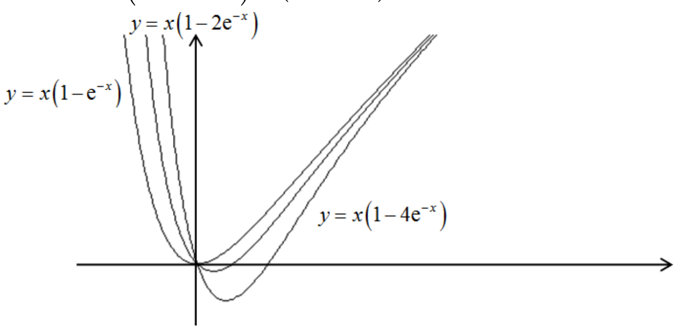
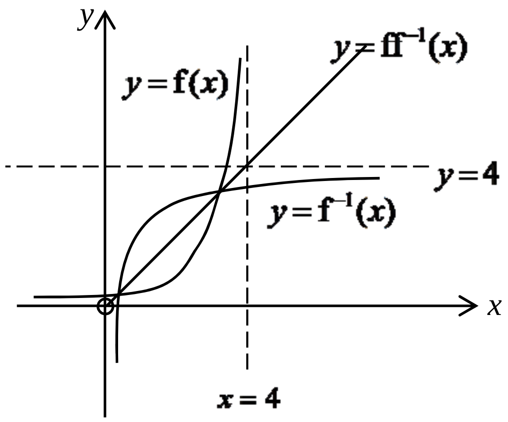
Prelim 2 Paper 2 Suggested Solutions

Qn	Suggested Solutions
1(i)	$y=\left(\cos^{-1}x\right)^2$ $\dfrac{dy}{dx}=2\left(\cos^{-1}x\right).\dfrac{-1}{\sqrt{1-x^2}}$ $\sqrt{1-x^2}\dfrac{dy}{dx}=-2\cos^{-1}x$ diff wrt x, $\sqrt{1-x^2}\dfrac{d^2y}{dx^2}+\dfrac{-2x}{2\sqrt{1-x^2}}\dfrac{dy}{dx}=-2\left(\dfrac{-1}{\sqrt{1-x^2}}\right)$ $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2$ (shown) **Alternative** $y=\left(\cos^{-1}x\right)^2$ $\dfrac{dy}{dx}=2\left(\cos^{-1}x\right).\dfrac{-1}{\sqrt{1-x^2}}$ diff wrt x, $\dfrac{d^2y}{dx^2}=2.\dfrac{-1}{\sqrt{1-x^2}}.\dfrac{-1}{\sqrt{1-x^2}}+(-2).\left(\cos^{-1}x\right).\dfrac{-1}{2}\left(1-x^2\right)^{-\frac{3}{2}}(-2x)$ $\dfrac{d^2y}{dx^2}=\dfrac{2}{\left(1-x^2\right)}+(-2x).\left(\cos^{-1}x\right).\dfrac{1}{\left(1-x^2\right)\sqrt{1-x^2}}$ $\left(1-x^2\right)\dfrac{d^2y}{dx^2}=2+(-2).\left(\cos^{-1}x\right).\dfrac{x}{\sqrt{1-x^2}}$ $\left(1-x^2\right)\dfrac{d^2y}{dx^2}=2+x\dfrac{dy}{dx}$ $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2$ (shown)
(ii)	diff wrt x, $\left(1-x^2\right)\dfrac{d^3y}{dx^3}+(-2x)\dfrac{d^2y}{dx^2}-x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}=0$ $\left(1-x^2\right)\dfrac{d^3y}{dx^3}-(3x)\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}=0$

	when $x=0$, $y=\left(\frac{\pi}{2}\right)^2$, $\frac{dy}{dx}=-\pi$, $\frac{d^2y}{dx^2}=2$, $\frac{d^3y}{dx^3}=-\pi$, $y=\left(\frac{\pi}{2}\right)^2+(-\pi)x+\frac{2}{2!}x^2+\frac{-\pi}{3!}x^3+\ldots$ $=\frac{\pi^2}{4}-\pi x+x^2-\frac{\pi}{6}x^3+\ldots$
(iii)	$e^{\left(\cos^{-1}x\right)^2}=e^{\frac{\pi^2}{4}-\pi x+x^2-\frac{\pi}{6}x^3}=e^{\frac{\pi^2}{4}}.e^{-\pi x+x^2-\frac{\pi}{6}x^3}$ $=e^{\frac{\pi^2}{4}}.\left[1+\left(-\pi x+x^2-\frac{\pi}{6}x^3\right)+\frac{1}{2}\left(-\pi x+x^2-\frac{\pi}{6}x^3\right)^2+\ldots\right]$ $=e^{\frac{\pi^2}{4}}.\left[1+\left(-\pi x+x^2-\frac{\pi}{6}x^3\right)+\frac{1}{2}(-\pi x)^2\ldots\right]$ $=e^{\frac{\pi^2}{4}}.\left[1-\pi x+x^2-\frac{\pi}{6}x^3+\frac{1}{2}\pi^2x^2\ldots\right]$ $=e^{\frac{\pi^2}{4}}.\left[1-\pi x+\left(1+\frac{1}{2}\pi^2\right)x^2+\ldots\right]$ $\therefore a=-\pi$ and $b=1+\frac{1}{2}\pi^2$

Qn	**Suggested Solutions**
2	$y = vx$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$ $x\frac{dy}{dx} - y = x(x-y)$ $x\left(v + x\frac{dv}{dx}\right) - vx = x(x - vx)$ $v + x\frac{dv}{dx} - v = x - vx$ $x\frac{dv}{dx} = x(1-v)$ $\frac{dv}{dx} = 1 - v$ $\Rightarrow \int \frac{1}{1-v}\,dv = \int dx$ $-\ln\lvert 1-v\rvert = x + C$ $\lvert 1-v\rvert = e^{-x-C}$ $1 - v = \pm e^{-C}e^{-x}$ $v = 1 - Ae^{-x}$ $y = x(1 - Ae^{-x})$ (shown)  At the stationary point, $\frac{dy}{dx} = 0$. Hence, equation of locus is: $x(0) - y = x(x-y)$ $y = -x(x-y)$ $y = -x^2 + xy$ i.e., $y = \frac{x^2}{x-1}$

Qn	Suggested Solutions
3(i)	Let $y=\frac{5}{(x-4)^2}$. $(x-4)^2=\frac{5}{y}$ $x-4=\pm\sqrt{\frac{5}{y}}$ $x=\pm\sqrt{\frac{5}{y}}+4$ (reject positive sq root $\because x<4$) $x=4-\sqrt{\frac{5}{y}}$ $\therefore \mathrm{f}^{-1}(x)=4-\sqrt{\frac{5}{x}}$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=(0,\infty)$
(iii)	
(iv)	Solve $\mathrm{f}(x)=x$. $\frac{5}{(x-4)^2}=x$ $5=x(x-4)^2$ $5=x(x^2-8x+16)$ $x^3-8x^2+16x-5=0$ (shown) $x^3-8x^2+16x-5=0$ $(x-5)(x^2-3x+1)=0$ $x=5,\ x=\frac{3\pm\sqrt{9-4}}{2}$ $x=5,\ x=\frac{3+\sqrt{5}}{2}$ or $x=\frac{3-\sqrt{5}}{2}$ (rej as $x<4$)

4(i)	$\overrightarrow{OA} = -6\underset{\sim}{i} - 3\underset{\sim}{j} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix}, \overrightarrow{OC} = -\overrightarrow{OA} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}, \overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ $\therefore \overrightarrow{OM} = \frac{1}{2}\left[\begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}\right] = \begin{pmatrix} 3 \\ \frac{3}{2} \\ 3 \end{pmatrix}$ $\therefore \overrightarrow{AM} = \begin{pmatrix} 3 \\ \frac{3}{2} \\ 3 \end{pmatrix} - \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ \frac{9}{2} \\ 3 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ Hence, equation of the line *AM* is $\mathbf{r} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}$
(ii)	$\overrightarrow{AB} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$ Length of projection of $\overrightarrow{AB}$ onto the line *AM*, $AN = \frac{\left\lvert \overrightarrow{AB} \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \right\rvert}{\sqrt{6^2 + 3^2 + 2^2}}$ $= \frac{\left\lvert \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \right\rvert}{7} = \frac{72}{7}$ Perpendicular distance from *P* to the line *AM* $= \sqrt{(AB)^2 - (AN)^2}$ $= \sqrt{12^2 - \left(\frac{72}{7}\right)^2} = 6.18 \text{ (3 s.f.)}$

	<u>Alternative Method</u> Perpendicular distance from P to the line AM $= \dfrac{\left\| \overrightarrow{AB} \times \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \right\|}{\sqrt{6^2+3^2+2^2}} = \dfrac{\left\| \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \right\|}{\sqrt{49}}$ $= \dfrac{12 \left\| \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \right\|}{7} = \dfrac{12\sqrt{4+9}}{7}$ $= 6.18 \ (3 \text{ s.f.})$
(iii)	A normal vector to the plane AMB $= \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = 12 \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$ $\overrightarrow{DV} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} -6 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ Let θ be that angle betw. the line DV and plane AMB. $\sin\theta = \dfrac{\left\| \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \right\|}{\sqrt{4+1+4}\sqrt{4+9}}$ $= \dfrac{2+6}{3\sqrt{13}}$ $\theta = 47.7^\circ \ (1 \text{ d.p.})$
(iv)	If the 3 planes AMB, AMD and Π do not have a common point, the line AM is parallel to Π but does not lie in Π. $\therefore \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ a \end{pmatrix} = 0$ $\Rightarrow -6+12+2a = 0$ $\Rightarrow a = -3$ Note that $\begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ a \end{pmatrix} = 6-12 \neq 4$, Therefore point A does not lie in Π. Hence the line AM does not lie in Π.

Qn	Suggested Solutions
5	Let X be the random variable denoting the volume of hot chocolate. $\mathrm{E}(X)=55$, $\mathrm{Var}(X)=10^2$ By Central Limit Theorem, $\overline{X} \sim N\left(55, \frac{10^2}{n}\right)$ approximately. $P\left(\overline{X} > 54\right) > 0.77$ $P\left(Z > \frac{54-55}{10/\sqrt{n}}\right) > 0.77$ $P\left(Z < \frac{-\sqrt{n}}{10}\right) < 0.23$ $\frac{\sqrt{n}}{10} > 0.7388$ $\sqrt{n} > 10(0.7388) = 7.388$ $n > 54.59$ $\therefore$ the least $n = 55$ Alternative Method

Qn	Suggested Solutions
6(i)	• Obtain a list of households, in order of surnames in that particular constituency. • Randomly select a starting point in the first 20 households on the list and thereafter select every 20^{th} household on the list to be interviewed to get their responses.
6(iii)	A better sampling method is stratified sampling. Obtain a list of all households and divide the households according to the different types of housing. Select a random sample from each stratum such that the sample size is proportional to the relative size of each type of housing. Thus, the sample obtained is more representative of the population.

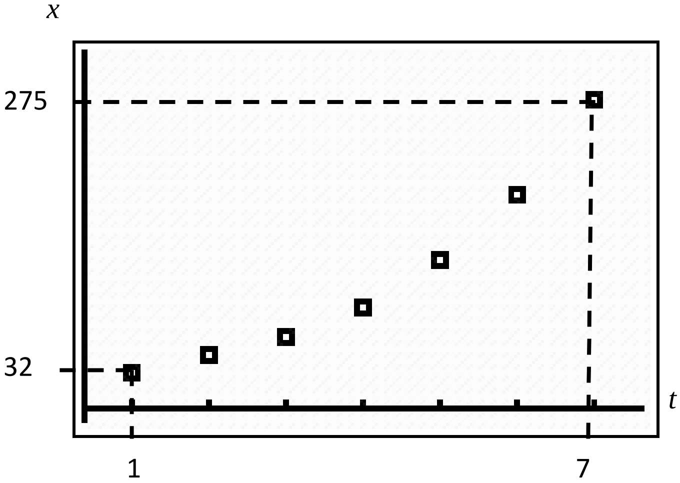
Qn	Suggested Solutions
7(i)	Let X be the random variable denoting the number of visitors to the blog in two days. $X \sim \text{Po}(6.4)$ $\mathrm{P}(5 \le X < 10)$ $= \mathrm{P}(X \le 9) - \mathrm{P}(X \le 4)$ $= 0.650729 = 0.651\ (3\text{sf})$

(ii)	Let Y be the random variable denoting the number of visitors to the blog in a week. $Y \sim \text{Po}(22.4)$ Since $\lambda = 22.4 > 10$ $Y \sim \text{N}(22.4, 22.4)$ approx $\text{P}(Y > 30) = \text{P}(Y > 30.5)$ with cc $= 0.043500 = 0.0435$ (3 sf)
(iii)	The mean no. of visitors may not be the same for each day, e.g. weekends or during school holidays, mean no. of visitors may be higher.

Qn	**Suggested Solutions**
8 (i)	no. of ways the married couple seated together = 2! x 6! = 1440 Total no. of arrangement, without restriction = 7! P(no. of ways the married couple seated together) = 1440/7!=2/7
(ii)	P(all women sat together | couple sat together) $= \dfrac{\text{P(all women sat together and couple sat together)}}{\text{P(couple sat together)}}$ $= \dfrac{(6)3!2!}{6!2!}$ $= \dfrac{1}{20}$
(iii)	no. of ways men and women alternate $= 4! \times 3! = 144$ P(men and women alternate)= $\dfrac{144}{7!} = \dfrac{1}{35}$
(iv)	no. of ways to arrange around a table with no restriction = (7-1)! = 720 no. of ways to arrange around a table with one particular woman must sit between two men = ${}^{3}P_{2}$ x (5-1)! = 144 P(no. of ways to arrange around a table with one particular woman must sit between two men) = 144/720=1/5

Qn	**Suggested Solutions**
9(i)	Let X be the random variable denoting the number of students who are awarded distinction out of 23 students. $X \sim \mathrm{B}(23, 0.2)$ $\mathrm{E}(X) = np = 4.6$
(ii)	Let Y be the random variable denoting the number of students who are awarded distinction out of n students. $Y \sim \mathrm{B}(n, 0.2)$ $\mathrm{P}(Y \geq 9) > 0.7$ $1 - \mathrm{P}(Y \leq 8) > 0.7$ $\mathrm{P}(Y \leq 8) < 0.3$ From GC, when $n = 50$, $\mathrm{P}(Y \leq 8) = 0.30733$ $n = 51$, $\mathrm{P}(Y \leq 8) = 0.28395$ $\therefore$ smallest $n = 51$
(iii)	Let S be the random variable denoting the number of students who pass out of 60 students. $S \sim \mathrm{B}(60, 0.93)$ $n = 60$ is large, $np = 55.8 > 5$, $nq = 4.2 < 5$ $S' \sim \mathrm{B}(60, 0.07)$ Since $n = 60$ is large, $np = 4.2 < 5$, $S' \sim \mathrm{Po}(4.2)$ approx. $\mathrm{P}(S > 55) = \mathrm{P}(S' < 5)$ $= \mathrm{P}(S' \leq 4) = 0.58982 = 0.590$

Qn	**Suggested Solutions**
10	Let A and C be the random variables denoting the mass of a randomly chosen almond cookie and chocolate chip cookie respectively. $A \sim \text{N}\left(32, 3.0^2\right)$ $C \sim \text{N}\left(28, 1.8^2\right)$
(i)	$\text{E}(C_1+C_2+C_3+C_4) = (4)(28) = 112$ $\text{Var}(C_1+C_2+C_3+C_4) = (4)(1.8^2)$ $C_1+C_2+C_3+C_4 \sim \text{N}\left(112, 12.96\right)$ $P\left(100 < C_1+C_2+C_3+C_4 < 120\right)$ $= 0.9864368$ $= 0.986\left(3sf\right)$
(ii)	$\text{E}(C_1+C_2+C_3+C_4-2A) = 4E(C)-2E(A) = (4)(28)-2(32) = 48$ $\text{Var}(C_1+C_2+C_3+C_4-2A) = 4Var(C)+2^2Var(A) = (4)(1.8^2)+2^2(3.0^2) = 48.96$ $C_1+C_2+C_3+C_4-2A \sim \text{N}\left(48, 48.96\right)$ $\text{P}\left(-50 < C_1+C_2+C_3+C_4-2A < 50\right)$ $= 0.6124961464 = 0.612\left(3\text{sf}\right)$
(iii)	Let $X = \frac{6}{100}(A_1+A_2+A_3+A_4)+\frac{5}{100}\left(C_1+C_2+C_3+C_4\right)$ $-\frac{5}{100}\left(C_{11}+C_{12}+\ldots+C_{20}\right)$ $\text{E}(X) = \frac{6}{100}(4)\text{E}(A)-\frac{5}{100}(6)\text{E}(C) = -0.72$ $\text{Var}(X) = \left(\frac{6}{100}\right)^2(4)\text{Var}(A)+\left(\frac{5}{100}\right)^2(14)\text{Var}(C) = 0.243$ $X \sim \text{N}\left(-0.72, 0.243\right)$ $P\left(X > 0\right) = 0.0720635555 = 0.0721\left(3sf\right)$

Qn	Suggested Solutions
11 (i)	 From the scatter plot, over time, there is an increase in the rate of increase of the no. of mosquitoes. Hence it shows a curvilinear relationship between t and x. A linear model will not be appropriate.
(ii)	$x = ae^{bt}$ $\ln x = \ln a + bt$ From GC, $\ln a = 3.114769038$ $b = 0.3555454706 = 0.356$ (3sf) $\ln x = 3.11 + 0.356\ t$ $\therefore a = e^{3.114769038} = 22.52822659 = 22.5$ (3sf)
(iii)	When $t = 30$, $\ln x = 13.78113316$ $x = 966207$
(iv)	The calculated value for x is unreliable since $t = 30$ is outside the data range of t.

Qn	Suggested Solutions
12(i)	unbiased estimate of population mean, $\bar{x} = \frac{\sum(x-1200)}{120} + 1200 = \frac{-60}{120} + 1200 = 1199.5$ unbiased estimate of population variance, $s^2 = \frac{1}{119}\left[2014 - \frac{(-60)^2}{120}\right] \approx 16.672 = 16.7 \text{ (3 s.f.)}$
(ii)	Let μ hrs be the population mean lifetime of the bulbs. $H_0 : \mu = 1200$ $H_1 : \mu < 1200$ (manufacturer's claim invalid) Level of significance: 10% Since population variance is unknown and sample size n = 120 is large, by Central Limit Theorem, $\bar{X} \sim N\left(\mu, \frac{s^2}{n}\right)$ approximately. Under H_0, test statistics, $Z = \frac{\bar{X} - 1200}{s/\sqrt{120}} \sim N(0,1)$ approximately From GC, $p-$value = 0.089878 = 0.0899 (3 s.f.) Since p-value = 0.0899 (< 0.1), reject H_0 and conclude that there is sufficient evidence at the 10% level of significance that the manufacturer's claim is not valid.
(iii)	Assume that the lifetime of the light bulbs produced by the improved process follows a normal distribution. $H_0 : \mu = 1200$ $H_1 : \mu < 1200$ Level of significance = 5% Under H_0, test statistic: $T = \frac{\bar{X} - 1200}{\frac{S}{\sqrt{20}}} \sim t_{(19)}$ Critical region : Reject H_0 if $t \leq$ -1.72913. Given sample mean = k and sample std. dev. = 9.8 $\therefore$ unbiased estimate of population variance, $s^2 = \frac{n}{n-1} \times (\text{sample variance}) = \frac{20}{19}(9.8)^2 = 101.095$ For the test to indicate that the manufacturer's claim is valid for this improved

	process, we do not reject H_0. Hence, $\dfrac{k-1200}{\frac{\sqrt{101.095}}{\sqrt{20}}} > -1.72913$ $k > 1196.11243$ The least possible value of k is 1196.113 (or 1196.112) (3 d.p.)