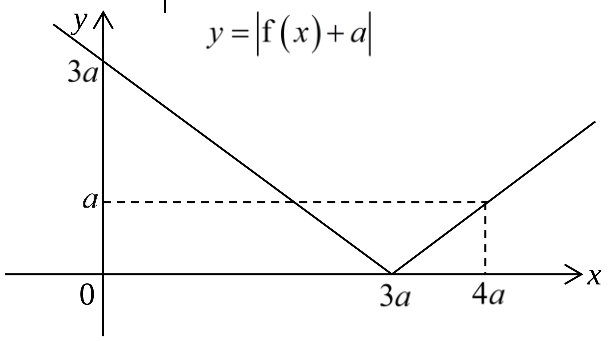
1. Translation by -1 unit in the direction of x-axis.
2. Scaling parallel to the x-axis with a factor of $\frac{1}{2}$.
3. Reflection about the y-axis.

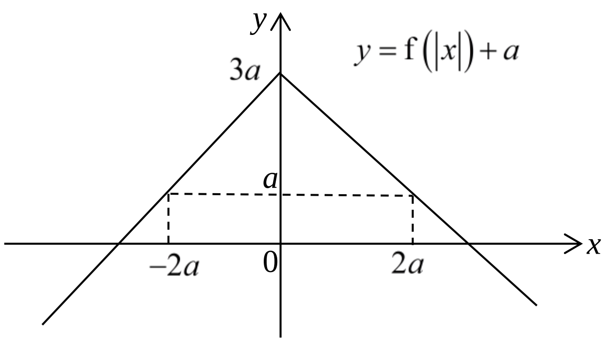
> Note: Can also do step 3 first followed by step 2. In fact there are many other possible ways.

(b)









$$\int_0^{4a} |\mathrm{f}(x) + a| \,\mathrm{d}x = \frac{1}{2}(3a)(3a) + \frac{1}{2}(a)(a) = 5a^2$$

$$\int_{-2a}^{2a} \left(\mathrm{f}(|x|) + a\right) \mathrm{d}x = 2\left(\frac{1}{2}(3a + a)(2a)\right)$$

$$= 8a^2$$

Therefore,

$$5a^2 = k\left(8a^2\right)$$

$$\therefore k = \frac{5}{8}$$

Q5

(i)
$\mathbf{a}$ and $2\mathbf{a}+\mathbf{b}$ are perpendicular

$\Rightarrow \mathbf{a}\cdot(2\mathbf{a}+\mathbf{b})=0$

$2\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}=0$

$2|\mathbf{a}|^2+\mathbf{a}\cdot\mathbf{b}=0$

$\mathbf{a}\cdot\mathbf{b}=-2|\mathbf{a}|^2=-8$

$|\mathbf{a}|\,|\mathbf{b}|\cos\left(\frac{3\pi}{4}\right)=-8$

$2|\mathbf{b}|\left(-\frac{1}{\sqrt{2}}\right)=-8$

$|\mathbf{b}|=4\sqrt{2}$

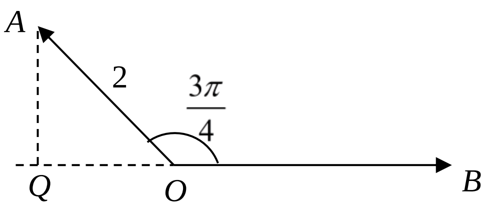
(ii)
Length of projection of $\mathbf{a}$ onto $\mathbf{b}$

$= |\mathbf{a}\cdot\hat{\mathbf{b}}|$

$= \left|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right|$

$= \left|\frac{-8}{4\sqrt{2}}\right|=\sqrt{2}$

Alternative method



Length of projection of $\mathbf{a}$ onto b

= OQ = $2\cos\frac{\pi}{4}=\sqrt{2}$

(iii)
$\overrightarrow{PB}=\frac{2}{5}\mathbf{b}$

Area of triangle APB = $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{PB}\right|$

$= \frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times\frac{2}{5}\mathbf{b}\right|$

$= \frac{1}{2}\left|\frac{2}{5}\mathbf{b}\times\mathbf{b}-\frac{2}{5}\mathbf{a}\times\mathbf{b}\right|$

$= \frac{1}{5}|\mathbf{a}\times\mathbf{b}|$

$= \frac{1}{5}|\mathbf{a}||\mathbf{b}|\sin\left(\frac{3\pi}{4}\right)$

$= \frac{1}{5}(2)(4\sqrt{2})(\frac{1}{\sqrt{2}})$

$= \frac{8}{5}$

Q6

(i)
Let the height of the shortest Singa sculpture be x.

Total height of all the Singa sculptures = $2\times\frac{14}{2}\left[2x+(14-1)(10)\right]+\left[x+(15-1)(10)\right]=3120$

$28x+1820+x+140=3120$
$29x=1160$
$x=40$

Height of the shortest Singa sculpture is 40 cm

Height of the tallest Singa sculpture = $40+(15-1)(10)=180$ cm

Alternative
Let y be the height of the tallest Singa sculpture.

Total height of all the Singa sculptures = $2\times\frac{14}{2}\left[2(y-10)+(14-1)(-10)\right]+y=3120$

$28y-280-1820+y=3120$
$29y=5220$
$y=180$
The height of the tallest Singa sculpture is 180 cm.

Height of shortest Singa sculpture = $180+(15-1)(-10)=40$ cm
or
Height of shortest Singa sculpture = $170+(14-1)(-10)=40$ cm

Alternative
Let the height of the shortest Singa sculpture be x.

Total height of all the Singa sculptures = $\frac{14}{2}\left[2x+(14-1)(10)\right]+\frac{15}{2}\left[2x+(15-1)(10)\right]=3120$

$x=40$
Height of the shortest Singa sculpture is 40 cm
Height of the tallest Singa sculpture = $40+(15-1)(10)=180$ cm

(ii)

$T_n < 35$

$210(0.95)^{n-1} < 35$

$(0.95)^{n-1} < \frac{1}{6}$

$(n-1)\ln 0.95 < \ln\left(\frac{1}{6}\right)$

$n-1 > \frac{\ln\left(\frac{1}{6}\right)}{\ln 0.95}$

$n > 35.93$

$n = 36$

The number of sports contested at the 2015 SEA Games is 36.

(iii)

$T_{36} = 210(0.95)^{36-1} = 34.88$

The height of the shortest Nila sculpture is 34.88 cm.

$S_{36} = \frac{210\left[1-(0.95)^{36}\right]}{1-0.95} = 3537.33$

The total height of all the Nila sculptures is 3537.33 cm.

Q7

(i)

$z = re^{i\theta}$ is a root $\Rightarrow$ $z = re^{-i\theta}$ is another root since P(z) has real coefficients.

A quadratic factor of $\mathrm{P}(z)$

$= \left(z - re^{i\theta}\right)\left(z - re^{-i\theta}\right)$

$= z^2 - zre^{-i\theta} - zre^{i\theta} + r^2$

$= z^2 - zr\left(e^{i\theta} + e^{-i\theta}\right) + r^2$

$= z^2 - zr\left(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)\right) + r^2$

$= z^2 - zr\left(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta\right) + r^2$

$= z^2 - 2rz\cos\theta + r^2$ (shown)

(ii)

$z^4 = -625 = 5^4 e^{i(\pi + 2n\pi)}$

$z = 5e^{i\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)}, n = 0, \pm 1, -2$

so $z = 5e^{i\left(-\frac{3\pi}{4}\right)}, 5e^{i\left(-\frac{\pi}{4}\right)}, 5e^{i\left(\frac{\pi}{4}\right)}, 5e^{i\left(\frac{3\pi}{4}\right)}$

(iii)

$$z^4+625=(z-5e^{i(\frac{\pi}{4})})(z-5e^{i(-\frac{\pi}{4})})(z-5e^{i(\frac{3\pi}{4})})(z-5e^{i(-\frac{3\pi}{4})})$$

$$=[z^2-(2)(5)z\cos\left(\frac{\pi}{4}\right)+5^2][z^2-(2)(5)z\cos\left(\frac{3\pi}{4}\right)+5^2]$$

$$=(z^2-5\sqrt{2}z+25)(z^2+5\sqrt{2}z+25)$$

Q8

(i)

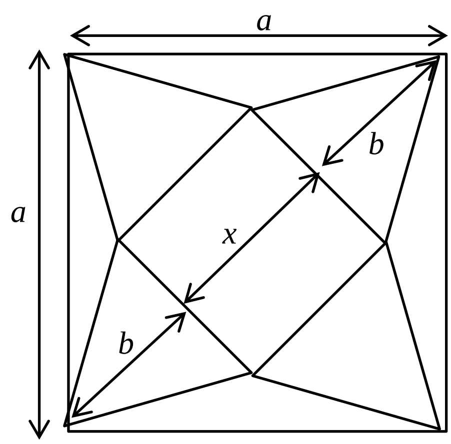
Consider the diagonal length of the card in Fig 1.

Diagonal length $=\sqrt{a^2+a^2}=\sqrt{2}a$

Let the height of the triangle side of the pyramid be b.

$$2b+x=\sqrt{2}a$$

$$b=\frac{\sqrt{2}a-x}{2}$$



Let the height of the pyramid be h.

By Pythagoras Theorem,

$$h^2+\left(\frac{x}{2}\right)^2=\left(\frac{\sqrt{2}a-x}{2}\right)^2$$

$$\Rightarrow h=\sqrt{\left(\frac{\sqrt{2}a-x}{2}\right)^2-\left(\frac{x}{2}\right)^2}$$

$$V=\frac{1}{3}x^2h$$

$$=\frac{x^2}{3}\sqrt{\left(\frac{\sqrt{2}a-x}{2}\right)^2-\left(\frac{x}{2}\right)^2}$$

$$=\frac{x^2}{3}\sqrt{\left(\frac{2a^2-2\sqrt{2}ax+x^2}{4}\right)-\left(\frac{x^2}{4}\right)}$$

$$=\frac{x^2}{3}\sqrt{\frac{a^2-\sqrt{2}ax}{2}}$$

(ii)

<u>Method 1</u>

$$\frac{dV}{dx}=\frac{2x}{3}\sqrt{\frac{a^2-\sqrt{2}ax}{2}}+\frac{x^2}{3}\frac{1}{2}\left(\frac{a^2-\sqrt{2}ax}{2}\right)^{-\frac{1}{2}}\left(-\frac{\sqrt{2}a}{2}\right)$$

For stationary value of V, $\frac{dV}{dx}=0$

$$\frac{2x}{3}\left(\frac{a^2-\sqrt{2}ax}{2}\right)=\frac{x^2}{3}\frac{1}{2}\left(\frac{\sqrt{2}a}{2}\right)$$

$a-\sqrt{2}x=\frac{\sqrt{2}}{4}x$ *or* $x=0$ (N.A as $x\neq 0$)

$$\frac{a}{x}=\frac{5\sqrt{2}}{4}$$

Method 2

$$V^2=\frac{x^4}{9}\left(\frac{a^2-\sqrt{2}ax}{2}\right)$$

$$=\frac{a^2x^4-\sqrt{2}ax^5}{18}$$

$$2V\frac{dV}{dx}=\frac{4a^2x^3-5\sqrt{2}ax^4}{18}$$

When $\frac{dV}{dx}=0$,

$$4a^2x^3-5\sqrt{2}ax^4=0$$

$$ax^3\left(4a-5\sqrt{2}x\right)=0$$

$ax^3 4a-5\sqrt{2}x=0$ or $ax^3=0$ (NA as $x\neq 0$)

$$4a=5\sqrt{2}x$$

$$\frac{a}{x}=\frac{5\sqrt{2}}{4}$$

Q9

(a)(i)

Consider $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}\right)=2x\mathrm{e}^{x^2}$

Therefore, $\int x\mathrm{e}^{x^2}\,\mathrm{d}x=\frac{1}{2}\mathrm{e}^{x^2}+C$

(ii)

$$\int x^3\mathrm{e}^{x^2}\,\mathrm{d}x$$

$$=\int x^2\left(x\mathrm{e}^{x^2}\right)\mathrm{d}x$$

$$=\frac{1}{2}x^2\mathrm{e}^{x^2}-\int xe^{x^2}\,\mathrm{d}x$$

$$=\frac{1}{2}x^2\mathrm{e}^{x^2}-\frac{1}{2}\mathrm{e}^{x^2}+C$$

$$=\frac{1}{2}x^2\mathrm{e}^{x^2}-\frac{1}{2}\mathrm{e}^{x^2}+C$$

Let $u=x^2$ $\quad\frac{\mathrm{d}v}{\mathrm{d}x}=x\mathrm{e}^{x^2}$

$\frac{\mathrm{d}u}{\mathrm{d}x}=2x$ $\quad v=\frac{1}{2}\mathrm{e}^{x^2}$

(b)

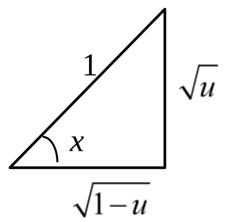
$u = \sin^2 x$

$\dfrac{du}{dx} = 2\sin x \cos x$

$$\int \sqrt{\frac{1-u}{u}}\ du$$
$$= \int \sqrt{\frac{1-\sin^2 x}{\sin^2 x}}\left(2\sin x\cos x\right)\ dx$$
$$= \int \frac{\cos x}{\sin x}\left(2\sin x\cos x\right)\ dx$$
$$= \int 2\cos^2 x\, dx$$
$$= \int \cos 2x + 1\, dx$$
$$= \frac{\sin 2x}{2} + x + C$$
$$= \sin x\cos x + x + C$$
$$= \sqrt{u}\sqrt{1-u} + \sin^{-1}\sqrt{u} + C$$
$$= \sqrt{u-u^2} + \sin^{-1}\sqrt{u} + C$$

$u = \sin^2 x \Rightarrow \sin x = \sqrt{u}$

$\Rightarrow x = \sin^{-1}\sqrt{u}$



Q10

Let P_n be the statement $\displaystyle\sum_{r=1}^{n}\sin(2r\theta) = \frac{\cos\theta - \cos\left[(2n+1)\theta\right]}{2\sin\theta}$, $n = 1, 2, 3, \ldots$

$n = 1$:

$$\text{LHS} = \sum_{r=1}^{1}\sin(2r\theta) = \sin(2\theta)$$

$$\text{RHS} = \frac{\cos\theta - \cos(3\theta)}{2\sin\theta}$$
$$= \frac{-2\sin\left[\frac{1}{2}(\theta+3\theta)\right]\sin\left[\frac{1}{2}(\theta-3\theta)\right]}{2\sin\theta}$$
$$= \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin\theta}$$
$$= \sin(2\theta)$$

$\text{LHS} = \text{RHS}$, $\therefore P_1$ is true.

Assume P_k is true for some $k = 1, 2, 3\ldots\ldots$.

i.e. $\displaystyle\sum_{r=1}^{k}\sin(2r\theta) = \frac{\cos\theta - \cos\left[(2k+1)\theta\right]}{2\sin\theta}$

To show P_{k+1} is true.

i.e. $\displaystyle\sum_{r=1}^{k+1}\sin(2r\theta) = \frac{\cos\theta - \cos\left[(2k+3)\theta\right]}{2\sin\theta}$

$$\text{LHS} = \sum_{r=1}^{k+1} \sin\left(2r\theta\right)$$

$$= \sum_{r=1}^{k} \sin\left(2r\theta\right) + \sin\left[\left(2k+2\right)\theta\right]$$

$$= \frac{\cos\theta - \cos\left[\left(2k+1\right)\theta\right]}{2\sin\theta} + \sin\left[\left(2k+2\right)\theta\right]$$

$$= \frac{\cos\theta - \cos\left[\left(2k+1\right)\theta\right] + 2\sin\left[\left(2k+2\right)\theta\right]\sin\theta}{2\sin\theta}$$

$$= \frac{\cos\theta - \cos\left[\left(2k+1\right)\theta\right] - \cos\left[\left(2k+3\right)\theta\right] + \cos\left[\left(2k+1\right)\theta\right]}{2\sin\theta}$$

$$= \frac{\cos\theta - \cos\left[\left(2k+3\right)\theta\right]}{2\sin\theta} = \text{RHS}$$

$\therefore P_{k+1}$ is true

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is also true, by the principle of Mathematical Induction, P_n is true for all $n = 1, 2, 3.......$

$$\sum_{r=1}^{n} \sin(2r\theta) = \frac{\cos\theta - \cos(2n+1)\theta}{2\sin\theta}$$

$$\sum_{r=1}^{n} 2\sin(r\theta)\cos(r\theta) = \frac{\cos\theta - \cos(2n+1)\theta}{2\sin\theta}$$

$$2(\sin\theta\cos\theta + \sin 2\theta\cos 2\theta + ... + \sin n\theta\cos n\theta) = \frac{\cos\theta - \cos(2n+1)\theta}{2\sin\theta}$$

$$\sin\theta\cos\theta + \sin 2\theta\cos 2\theta + ... + \sin n\theta\cos n\theta = \frac{\cos\theta - \cos(2n+1)\theta}{4\sin\theta}$$

Note:

$2\sin\theta\sin(2k+2)\theta$

$= -\cos\left[\left(2k+3\right)\theta\right] + \cos\left[\left(-2k-1\right)\theta\right]$

$= -\cos\left[\left(2k+3\right)\theta\right] + \cos[-(2k+1)\theta]$

$= -\cos\left[\left(2k+3\right)\theta\right] + \cos[(2k+1)\theta]$

Factor formula should be applied on $2\sin\theta\sin(2k+2)\theta$ to avoid the negative angle.

Q11

(i)

Using $u_{r+1} = u_r + \dfrac{1}{(\ln k)^r}, \quad r \geq 1$

$$\sum_{r=1}^{n-1}\left(u_{r+1} - u_r\right) = \sum_{r=1}^{n-1}\frac{1}{(\ln k)^r}$$

$$\left\{\begin{array}{l} \cancel{u_2} - u_1 \\ +\cancel{u_3} - \cancel{u_2} \\ +\cancel{u_4} - \cancel{u_3} \\ \quad\vdots \\ +\cancel{u_{n-2}} - \cancel{u_{n-3}} \\ +\cancel{u_{n-1}} - \cancel{u_{n-2}} \\ +u_n - \cancel{u_{n-1}} \end{array}\right. = \frac{\dfrac{1}{\ln k}\left(1-\left(\dfrac{1}{\ln k}\right)^{n-1}\right)}{1-\dfrac{1}{\ln k}}$$

$$u_n - 1 = \frac{\dfrac{1}{\ln k} - \left(\dfrac{1}{\ln k}\right)^n}{1-\dfrac{1}{\ln k}}$$

$$u_n = \frac{\dfrac{1}{\ln k} - \left(\dfrac{1}{\ln k}\right)^n}{1-\dfrac{1}{\ln k}} + 1$$

$$= \frac{\dfrac{1}{\ln k} - \left(\dfrac{1}{\ln k}\right)^n + 1 - \dfrac{1}{\ln k}}{1-\dfrac{1}{\ln k}}$$

$$= \frac{1-\left(\dfrac{1}{\ln k}\right)^n}{1-\dfrac{1}{\ln k}}$$

(ii)
(a)

$$\therefore u_n \to \frac{1}{1-\dfrac{1}{\ln k}} = \frac{\ln k}{\ln k - 1}$$

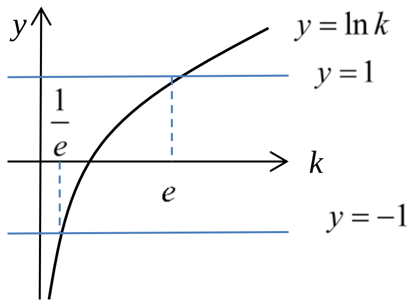
$$\lim_{n\to\infty} u_n = \frac{\ln k}{\ln k - 1}$$

Note: As $n \to \infty, \left(\dfrac{1}{\ln k}\right)^n \to 0$,

therefore sequence converges and limit exists.

(b)
Since u_n converges,

$$\left|\frac{1}{\ln k}\right|<1$$
$$1<\left|\ln k\right|$$
$$\ln k<-1 \qquad \text{or} \qquad \ln k>1$$
$$0<k<\frac{1}{e} \qquad\qquad k>e$$



Q12

(i)

Subst coordinates of A into the Cartesian equations of p_1 and p_2
$2\lambda-\mu=3$ ….. (1)
$\lambda-3=-3$ ….(2)
From (2), $\lambda=0 \Rightarrow \mu=-3$

(ii)

Direction vector of l is given by $\begin{pmatrix}2\\0\\-1\end{pmatrix}\times\begin{pmatrix}1\\-3\\0\end{pmatrix}=\begin{pmatrix}-3\\-1\\-6\end{pmatrix}=-\begin{pmatrix}3\\1\\6\end{pmatrix}$

Point $A\left(0,1,-3\right)$ lies on both p_1 and p_2 and hence on l.

Vector equation of line l is $\mathbf{r}=\begin{pmatrix}0\\1\\-3\end{pmatrix}+t\begin{pmatrix}3\\1\\6\end{pmatrix}, \quad t\in\mathbb{R}$.

(iii)
Since the 3 planes have no common point, l must be parallel to p_3 but not contained in p_3.

$$\therefore \quad \begin{pmatrix}\alpha\\1\\0\end{pmatrix}\cdot\begin{pmatrix}3\\1\\6\end{pmatrix}=0\Rightarrow 3\alpha+1=0$$

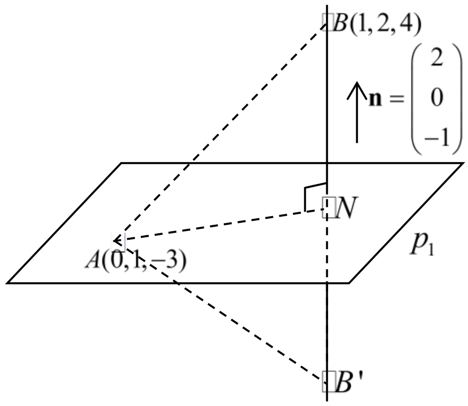
$$\alpha=-\frac{1}{3}$$

$$\begin{pmatrix}0\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}\alpha\\1\\0\end{pmatrix}\neq\beta$$

$$\therefore \quad \beta\neq1$$

(ii)

Equation of line through B and perpendicular to p_1 is



$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + s\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \quad s \in \mathbb{R}$$

This line intersects p_1 at N. Hence

$$\begin{pmatrix} 1+2s \\ 2 \\ 4-s \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = 3$$

$$2 + 4s - 4 + s = 3$$

$$s = 1$$

$$\therefore \quad \overrightarrow{ON} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

Let B' be the image of B upon reflection in p_1

By ratio theorem

$$\overrightarrow{ON} = \frac{\overrightarrow{OB} + \overrightarrow{OB'}}{2}$$

$$\overrightarrow{OB'} = 2\overrightarrow{ON} - \overrightarrow{OB} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB'} = \overrightarrow{OB'} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$$

Equation of reflected line is $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \phi\begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}, \quad \phi \in \mathbb{R}$

Alternative method

By ratio theorem'

$$\overrightarrow{AN} = \frac{\overrightarrow{AB} + \overrightarrow{AB'}}{2}$$

$$\overrightarrow{AB'} = 2\overrightarrow{AN} - \overrightarrow{AB} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$$

Equation of reflected line is

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \phi\begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}, \quad \phi \in \mathbb{R}$$