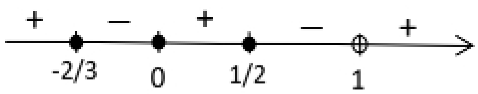
1. Using an algebraic method, solve the inequality $\frac{5}{1-x} \le 6x^2+7x+5$ leaving your answers in exact form. [3]

Qn	Suggested Solution
1	$\frac{5}{1-x} \le 6x^2+7x+5$ ← Do not cross multiply! $\frac{(1-x)(6x^2+7x+5)-5}{1-x} \ge 0$ $\frac{6x^3+x^2-2x}{x-1} \ge 0$ $\frac{x(2x-1)(3x+2)}{x-1} \ge 0$ ← Find critical values and use number lines  $\Rightarrow x \le -\frac{2}{3}$ or $0 \le x \le \frac{1}{2}$ or $x > 1$ ← Do not use "comma"

2. A function is defined as $\mathrm{f}(x) = 3x^2 - 6x - 1$. By completing the square, describe a sequence of transformations that transforms the graph of $y = x^2$ onto the graph of $y = \mathrm{f}(x)$. [3]

Qn	Suggested Solution
2	$\mathrm{f}(x) = 3x^2 - 6x - 1 = 3(x-1)^2 - 4$ $y = x^2$ ↓ Scaling with factor 3 parallel to the y-axis $\frac{y}{3} = x^2$ $\therefore y = 3x^2$ ↓ Translating 1 unit in the positive x-axis direction $y = 3(x-1)^2$ ↓ Translating 4 units in the negative y-axis direction $y + 4 = \mathrm{f}(x) = 3(x-1)^2$

3 A telecommunication company, StarPub charges their mobile prepaid card users based on the usage for local calls, text messaging and data. Local calls are charged per minute, while text messaging is charged per SMS sent and data usage is charged per megabytes (MB) used or part thereof. For instance, a data usage in the interval of $(1000,1001]$ MB will be charged the amount for the usage of 1001 MB of data.

Below are the monthly mobile prepaid card bill statements for Mr Wee from May to August 2020.

StarPub Mobile Bill (May)

Mr Wee
Hock Guan Biscuit Mansion…

Current month charges

Local calls	124 mins	**.**
SMS	18 SMS	**.**
Data	6234.3 MB	**.**
Total		**$48.16**

StarPub Mobile Bill (June)

Mr Wee
Hock Guan Biscuit Mansion…

Current month charges

Local calls	156 mins	**.**
SMS	12 SMS	**.**
Data	5839.5 MB	**.**
Total		**$52.04**

StarPub Mobile Bill (July)

Mr Wee
Hock Guan Biscuit Mansion…

Current month charges

Local calls	108 mins	**.**
SMS	10 SMS	**.**
Data	8625.0 MB	**.**
Total		**$57.30**

StarPub Mobile Bill (August)

Mr Wee
Hock Guan Biscuit Mansion …

Current month charges

Local calls	88 mins	**.**
SMS	15 SMS	**.**
Data	?? MB	**.**
Total		**$47.10**

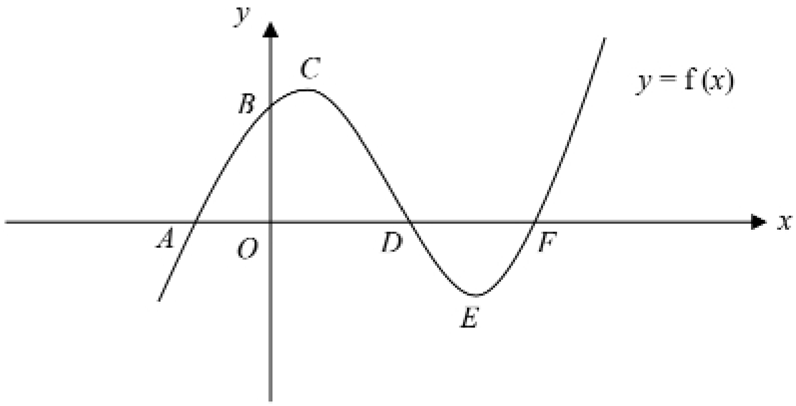
It is known that the unit cost for local calls and text messaging remained unchanged from May to August 2020. The unit cost for data usage remained unchanged for May and June 2020 but was increased by 8% with effect from 1 July 2020.

(i) Find the unit cost for local calls, text messaging and data usage for June 2020 in dollars, giving your answers correct to the nearest 3 decimal places. [2]

(ii) The data usage for August 2020 printed on the bill has faded away. Using your answers in part **(i)**, find the data usage for August 2020 to the nearest MB. [2]

Qn	Suggested Solution
3(i)	Let \$$L$, \$$T$ and \$$D$ be the unit cost of local calls, text messaging and data, respectively. $124L + 18T + 6235D = 48.16$ $156L + 12T + 5840D = 52.04$ $108L + 10T + 8625(1.08)D = 57.30$ Using G.C, $L = 0.17954 = 0.180,\ T = 0.04620 = 0.046,\ \ D = 0.00402 = 0.004$
(ii)	Let d be the data usage for August 2020. $0.180(88) + 0.046(15) + 0.004(1.08)d = 47.10$ $0.00432d = 30.57$ $d = 7076$ (nearest MB)

4 The diagram below shows the curve $y = \mathrm{f}(x)$. The curve cuts the x–axis at $A(-1.5,\ 0)$, $D(3,\ 0)$ and $F(5,\ 0)$ as well as the y–axis at $B(0,\ 2.5)$. It has a maximum point at $C(1,\ 3)$ and a minimum point at $E(4,\ -1.5)$.



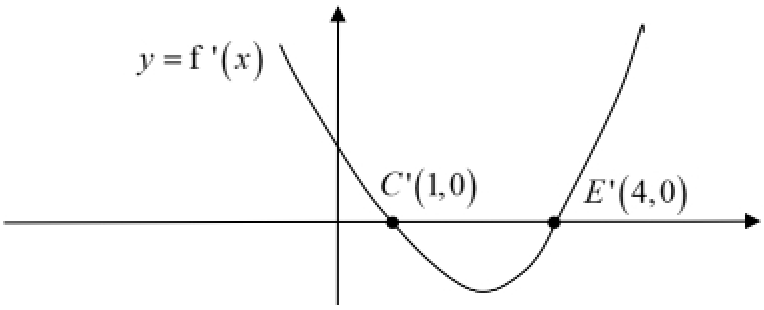
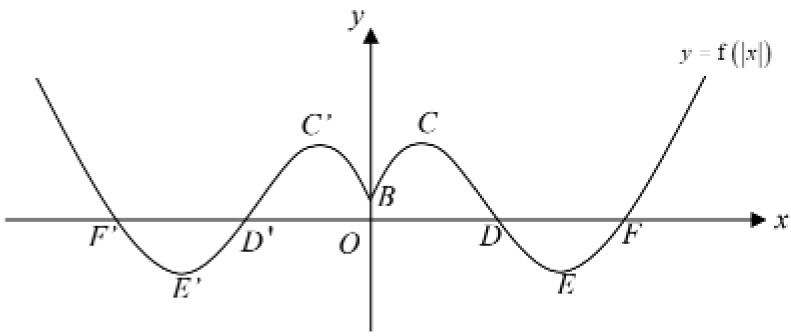
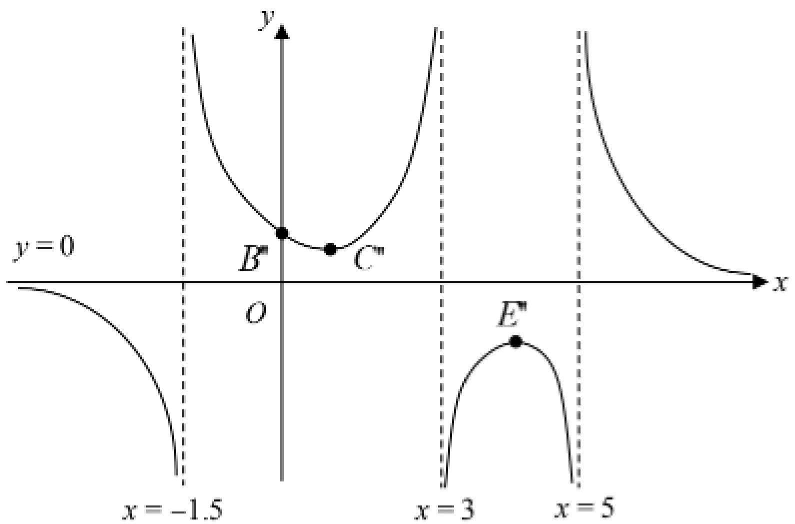
Sketch, on separate diagrams, the following graphs. State clearly the equations of any asymptotes and the coordinates of the points corresponding to A, B, C, D, E and F (if any).

(i) $y = \mathrm{f}\,'(x)$, [2]

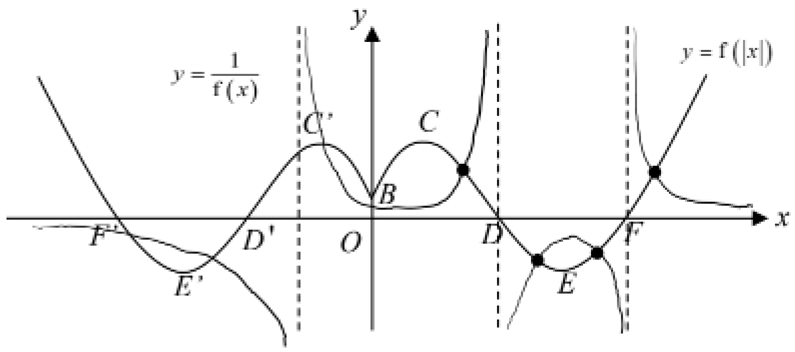
(ii) $y = \mathrm{f}\left(|x|\right)$, [2]

(iii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

(iv) Deduce the number of distinct positive solutions for the equation of $\mathrm{f}\left(|x|\right) = \dfrac{1}{\mathrm{f}(x)}$. [1]

Qn	Suggested Solution
4(i)	
4(ii)	 $B(0,\ 2.5)$, $C(1,\ 3)$, $D(3,\ 0)$, $E(4,\ -1.5)$, $F(5,\ 0)$, $C'(-1,\ 3)$, $D'(-3,\ 0)$, $E'(-4,\ -1.5)$, $F'(-5,\ 0)$
(iii)	 $B''(0,\ 0.4)$, $C''\left(1,\ \frac{1}{3}\right)$, $E''\left(4,\ -\frac{2}{3}\right)$

(iv)



For the equation of $f(|x|)=\frac{1}{f(x)}$, there are 4 **positive** distinct solutions.

5 **(i)** Using the method of differences, show that

$$\sum_{r=2}^{n}\frac{1}{(r-1)(r+2)}=\frac{1}{3}\left[A+\frac{B}{n}+\frac{C}{n+1}+\frac{D}{n+2}\right]$$

where A, B, C and D are constants to be determined. [4]

(ii) Using the result in part **(i)**, deduce the exact value of $\sum_{r=2}^{\infty}\frac{1}{r(r+3)}$.

Hence show that $\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots>\frac{13}{36}$. [4]

Qn	Suggested Solution
5(i)	$\sum_{r=2}^{n}\frac{1}{(r-1)(r+2)}$ $=\frac{1}{3}\sum_{r=2}^{n}\left[\frac{1}{r-1}-\frac{1}{r+2}\right]$ $=\frac{1}{3}\left[\begin{array}{ll}\frac{1}{1} & -\frac{1}{4}\\ +\frac{1}{2} & -\frac{1}{5}\\ +\frac{1}{3} & -\frac{1}{6}\\ +\frac{1}{4} & -\frac{1}{7}\\ \dots\dots & \\ +\frac{1}{n-4} & -\frac{1}{n-1}\\ +\frac{1}{n-3} & -\frac{1}{n}\\ +\frac{1}{n-2} & -\frac{1}{n+1}\\ +\frac{1}{n-1} & -\frac{1}{n+2}\end{array}\right]$ $=\frac{1}{3}\left[1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\right]$ $=\frac{1}{3}\left[\frac{11}{6}-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\right]$
(ii)	$\sum_{r=2}^{n}\frac{1}{r(r+3)}=\sum_{r-1=2}^{r-1=n}\frac{1}{(r-1)(r-1+3)}$ (replace r with $r-1$) $=\sum_{r=3}^{n+1}\frac{1}{(r-1)(r+2)}$ $=\sum_{r=2}^{n+1}\frac{1}{(r-1)(r+2)}-\frac{1}{(2-1)(2+2)}$ $=\frac{1}{3}\left[\frac{11}{6}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right]-\frac{1}{4}$ As $n\to\infty$, $\frac{1}{n+1}\to 0$, $\frac{1}{n+2}\to 0$, $\frac{1}{n+3}\to 0$. $\therefore \sum_{r=2}^{\infty}\frac{1}{r(r+3)}=\frac{13}{36}$

	$\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots=\sum_{r=2}^{\infty}\frac{1}{r^2}$ $r^2<r(r+3)$ Since $r>0$, $\frac{1}{r^2}>\frac{1}{r(r+3)}$ $\sum_{r=2}^{\infty}\frac{1}{r^2}>\sum_{r=2}^{\infty}\frac{1}{r(r+3)}=\frac{13}{36}$ (shown)

6 Do not use a calculator in answering this question.

Show that $\frac{x-1}{x^2-2x+3}-\frac{2-2x}{2x-x^2+1}=\frac{(1-x)(x^2-2x+7)}{(x^2-2x+3)(x^2-2x-1)}$. [1]

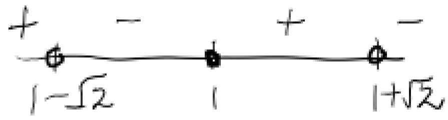
Solve the inequality $\frac{x-1}{x^2-2x+3}\geq\frac{2-2x}{2x-x^2+1}$. [3]

Hence solve the following exactly.

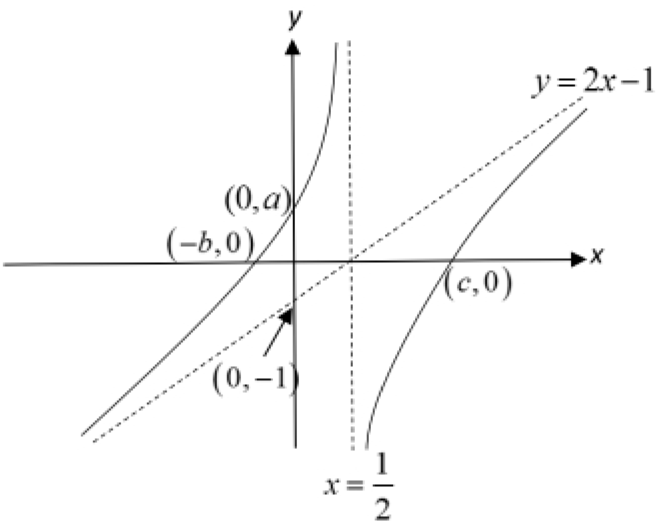
(i) $\frac{2x-1}{4x^2-4x+3}\geq\frac{2-4x}{4x-4x^2+1}$. [1]

(ii) $\frac{\tan x}{\tan^2 x+2}\geq\frac{-2\tan x}{2-\tan^2 x}$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$. [3]

Qn	Suggested Solution
6	$\frac{x-1}{x^2-2x+3}-\frac{2-2x}{2x-x^2+1}$ $=\frac{(x-1)(2x-x^2+1)-(2-2x)(x^2-2x+3)}{(x^2-2x+3)(2x-x^2+1)}$ $=\frac{2x^2-x^3+x-2x+x^2-1-2x^2+4x-6+2x^3-4x^2+6x}{(x^2-2x+3)(2x-x^2+1)}$ $=\frac{-x^3+3x^2-9x+7}{(x^2-2x+3)(x^2-2x-1)}$ $=\frac{(1-x)(x^2-2x+7)}{(x^2-2x+3)(x^2-2x-1)}$
	$\frac{x-1}{x^2-2x+3}\geq\frac{2-2x}{2x-x^2+1}$ $\frac{(1-x)(x^2-2x+7)}{(x^2-2x+3)(x^2-2x-1)}\geq 0$ $x^2-2x+3=(x-1)^2+2>0$ & $x^2-2x+7=(x-1)^2+6>0\ \forall x\in\mathbb{R}$

	$\dfrac{1-x}{x^2-2x-1} \geq 0$ $\dfrac{1-x}{\left(x-\left(1+\sqrt{2}\right)\right)\left(x-\left(1-\sqrt{2}\right)\right)} \geq 0$  $x < 1-\sqrt{2}$ or $1 \leq x < 1+\sqrt{2}$
(i)	Substitute x by $2x$: $\dfrac{2x-1}{(2x)^2-2(2x)+3} \geq \dfrac{2-2(2x)}{2(2x)-(2x)^2+1}$ $\dfrac{2x-1}{4x^2-4x+3} \geq \dfrac{2-4x}{4x-4x^2+1}$ $2x < 1-\sqrt{2}$ or $1 \leq 2x < 1+\sqrt{2}$ $x < \dfrac{1-\sqrt{2}}{2}$ or $\dfrac{1}{2} \leq x < \dfrac{1+\sqrt{2}}{2}$
(ii)	$\dfrac{x-1}{x^2-2x+3} \geq \dfrac{2-2x}{2x-x^2+1}$ Substitute x by $\tan x+1$: $\dfrac{\tan x}{\tan^2 x+2} \geq \dfrac{-2\tan x}{2-\tan^2 x}$ $\tan x+1 < 1-\sqrt{2}$ or $1 \leq \tan x+1 < 1+\sqrt{2}$ $\tan x < -\sqrt{2}$ or $0 \leq \tan x < \sqrt{2}$ Since $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, $-\dfrac{\pi}{2} < x < -\tan^{-1}\sqrt{2}$ or $0 \leq x < \tan^{-1}\sqrt{2}$ Observe from graph of $y = \tan x$

7 The diagram shows the graph of $y = \mathrm{f}(x)$. It has asymptotes $x = \dfrac{1}{2}$ and $y = 2x-1$ and it passes through the points $(0,\ a)$, $(-b,\ 0)$ and $(c, 0)$, where a, b and c are positive constants.



Function g is defined by

$$g(x) = f(x), \qquad x > k,$$

where k is a constant.

(i) State the minimum value of k for the function g^{-1} to exist. [1]

In the rest of the question, let k be the value stated in **(i)**.

(ii) On the same diagram, sketch the graphs of g and g^{-1}, stating, in terms of a, b and/or c, the coordinates of the points of intersection with the axes, and the equations of the asymptotes. [3]

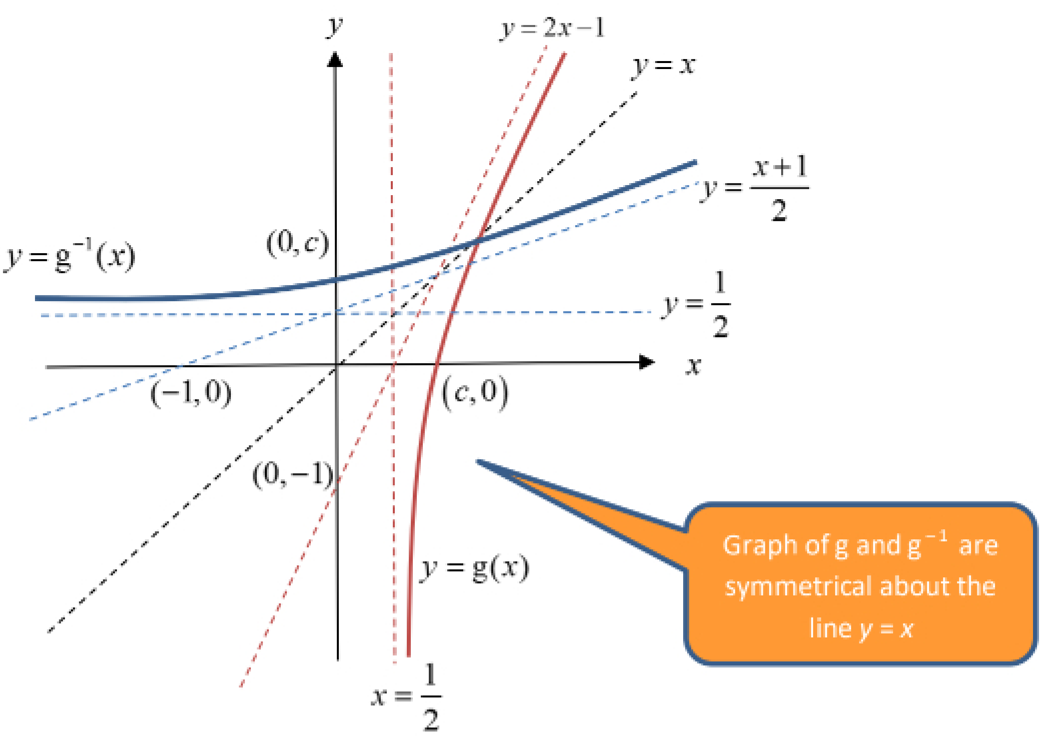
(iii) By considering the respective ranges, show that it is not possible for the range of gg^{-1} and the range of $g^{-1}g$ to be the same. [2]

Given that f is defined by

$$f : x \mapsto \frac{4x^2 - 4x - a}{2x-1}, \qquad x \in \mathbb{R}, x \neq \frac{1}{2}.$$

(iv) Find, in terms of a, the value(s) of x when $g(x) = g^{-1}(x)$. [3]

(v) It is given that $h(x) = g(x)$. For $a = 0$, state a possible domain of h such that the range of hh^{-1} is the same as the range of $h^{-1}h$. [1]

7(i)	Minimum value of $k = \frac{1}{2}$
(ii)	
(iii)	$R_{gg^{-1}} = D_{gg^{-1}} = D_{g^{-1}} = R_g = \mathbb{R}$ $R_{g^{-1}g} = D_{g^{-1}g} = D_g = (0.5, \infty)$

	Hence, it is not possible for range of gg^{-1} and range of $g^{-1}g$ to be the same. Alternatively, $R_{gg^{-1}}\ :\ D_{g^{-1}} \xrightarrow{g^{-1}} R_{g^{-1}} = D_g \xrightarrow{g} R_g = \mathbb{R}$ $R_{g^{-1}g}\ :\ D_g \xrightarrow{g} R_g = D_{g^{-1}} \xrightarrow{g^{-1}} R_{g^{-1}} = D_g = (0.5, \infty)$ Hence, it is not possible for range of gg^{-1} and range of $g^{-1}g$ to be the same.
(iv)	$g(x) = g^{-1}(x) \quad \Rightarrow \quad g(x) = x$ Graphs of g and g^{-1} intersect at the line $y = x$ $\Rightarrow \quad \dfrac{4x^2 - 4x - a}{2x-1} = x$ $\Rightarrow \quad 4x^2 - 4x - a = 2x^2 - x$ $\Rightarrow \quad 2x^2 - 3x - a = 0$ $\Rightarrow \quad x = \dfrac{3 \pm \sqrt{9 - 8(-a)}}{2(2)} = \dfrac{3 \pm \sqrt{9+8a}}{4}$ Since the graphs intersect at $x > \dfrac{1}{2}$, $x = \dfrac{3 + \sqrt{9+8a}}{4}$
(v)	When $a = 0$, we have $x = 1.5$. Possible domain of h $= [1.5, \infty)$ or $\{1.5\}$ or $(1.5, \infty)$

8 **(a)** Find $\int \sin px \sin qx \, dx$, where p and q are real constants. [2]

(b) Find the exact value of $\int_{-\ln 2}^{0} \left(\dfrac{e^{2x}}{e^{2x}+1} \right) dx$, giving your answer as a single logarithm. [2]

(c) Find $\int \dfrac{x}{\sqrt{8 - 2x - x^2}} \, dx$. [3]

(d) Find $\dfrac{d}{dx}\left(\tan^3 x\right)$. [1]

Hence find $\int \sec^4 x \, dx$. [2]

Q8	**[Solution]**
(a)	$\int \sin px \sin qx \, dx$ $= -\dfrac{1}{2} \int \cos(p+q)x - \cos(p-q)x \, dx$ $= -\dfrac{1}{2}\left[\dfrac{\sin(p+q)x}{p+q} - \dfrac{\sin(p-q)x}{p-q} \right] + c$
(b)	$\int_{-\ln 2}^{0} \left(\dfrac{e^{2x}}{e^{2x}+1} \right) dx = \dfrac{1}{2} \int_{-\ln 2}^{0} \left(\dfrac{2e^{2x}}{e^{2x}+1} \right) dx$

	$=\frac{1}{2}\left[\ln\left(e^{2x}+1\right)\right]_{-\ln 2}^{0}$ $=\frac{1}{2}\left[\ln\left(e^{0}+1\right)-\left(e^{-2\ln 2}+1\right)\right]$ $=\frac{1}{2}\left(\ln 2-\ln\left(\frac{1}{4}+1\right)\right)$ $=\frac{1}{2}\ln\frac{8}{5}$
(c)	$\int \frac{x}{\sqrt{8-2x-x^2}}\,dx=-\frac{1}{2}\int\frac{-2x-2}{\sqrt{8-2x-x^2}}\,dx-\int\frac{1}{\sqrt{8-2x-x^2}}\,dx$ $=-\sqrt{8-2x-x^2}-\int\frac{1}{\sqrt{9-(x+1)^2}}\,dx$ $=-\sqrt{8-2x-x^2}-\sin^{-1}\left(\frac{x+1}{3}\right)+c$
(d)	$\frac{d}{dx}\left(\tan^3 x\right)=3\tan^2 x\sec^2 x$ $\int\sec^4 x\,dx=\int\sec^2 x\left(1+\tan^2 x\right)\,dx$ $=\int\left(\sec^2 x+\sec^2 x\tan^2 x\right)\,dx$ $=\tan x+\frac{1}{3}\tan^3 x+c$

9 The function f is defined by

$$f: x \mapsto x^2-4x+e^x,\ x\geq a$$

where a is a positive integer.

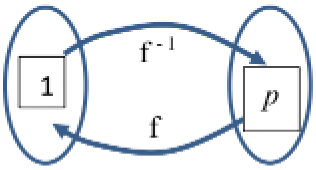
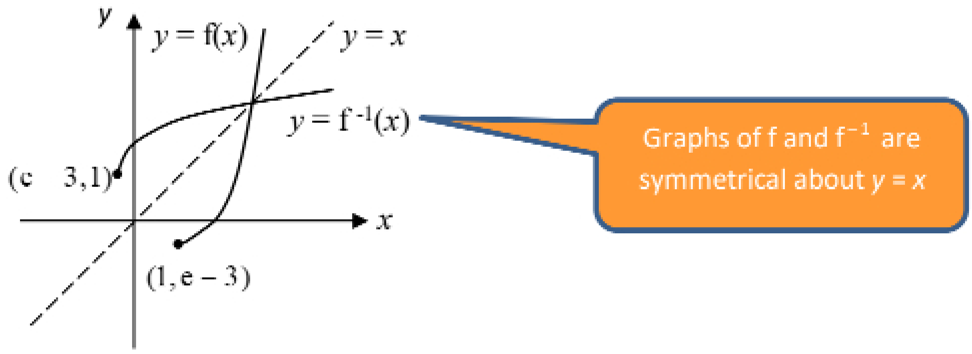
(i) State the least value of a for the inverse function of f to exist. Hence find the value of $f^{-1}(1)$. You may leave your answer correct to 3 decimal places. [3]

Use the least value of a found in part **(i)** for the remaining parts of the question.

(ii) Sketch, on the same diagram, the graphs of $y=f(x)$ and $y=f^{-1}(x)$, showing the graphical relationship between the two graphs. [3]

(iii) Explain why the solution(s) to the equation $f(x)=f^{-1}(x)$ can be obtained by solving the equation $x^2-5x+e^x=0$. [1]

(iv) It is given that the gradient of the tangent to the curve with equation $y = \mathrm{f}^{-1}(x)$ is $\frac{1}{3}$ at the point with $x = m$. Find the value of m, giving your answer correct to 3 decimal places. [3]

Qn	Suggested Solution
9(i)	From GC, least value of $a = 1$ Let $p = \mathrm{f}^{-1}(1)$,  $\mathrm{f}(p) = 1$ $p^2 - 4p + \mathrm{e}^p = 1$ From GC, $p = 1.572$
(ii)	
(iii)	The graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = x$ intersect at the same point. $\therefore$ The solution can be found by finding the intersection point between the graphs of $y = \mathrm{f}(x)$ and $y = x$, thus solving the equation : $\mathrm{f}(x) = x$ $x^2 - 4x + \mathrm{e}^x = x$ $x^2 - 5x + \mathrm{e}^x = 0$ (shown)
(iv)	Since gradient of tangent to $y = \mathrm{f}^{-1}(x)$ is $\frac{1}{3}$ at $x = m$ and we let $m = \mathrm{f}(b)$, gradient of tangent at $y = \mathrm{f}(x)$ is 3 at $x = b$. Given that $\mathrm{f}(x) = x^2 - 4x + \mathrm{e}^x$, $\mathrm{f}'(x) = 2x - 4 + \mathrm{e}^x$ $2b - 4 + \mathrm{e}^b = 3$ From GC, $b = 1.4237$ $m = b^2 - 4b + \mathrm{e}^b = 0.485$ (3 dp)

10 A curve C has parametric equations

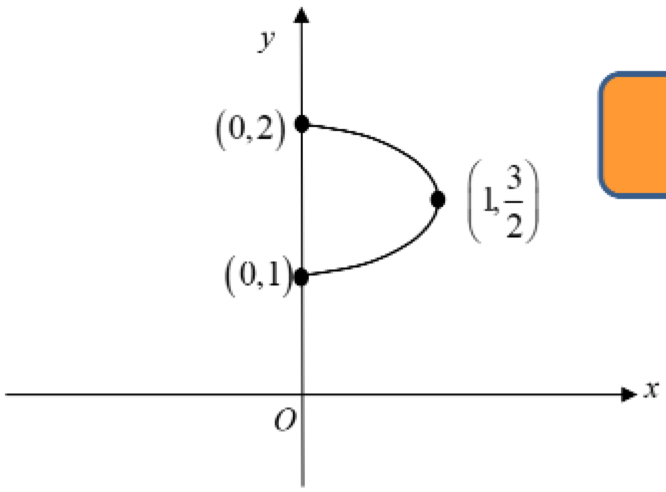
$$x = \sin 2t, \quad y = \cos^2 t + 1, \quad 0 \le t \le \tfrac{\pi}{2}.$$

(i) Sketch C, showing clearly the axial-intercepts and the vertex. [3]

(ii) Express $\frac{dy}{dx}$ in the form $a \tan bt$, where a and b are constants to be determined. [2]

(iii) The tangents to the points P and Q on C are such that these two tangents meet at the point $\left(2, \frac{3}{2}\right)$. Find the coordinates of P and Q, giving your answers correct to 3 decimal places. [4]

(iv) Find the Cartesian equation of C. [3]

Qn	Suggested Solution
10(i)	 Take note that $0 \le t \le \frac{\pi}{2}$
(ii)	$\frac{dx}{dt} = 2\cos 2t, \quad \frac{dy}{dt} = -2\cos t \sin t$ $\frac{dy}{dx} = \frac{-2\cos t \sin t}{2\cos 2t}$ $= \frac{-\sin 2t}{2\cos 2t}$ $= -\frac{1}{2}\tan 2t \quad$ (where $a = -\frac{1}{2}$, $b = 2$)
(iii)	Equation of tangent at any point on C: $\frac{y - \left(\cos^2 t + 1\right)}{x - \sin(2t)} = -\frac{1}{2}\tan 2t$ $\left(2, \frac{3}{2}\right)$ must satisfy the equations of the tangents at P and Q.

	$\dfrac{\frac{3}{2}-\left(\cos^2 t+1\right)}{2-\sin(2t)}=-\frac{1}{2}\tan 2t$ From GC, $t = 0.26180$ or 1.30900 Coordinates of P and Q are (0.500, 1.933) and (0.500,1.067) (3 dp).
(iv)	$x=\sin 2t ---(1)$ $y=\cos^2 t+1=\frac{1}{2}\left(\cos 2t+1\right)+1$ $2\left(y-\frac{3}{2}\right)=\cos 2t ---(2)$ Using $\sin^2 2t+\cos^2 2t=1$, $x^2+4\left(y-\frac{3}{2}\right)^2=1, \quad 0\le x\le 1$ Don't forget to include the range of values of x, given that $0\le t\le \frac{\pi}{2}$

11 **(a)** Given that U_n is a linear polynomial in terms of n, explain why $\{U_n\}$ is an arithmetic sequence. [2]

(b) Edward plans to take up a study loan of \$30000 with a local bank. For this type of loan, the bank will only start to compound the interest the year after Edward's graduation. The bank compounds interest on 1 Jan, at a rate of 5% per annum on the outstanding amount on 31 Dec of the preceding year.

Edward is going to graduate in 2024. He plans to make a repayment of \$200 at the end of every month, starting from Jan 2025.

(i) Show that the outstanding loan amount is \$28155 on 31 Dec 2026. [1]

(ii) Taking 2025 as the first year, show that the outstanding loan amount at the end of the nth year is $6000\left[8-3(1.05)^n\right]$. [3]

(iii) Find the year and the month in which Edward will finish servicing his loan. Hence determine the total interest he paid, giving your answer to the nearest cent. [4]

(iv) Edward intends to finish servicing the loan in at most 60 monthly repayments. Find the minimum monthly repayment needed, correct to the nearest dollar. [2]

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>11(a)</td><td>For $U_n = an+b$,
then $U_n - U_{n-1} = (an+b) - [a(n-1)+b] = a$, which is a constant.
Thus $\{U_n\}$ is an arithmetic sequence.</td></tr>
<tr><td>11
(b) (i)</td><td>
<table>
<tr><th>Year</th><th>Amt at the Start</th><th>Amt at the End</th></tr>
<tr><td>1
(2025)</td><td>$1.05(30000)$</td><td>$1.05(30000)-12(200)$</td></tr>
<tr><td>2
(2026)</td><td>$1.05[1.05(30000)-12(200)]$
$=1.05^2(30000)-1.05(2400)$</td><td>$1.05^2(30000)-1.05(2400)-2400$
$=28155$</td></tr>
<tr><td></td><td>⋮</td><td>⋮</td></tr>
<tr><td>n</td><td>$1.05^n(30000)-1.05^{n-1}(2400)$
$-1.05^{n-2}(2400)-\ldots-(1.05)2400$</td><td>$1.05^n(30000)-1.05^{n-1}(2400)$
$-1.05^{n-2}(2400)-\ldots(1.05)2400-2400$</td></tr>
</table>

Outstanding loan amount on 31 Dec 2026
$=1.05[1.05(30000)-12(200)]-12(200)$
$=\$28155$ (shown)</td></tr>
<tr><td>(ii)</td><td>Outstanding loan amount at the end of the nth year
$=1.05^n(30000)-1.05^{n-1}(2400)-1.05^{n-2}(2400)-\ldots(1.05)2400-2400$
$=1.05^n(30000)-2400(1+1.05+\ldots+1.05^{n-1})$
$=1.05^n(30000)-2400\left(\frac{1.05^n-1}{1.05-1}\right)$
$=1.05^n(30000)-48000(1.05^n-1)$
$=48000-18000(1.05)^n$
$=6000\left[8-3(1.05)^n\right]$ (shown)</td></tr>
<tr><td>(iii)</td><td>Solving $6000\left[8-3(1.05)^n\right] \le 0$
$(1.05)^n \ge \frac{8}{3}$
$n\ln(1.05) \ge \ln\left(\frac{8}{3}\right)$
$n \ge \frac{\ln\frac{8}{3}}{\ln 1.05} = 20.1$
[Note : Yr 1 = 2025 → Yr 20 = 2044]
Outstanding amount at the end of 20th year $=6000\left[8-3(1.05)^{20}\right]=\$240.64$
Thus Edward will finish servicing his loan in <u>Feb 2045</u>.
Total interest paid = $20(2400)+1.05(240.6413)-30000=\$18252.67$</td></tr>
<tr><td>(iv)</td><td>Let the monthly repayment be $\$m$. 60 months = 5 years.</td></tr>
</table>

	$1.05^5(30000)-12m\left(\dfrac{1.05^5-1}{1.05-1}\right)\le 0$ $66.308m \ge 1.05^5(30000)$ $m \ge 577.43$ Minimum monthly repayment needed = \$578 (nearest dollar)

12 [It is given that the volume of a circular cone with base radius r and vertical height h is $\frac{1}{3}\pi r^2 h$.]

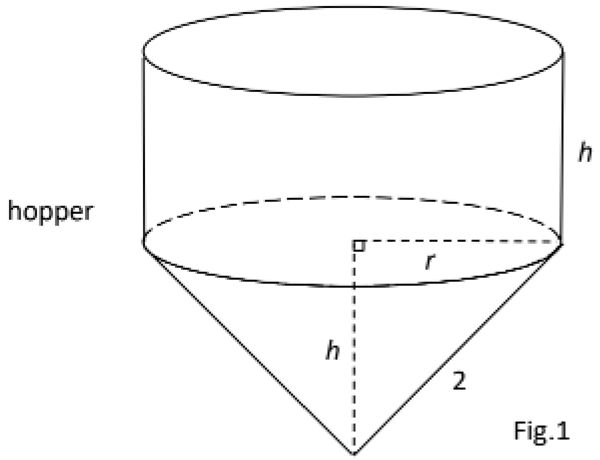


Fig.1

A hopper consists of an open cylinder of height h m joined to an open cone of radius r m and height h m (see Fig.1). The slant edge of the cone has a fixed length of 2 m.

(i) As r and h vary, use differentiation to find the exact maximum volume of the hopper. [6]

A coffee manufacturer uses such a hopper that has a cone of radius 1.2 m. For each batch, the coffee drink is initially filled to the brim of the hopper and mixed thoroughly before it is transferred to the packing chamber at a constant rate of 0.2 m^3/s.

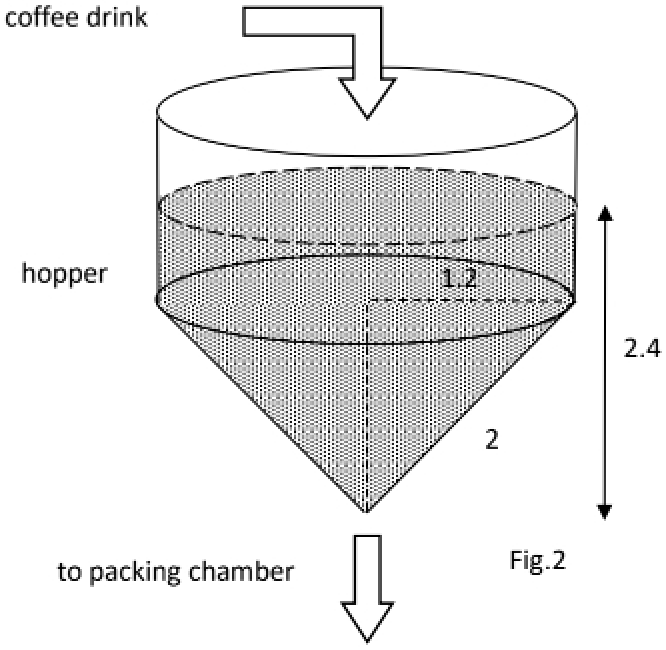


Fig.2

(ii) At a particular point in time, the height of the coffee drink in the cylindrical section is 2.4 m from the bottom of the cone (see Fig.2 shaded region). Find the rate at which the height of the coffee drink is changing at this instant. [2]

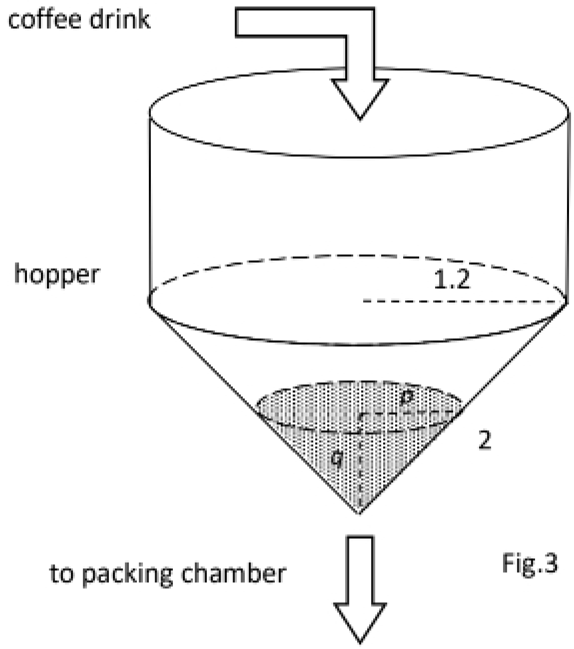
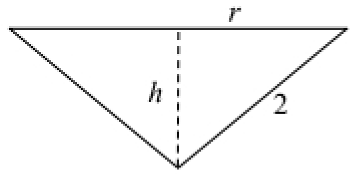
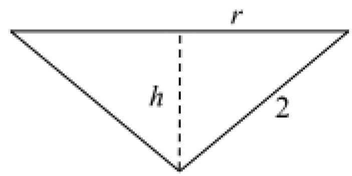


Fig.3

(iii) After some time has passed in the transfer process, there remains some coffee drink in the conical section with radius p m and height q m (see Fig.3 shaded region). Find p in terms of q and hence calculate the rate of decrease of q when $q = 0.5$. [4]

Qn	Suggested Solution
12(i)	**Method 1** $h^2 + r^2 = 2^2$ $r^2 = 4 - h^2$  Volume of hopper, $V = \frac{1}{3}\pi r^2 h + \pi r^2 h$ $= \frac{4}{3}\pi r^2 h = \frac{4}{3}\pi h(4-h^2)$ $= \frac{4}{3}\pi(4h - h^3)$ Must express V in terms of 1 variable only, either in terms of r or h $\frac{dV}{dh} = \frac{4}{3}\pi(4-3h^2)$ At stationary values, $\frac{dV}{dh} = \frac{4}{3}\pi(4-3h^2) = 0$ $h = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} \quad (\because h > 0)$ $\frac{d^2V}{dh^2} = \frac{4}{3}\pi(-6h) = -8\pi h$ When $h = \frac{2}{\sqrt{3}}$, $\frac{d^2V}{dh^2} < 0$ Maximum volume of the hopper, $V = \frac{4}{3}\pi\left(\frac{8}{\sqrt{3}} - \frac{8}{3\sqrt{3}}\right) = \frac{64}{9\sqrt{3}}\pi$
	Method 2 $h^2 + r^2 = 2^2$ $h = \sqrt{4 - r^2}$  Volume of hopper, $V = \frac{1}{3}\pi r^2 h + \pi r^2 h = \frac{4}{3}\pi r^2 h$ $V = \frac{4}{3}\pi r^2 \sqrt{4 - r^2}$

$$\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{4}{3}\pi\left[(2r)\sqrt{4-r^2}+r^2\left(\tfrac{1}{2}\left(4-r^2\right)^{-\frac{1}{2}}\right)(-2r)\right]$$

$$=\frac{4}{3}\pi\left[(2r)\sqrt{4-r^2}-\frac{r^3}{\sqrt{4-r^2}}\right]$$

$$=\frac{4}{3}\pi\left[\frac{(2r)\left(4-r^2\right)-r^3}{\sqrt{4-r^2}}\right]$$

$$=\frac{4}{3}\pi\left[\frac{8r-3r^3}{\sqrt{4-r^2}}\right]$$

$$=\frac{4}{3}\pi\left[\frac{r(8-3r^2)}{\sqrt{4-r^2}}\right]$$

At stationary values,

$$\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{4}{3}\pi\left[\frac{r(8-3r^2)}{\sqrt{4-r^2}}\right]=0$$

$$r(8-3r^2)=0$$

$$r=\sqrt{\frac{8}{3}}\quad(\because r>0)$$

r	$\left(\sqrt{\frac{8}{3}}\right)^-$	$\left(\sqrt{\frac{8}{3}}\right)$	$\left(\sqrt{\frac{8}{3}}\right)^+$
Sign of $\frac{\mathrm{d}V}{\mathrm{d}r}$	+ve	0	−ve
Tangent	/	–	\

Maximum volume of the hopper,

$$V=\frac{4}{3}\pi r^2\sqrt{4-r^2}=\frac{64}{9\sqrt{3}}\pi$$

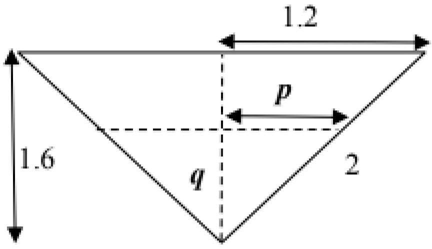
(ii) Since the coffee mixture is transferred at a constant rate of 0.2 $\mathrm{m^3/s}$ and the radius is constant at 1.2 m,

$$\therefore \text{ rate of change of height} = =\frac{-0.2}{\pi(1.2)^2}=-0.0442 \text{ m/s}$$

(iii) $\text{height}=\sqrt{4-1.2^2}=1.6$

By similar triangles,

$$\frac{p}{q}=\frac{1.2}{1.6}\Rightarrow p=\frac{3q}{4}$$



Let volume of coffee powder be V_2

	$V_2 = \frac{1}{3}\pi p^2 q = \frac{1}{3}\pi\left(\frac{3q}{4}\right)^2 q = \frac{3}{16}\pi q^3$ $\frac{\mathrm{d}V_2}{\mathrm{d}q} = \frac{9}{16}\pi q^2$ Given that $\frac{\mathrm{d}V_2}{\mathrm{d}t} = -0.2$ and $q = 0.5$ Using $\frac{\mathrm{d}V_2}{\mathrm{d}t} = \frac{\mathrm{d}V_2}{\mathrm{d}q} \times \frac{\mathrm{d}q}{\mathrm{d}t}$ $-0.2 = \frac{9}{16}\pi(0.5)^2 \times \frac{\mathrm{d}q}{\mathrm{d}t}$ $\frac{\mathrm{d}q}{\mathrm{d}t} = -0.453$ $\therefore$ rate of decrease of $q = 0.453$ m/s