Chapter 2 (Statistics): Probability (Teacher's copy)

Objectives

At the end of the chapter, you should be able to

(a) understand that the probability of an event measures how likely the event will occur

(b) construct a table of outcomes to calculate probabilities, and understand that the total probability of all possible outcomes is equal to 1

(c) calculate probabilities using addition and multiplication principles

(d) use a Venn diagram to interpret probabilities such as $\mathrm{P}(A')$, $\mathrm{P}(A\cup B)$, $\mathrm{P}(A\cap B)$, $\mathrm{P}(A\,|\,B)$

(e) understand the meaning of mutually exclusive events, and recognize events that are, or are not, mutually exclusive through practical examples; and use the result $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)$ where A and B are mutually exclusive

(f) understand the meaning of independent events, and use the result $\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)$, where A and B are independent.

(g) construct a tree diagram and use it to interpret and calculate probabilities, including probabilities of combined events and conditional probabilities.

Contents

2.1 Basic Rules and Definitions
- 2.1.1 Basic Definitions
- 2.1.2 Classical (Theoretical) Definition of Probability
- 2.1.3 Basic Probability Rules

2.2 Tools for Finding Probabilities
- 2.2.1 Venn Diagram
- 2.2.2 Tree Diagram
- 2.2.3 Table of Outcomes

2.3 Conditional Probability

2.4 Mutually Exclusive Events

2.5 Independent Events

2.6 A Mix of Probabilities

"Life is a school of probability." --**Walter Bagehot**

2.1 Basic Rules and Definitions

2.1.1 Basic Definitions

Definition	Experiment 1	Experiment 2
An **experiment** or **trial** is a process that generates data.	Throw a six-sided fair die. 	Toss a fair coin 2 times. 
An **outcome** is the result of a single trial of an experiment.	Obtain a '2'	Obtain a 'Head', then a 'Tail' (HT)
The **sample space, *S*,** of an experiment is the **set of all possible outcomes.** The number of outcomes in the sample space is denoted by n(S).	{1, 2, 3, 4, 5, 6} n(S)= 6	{HH, HT, TH, TT} n(S)= 4
An **event** is a subset of the sample space. The number of outcomes in an event A is denoted by n(A).	Let A be the event "a prime number is obtained". A = {2, 3, 5} n(A)= 3	Let B be the event "2 heads is obtained". B = {HH} n(B)= 1
Probability is the measure of how likely an event is to occur.	$\text{P}(A)=\frac{\text{n}(A)}{\text{n}(S)}=\frac{3}{6}=\frac{1}{2}$	$\text{P}(B)=\frac{\text{n}(B)}{\text{n}(S)}=\frac{1}{4}$

2.1.2 Classical (Theoretical) Definition of Probability

For **equally likely outcomes** from a **finite** sample space S**,** the probability of an event A is defined as

$$\text{P}(A)=\frac{\text{Number of Ways Event } A \text{ can occur}}{\text{Total Number of Possible Outcomes}}=\frac{\text{n}(A)}{\text{n}(S)}$$

Example 1

A playing card is to be drawn at random from a pack of 52 cards.
Find the probability that (i) it will be red, (ii) it will be a heart or an ace.

Solution:

(i) Event A : a red card is obtained
n(A) = 26

n(S) = 52

(ii) Event B: card is a heart or ace
n(B) = total number of cards that will be a heart or an ace = 13 + 3 = 16

n(S) = 52

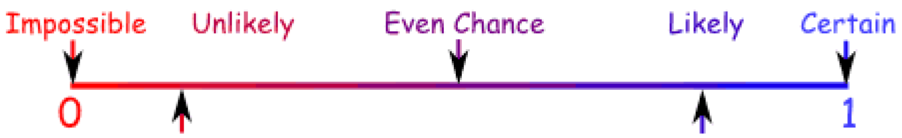
P(card will be red) = P(A) = $\frac{26}{52} = \frac{1}{2}$

P(card will be a heart or an ace)
= P(B) = $\frac{16}{52} = \frac{4}{13}$

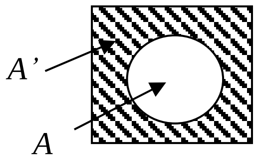
2.1.3 Basic Probability Rules

Suppose there are 2 events, A and B,

1. $0 \le \mathrm{P}(A) \le 1$
   If event A is impossible to happen, then P (A) = 0.
   If event A is an absolute certainty to happen, then P (A) = 1.
   A lot of the time, you'll be dealing with probabilities somewhere in between.

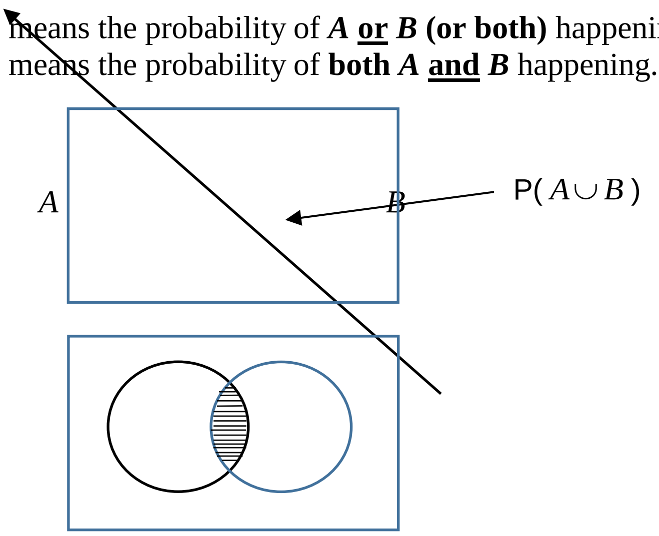


2. Complement of A means A does not happen.
   $\mathrm{P}(A') = 1 - \mathrm{P}(A)$



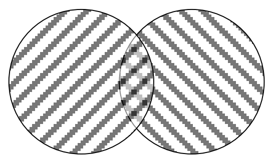
3. Union ($A \cup B$) and Intersection ($A \cap B$)

   $\mathrm{P}(A \cup B)$ means the probability of ***A* <u>or</u> *B* (or both)** happening.
   $\mathrm{P}(A \cap B)$ means the probability of **both *A* <u>and</u> *B*** happening.



P($A \cap B$)

P($A \cap B$)

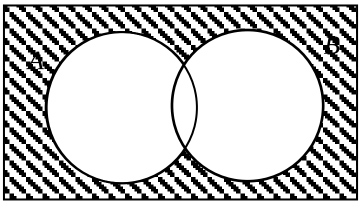
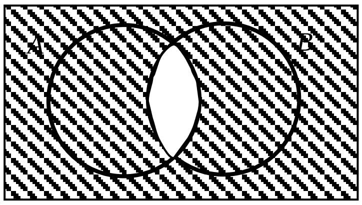
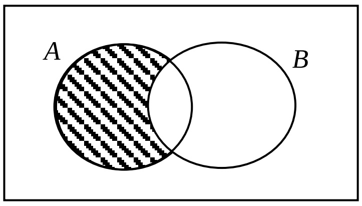
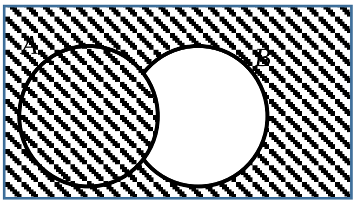


$$\mathrm{P}(A \text{ or } B) = \mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

2.2 Tools for finding Probabilities

2.2.1 Venn Diagram

A **diagram** that shows all possible logical relations between a finite collections of sets.

<table>
<tr><td colspan="2">Shade the Venn Diagram to produce these results:</td></tr>
<tr><td>1. $\mathrm{P}((A\cup B)') = \mathrm{P}(A'\cap B')$
</td><td>2. $\mathrm{P}((A\cap B)') = \mathrm{P}(A'\cup B')$
</td></tr>
<tr><td colspan="2">Note that case 1 & 2 above are representations of De Morgan's Laws.</td></tr>
<tr><td>3. $\mathrm{P}(A\cap B') = \mathrm{P}(A) - \mathrm{P}(A\cap B)$
</td><td>4. $\mathrm{P}(A\cup B')$
</td></tr>
</table>

Example 2

Events A and B are such that $\mathrm{P}(A)=0.3$, $\mathrm{P}(B)=0.4$ and $\mathrm{P}(A\cap B)=0.1$.

Find

(i) $\mathrm{P}(A\cup B)$ (ii) $\mathrm{P}(A\cap B')$

(iii) $\mathrm{P}(A'\cap B')$ (iv) $\mathrm{P}\left((A\cap B)'\right)$

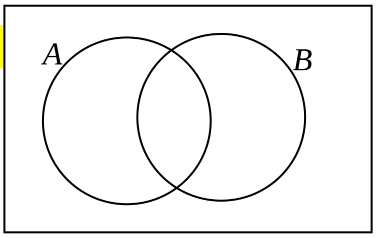
Solution:

(i) $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)=0.3+0.4-0.1=0.6$

(ii) $\mathrm{P}(A\cap B')=\mathrm{P}(A)-\mathrm{P}(A\cap B)=0.3-0.1=0.2$

(iii) $\mathrm{P}(A'\cap B')=1-\mathrm{P}(A\cup B)=1-0.6=0.4$

(iv) $\mathrm{P}\left((A\cap B)'\right)=1-\mathrm{P}\left(A\cap B\right)=1-0.1=0.9$



Example 3

Analysis of the results of a certain group of students who had taken examinations in both Mathematics and Economics produced the following information:

75% of the students passed Mathematics, 70% passed in Economics and 60% passed both subjects. Find

(i) the percentage of students who passed at least 1 subject;

(ii) the percentage of students who had passed exactly one of two subjects;

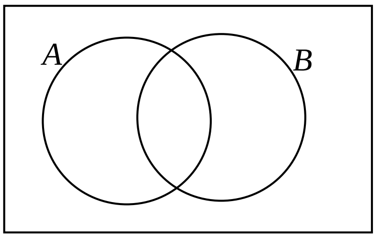
(iii) the probability that a student failed both subjects.

Solution:

Let Event A: a student passed Mathematics; Event B: a student passed Economics.

Given: $\mathrm{P}(A) =$ 0.75 , $\mathrm{P}(B) =$ 0.7 , $\mathrm{P}(A \cap B) =$ 0.6

(i) P(passed at least one subject)



$= \mathrm{P}(A \cup B) = 0.75 + 0.7 - 0.6 = 0.85$

Therefore. 85 % passed at least 1 subject.

(ii) P(passed exactly one of the two subjects)

$= \mathrm{P}(A \cap B') + \mathrm{P}(A' \cap B)$

$= 0.15 + 0.1 = 0.25$

Therefore, 25% of the students had passed exactly one of the two subjects.

(iii) P(a student failed both subjects)

$= \mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B)$

$= 1 - 0.85 = 0.15$

Example 4

In a race in which there are no ties, the probability that John wins is 0.3, the probability that Paul wins is 0.2 and the probability that Mark wins is 0.4. Find the probability that

(a) John or Mark wins,

(b) John or Paul or Mark wins,

(c) someone else wins.

Solution:

(a) P(John or Mark wins) = P(John wins)+ P(Mark wins) = 0.3 + 0.4 = 0.7

(b) P(John or Paul or Mark wins) = 0.3 + 0.2 + 0.4= 0.9

(c) P(someone else wins) = 1−0.9 =0.1

Exercise 1

1. Events C and D are such that $\mathrm{P}(C)=\frac{19}{30}$, $\mathrm{P}(D')=\frac{3}{5}$ and $\mathrm{P}(C\cup D)=\frac{4}{5}$.

   Find $\mathrm{P}(C\cap D)$. [Ans: $\frac{7}{30}$]

   Solution:

   $\mathrm{P}(D)=\frac{2}{5}$

   $\mathrm{P}(C\cup D)=\mathrm{P}(C)+\mathrm{P}(D)-\mathrm{P}(C\cap D)$

   $\therefore \mathrm{P}(C\cap D)=\mathrm{P}(C)+\mathrm{P}(D)-\mathrm{P}(C\cup D)=\frac{19}{30}+\frac{2}{5}-\frac{4}{5}=\frac{7}{30}$

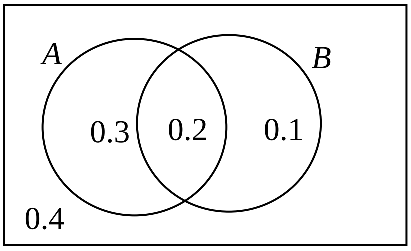
2. For the sample space S, it is given that $\mathrm{P}(A)=0.5$, $\mathrm{P}(A\cup B)=0.6$ and $\mathrm{P}(A\cap B)=0.2$. Find

   (i) $\mathrm{P}(B)$ (ii) $\mathrm{P}(A'\cap B)$

   (iii) $\mathrm{P}(A\cap B')$ (iv) $\mathrm{P}(A'\cap B')$

   [Ans: (i) 0.3 (ii) 0.1 (iii) 0.3 (iv) 0.4]

   Solution:



   (i) $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$

   $0.6=0.5+\mathrm{P}(B)-0.2$

   $\mathrm{P}(B)=0.3$

   (ii) $\mathrm{P}(A'\cap B)=\mathrm{P}(B)-\mathrm{P}(A\cap B)=0.3-0.2=0.1$

   (iii) $\mathrm{P}(A\cap B')=\mathrm{P}(A)-\mathrm{P}(A\cap B)=0.5-0.2=0.3$

   (iv) $\mathrm{P}(A'\cap B')=1-\mathrm{P}(A\cup B)=1-0.6=0.4$

   Question

   Are there any other ways to solve for 2(ii)?

   $\mathrm{P}(A'\cap B)=\mathrm{P}(A\cup B)-\mathrm{P}(A)=0.6-0.5=0.1$

3. In a survey, 15% of the participants said that they had never bought lottery tickets or premium bonds, 73% had bought lottery tickets and 49% had bought premium bonds. Find the probability that a participant chosen at random

   (a) had bought lottery tickets or premium bonds (or both),

   (b) had bought lottery tickets and premium bonds,

   (c) had bought lottery tickets only.

   [Ans (a) 0.85 (b) 0.37 (c) 0.36]

   Solution:

   (a) P(had bought lottery tickets or premium bonds) = 1 − 0.15 = 0.85

   (b) P(lottery tickets and premium bonds) = P(lottery tickets) + P(premium bonds) – P(lottery tickets or premium bonds) = 0.73 + 0.49 − 0.85 = 0.37

   (c) P(lottery tickets only) = P(lottery tickets) − P(lottery tickets and premium bonds) =0.73 − 0.37 = 0.36

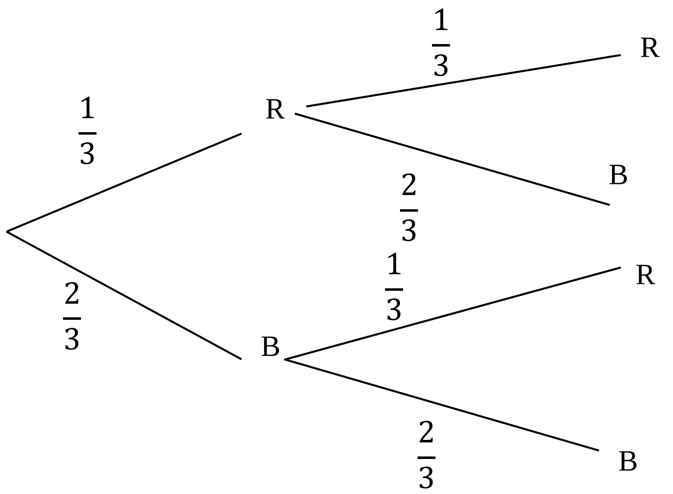
2.2.2 Tree Diagram

A useful way to tackle many probability problems is to draw a probability tree diagram.

Example 5

Put 2 red balls and 4 blue balls in a bag. Pick a first ball at random and **place it back in the bag**. Mix the balls well and pick a second ball randomly again.

First draw Second draw



Events
R: pick a red ball
B: pick a blue ball

To find P(two red balls drawn)= $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$

2.2.3 Table of Outcomes

A list of all the outcomes when two or more experiments take place at the same time. If the outcomes in each of these experiments are **equally likely**, then the combined outcomes will also be equally likely.

Example 6

Two fair dice are thrown together. Find the probability that the sum of the resulting number is
a) odd, b) a prime number.

Solution:
Construct the following table showing the sums:

Second die \ First die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a) Let A be the event that the sum is odd.
From the table, $n(A) = 18$.
$P(A) = \frac{18}{36} = \frac{1}{2}$

b) Let B be the event that the sum is a prime number. Count the number of 2, 3, 5, 7 and 11 in the table.
From the table, $n(B) = 15$
$P(B) = \frac{15}{36} = \frac{5}{12}$

Exercise 2

A unbiased coin and a fair die are tossed simultaneously. If a head appears, then the score is twice the number that appears on the die. If a tail appears, then the score is the same as the number that appears on the die. Find the probability that

(i) a head is obtained or the score is at least 6,

(ii) a head is obtained,

(iii) the score is at least 6,

(iv) a head is obtained and the score is at least 6,

(v) a tail is obtained and the score is at least 6.

[Ans: (i) $\frac{7}{12}$ (ii) $\frac{1}{2}$ (iii) $\frac{5}{12}$ (iv) $\frac{1}{3}$ (v) $\frac{1}{12}$]

Solution:

Coin \ Die	1	2	3	4	5	6
T	1	2	3	4	5	6
H	2	4	6	8	10	12

(i) P(head is obtained or the score is at least 6) = $\frac{6}{12}+\frac{5}{12}-\frac{4}{12}=\frac{7}{12}$

(ii) P(a head is obtained) = $\frac{6}{12}=\frac{1}{2}$

(iii) P(the score is at least 6) = $\frac{5}{12}$

(iv) P(a head is obtained and the score is at least 6) = $\frac{4}{12}=\frac{1}{3}$

(v) P(a tail is obtained and the score is at least 6) = $\frac{1}{12}$

2.3 Conditional Probability

If A and B are two events, where $\mathrm{P}(A) \neq 0$ and $\mathrm{P}(B) \neq 0$, then the probability of A, given that B has already occurred, is written as **P(*A* | *B*).**

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \text{ or } \mathrm{P}(A \mid B) = \frac{\mathrm{n}(A \cap B)}{\mathrm{n}(B)}$$

Example 7

A card is picked at random from a pack of 20 cards numbered 1,2,3,….. 20. Given that the card shows an even number, find the probability that it is a multiple of 4.

Solution:

Let event A be ’card is a multiple of 4’.

Let event B be ‘card is an even number’. Then

$$\mathrm{P}(A) = \frac{5}{20} = \frac{1}{4}\,; \qquad \mathrm{P}(B) = \frac{1}{2}$$

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \quad \text{means}$$

$$\mathrm{P}(\text{multiple of } 4 \mid \text{card is even}) = \frac{\mathrm{P}(\text{card is a multiple of 4 and is even})}{\mathrm{P}(\text{card is even})}$$

$$= \frac{\mathrm{P}(\text{card is a multiple of 4})}{\mathrm{P}(\text{card is even})} = \frac{1/4}{1/2} = \frac{1}{2}$$

ALT

$$\mathrm{P}(\text{multiple of } 4 \mid \text{card is even}) = \frac{\mathrm{n}(\text{card is a multiple of 4 and is even})}{\mathrm{n}(\text{card is even})}$$

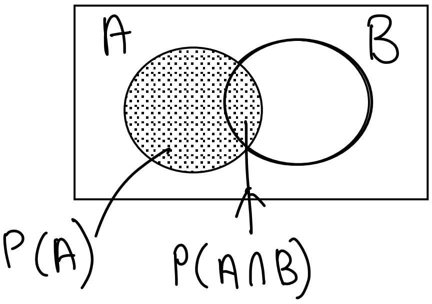
$$= \frac{\mathrm{n}(\text{card is a multiple of 4})}{\mathrm{n}(\text{card is even})} = \frac{5}{10} = \frac{1}{2}$$

Note to tutors: You may need to explain why

$$\mathrm{P}(\text{card is a multiple of 4 and is even}) \neq \mathrm{P}(\text{card is a multiple of 4}) \times \mathrm{P}(\text{card is even})$$

$$\frac{1}{4} \neq \left(\frac{1}{4}\right) \times \left(\frac{1}{2}\right).$$

Note:
In the Venn diagram representation, the possibility space for **P(*A* | *B*)** is reduced to just ***B***.
That is, **P(*A* | *B*) is the probability of A occurring by considering B as the sample space.**



Therefore, $\mathrm{P}(A\,|\,B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$.

Similarly, $\mathrm{P}(B\,|\,A)=\dfrac{\mathrm{P}(B\cap A)}{\mathrm{P}(A)}$

Some useful results:

(i) $\mathrm{P}(A\cap B)=\mathrm{P}(A\,|\,B)\mathrm{P}(B)=\mathrm{P}(B\,|\,A)\mathrm{P}(A)$

(ii) $\mathrm{P}(A'\,|\,B)=1-\mathrm{P}(A\,|\,B)$

Question: Are you able to prove Result (ii) above?

Proof of (ii):

$$\begin{aligned}&\mathrm{P}(A'\,|\,B)\\&=\frac{\mathrm{P}(A'\cap B)}{\mathrm{P}(B)}\\&=\frac{\mathrm{P}(B)-\mathrm{P}(A\cap B)}{\mathrm{P}(B)}\\&=1-\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}\\&=1-\mathrm{P}(A\,|\,B)\end{aligned}$$

Example 8

X and Y are two events such that $\mathrm{P}(X) = \frac{2}{5}$, $\mathrm{P}(X \mid Y) = \frac{1}{2}$ and $\mathrm{P}(Y \mid X) = \frac{2}{3}$.

Find (i) $\mathrm{P}(X \cap Y)$ (ii) $\mathrm{P}(Y)$ (iii) $\mathrm{P}(X \cup Y)$.

Solution:

(i)

Since $\mathrm{P}(Y \mid X) = \dfrac{\mathrm{P}(X \cap Y)}{\mathrm{P}(X)}$,

$$\mathrm{P}(X \cap Y) = \mathrm{P}(Y \mid X)\mathrm{P}(X) = \frac{2}{3}\left(\frac{2}{5}\right) = \frac{4}{15}$$

(ii)

Since $\mathrm{P}(X \mid Y) = \dfrac{\mathrm{P}(X \cap Y)}{\mathrm{P}(Y)}$,

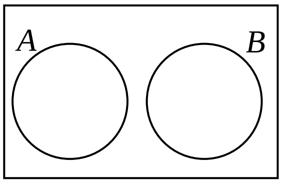
$$\mathrm{P}(Y) = \frac{\mathrm{P}(X \cap Y)}{\mathrm{P}(X \mid Y)} = \frac{4/15}{1/2} = \frac{8}{15}$$

(iii) $\mathrm{P}(X \cup Y) = \mathrm{P}(X) + \mathrm{P}(Y) - \mathrm{P}(X \cap Y) = \frac{2}{5} + \frac{8}{15} - \frac{4}{15} = \frac{2}{3}$

2.4 Mutually Exclusive Events

Event A and Event B are said to be mutually exclusive if they **cannot occur at the same time**. That is,

$$A \cap B = \varnothing \text{ or } \mathrm{P}(A \cap B) = 0.$$



Can these events happen at the same time?	
▪ Turn left and right at the same time	Yes / No
▪ Show '1' and '4' when you roll a die	Yes / No
▪ Show '1' and '4' when you roll 2 dice	Yes / No
▪ Show 'Ace' and 'Spade' when you pick a poker card	Yes / No

If your answer is **'No'** to the above questions, it means that 2 events A and B cannot happen at the same time. Then we say A and B are mutually exclusive events.

Since $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$, this also implies that

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$$

for mutually exclusive events.

Example 9

An ordinary die is thrown. Find the probability that the number obtained is
(a) a factor of 8 (b) a multiple of 3 (c) a factor of 8 or a multiple of 3.

Solution:
Event A: number obtained is a factor of 8
Event B: number obtained is a multiple of 3

A = {1,2,4} and B = {3,6}

(a) $\mathrm{P}(A) = \frac{3}{6} = \frac{1}{2}$

(b) $\mathrm{P}(B) = \frac{2}{6} = \frac{1}{3}$

(c) Since no number is both a factor of 8 and a multiple of 3, A and B are mutually exclusive events.

$\therefore \mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ (since $\mathrm{P}(A \cap B) = 0$)

2.5 Independent Events

Two events A and B are independent events if the occurrence of event A **does not affect** the occurrence of event B and vice versa.

Can the first event affect the chance of second event happening?	
▪ Show a '6' when you roll a die and show a 'tail' when you throw a coin.	Yes / No
▪ Choosing a "3" from a deck of cards, replacing it, **AND** then choosing an ace as the second card.	Yes / No
▪ Getting a black 'King' on the first card and getting a black card as the second card (without replacement).	Yes No

If your answer is **'No'** to the above questions, it means that 2 events can happen without affecting each other (and their probability). Then we say A and B are independent events.

Important Results

If A and B are independent events, then $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ since event A is not affected by event B. Similarly, $\mathrm{P}(B \mid A) = \mathrm{P}(B)$.

Check for Independence

Events A and B are independent if and only if

(1) $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ Or

(2) $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ Or

(3) $\mathrm{P}(B \mid A) = \mathrm{P}(B)$

Note:

Events A and B are independent $\Leftrightarrow$

(a) A and B' are independent

(b) A' and B are independent

(c) A' and B' are independent

Example 10

A die is thrown twice. Find the probability of obtaining a `4' on the first throw and an odd number on the second throw.

Solution:

Note that the events "obtaining 4 on the first throw" and "obtaining odd number on the second throw" are independent.

P(obtaining 4 on first throw and odd number on the second throw) $= \frac{1}{6} \times \frac{3}{6} = \frac{1}{12}$

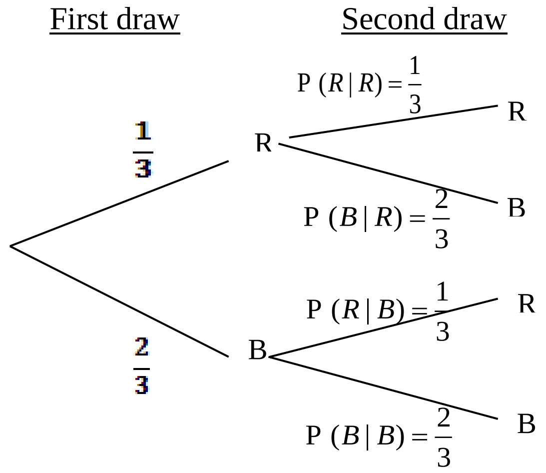
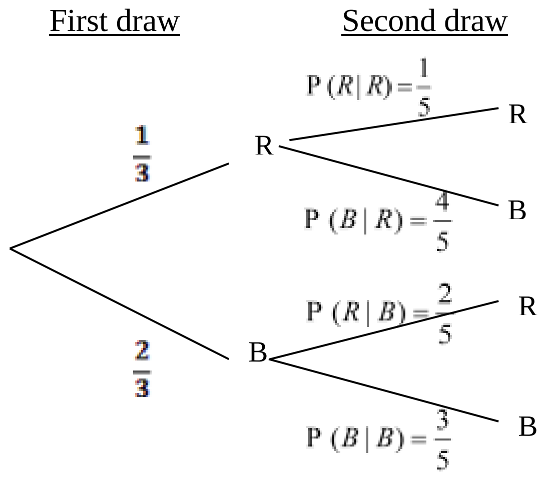
Can 2 events be independent and mutually exclusive at the same time?
<u>**Ans:** No *(unless one of the events have probability 0 of occurring)*</u>

If A and B are mutually exclusive, then $\mathrm{P}(A \cap B) = 0$

If A and B are independent, then $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

Thus if A and B are independent and mutually exclusive, then either P(A) = 0 or P(B) = 0 or both.

<u>Example 11</u>

If events are independent	If events are **NOT** independent
There are 2 red balls and 4 blue balls in a bag. Pick a ball at random and **replace it in the bag**. Mix the balls well and pick a second ball randomly again. What is the probability that both balls are red?	There are 2 red balls and 4 blue balls in a bag. Pick a ball at random and pick a second ball **without replacing the first one**. What is the probability that both balls are red?
	
P(both balls red) = $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$	P(both balls red) = $\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$

Tree diagrams, by their very nature, are closely linked to conditional probability!

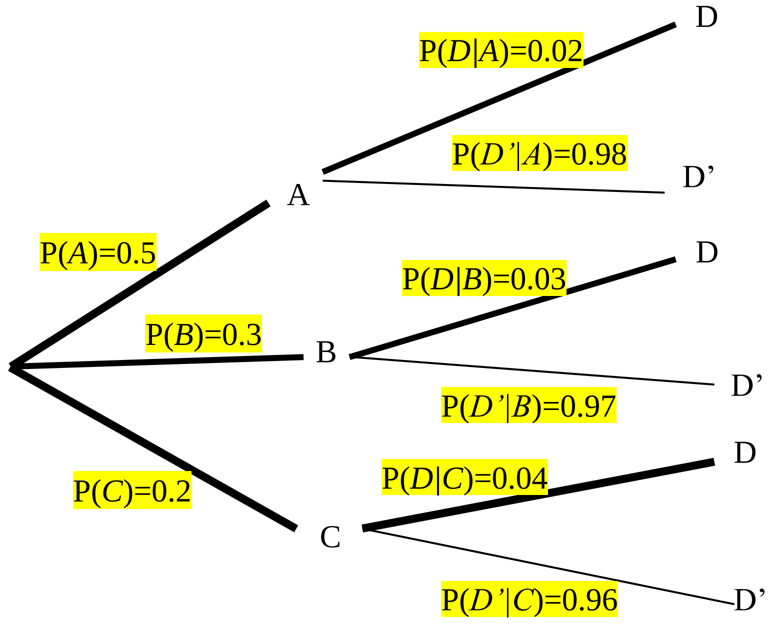
The advantage of tree diagrams is the limitation of Venn Diagrams:
Venn Diagrams usually work best for <u>concurrent</u> events (i.e. events that happen at the same time). Tree diagrams work better for <u>sequential</u> events (i.e. events that follow one after the other).

Example 12

A factory has three machines A, B, C producing large numbers of a certain item. Of the total daily production of the item, 50% are produced by A, 30% by B and 20% by C. Records show that 2% of the items produced by A are defective, 3% of items produced by B are defective and 4% of items produced by C are defective. The occurrence of a defective item is independent of all other items. One item is chosen at random from a day's total output.
Draw a tree diagram and find the probability of the item being defective.

Solution:

A : item produced by A
B : item produced by B
C : item produced by C
D : item is defective



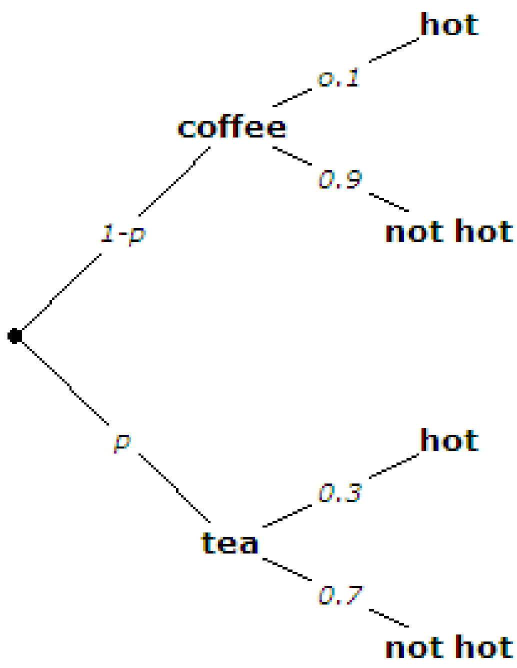
$$\begin{aligned} P(D) &= P(A\cap D)+P(B\cap D)+P(C\cap D) \\ &= P(A)P(D|A) + P(B)\,P(D|B) + P(C)\,P(D|C) \\ &= (0.5)\,(0.02) + (0.3)\,(0.03) + (0.2)\,(0.04) \\ &= 0.027 \end{aligned}$$

Remark:
This is known as the ***Law of Total Probability*** as it gives a way of finding the total probability of a particular event based on conditional probabilities.

Example 13

Susan only buys coffee or tea from the Shangri-La Café. The probability of Susan buying a cup of tea on any given day is p. The probability that a randomly chosen coffee Susan buys is hot is 0.1. The probability that a randomly chosen tea Susan buys is not hot is 0.7. Using a tree diagram, calculate, leaving your answers in terms of p, the probability that a hot drink bought by Susan is tea.

Solution:



$$\begin{aligned} P(\text{tea} \mid \text{hot}) &= \frac{P(\text{tea \& hot})}{P(\text{hot})} \\ &= \frac{0.3p}{(1-p)0.1+0.3p} \\ &= \frac{0.3p}{0.1+0.2p} \end{aligned}$$

Note to tutors: This question is from CT 2010. Many students could not interpret the question correctly (which is to find the conditional probability).

Exercise 3

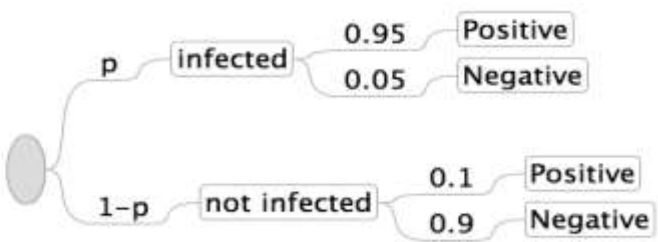
The medical test for a certain infection is not completely reliable: if an individual has the infection there is a probability of 0.95 that the test will prove positive, and if an individual does not have the infection there is a probability of 0.1 that the test will prove positive. In a certain population, the probability that an individual chosen at random will have the infection is p. Draw a tree diagram to represent this information.

(i) An individual is chosen at random and tested. Show that the probability of the test being positive is $0.1 + 0.85p$.

(ii) Express, in terms of p, the conditional probability that a randomly chosen individual whose test is positive has the infection. Given that this probability is 0.6, find the conditional probability that a randomly chosen individual whose test is negative does not have the infection.

[Ans: (ii) $\dfrac{0.95p}{0.1+0.85p}$; 0.991]

Solution:



(i) P(test positive)

$= p(0.95) + (1 - p)(0.1)$

$= 0.1 + 0.85p$ (shown)

(ii) P(randomly chosen person has the infection| test is positive)

$$= \frac{\text{P(random person is infected and test is positive)}}{\text{P(test is positive)}}$$

$$= \frac{0.95p}{0.1+0.85p}$$

If this probability is 0.6,

$$\frac{0.95p}{0.1+0.85p} = 0.6$$

$p = 0.13636$

P(randomly chosen person does not have the infection| test is negative)

$$= \frac{\text{P(random person is not infected and test is negative)}}{\text{P(test is negative)}}$$

$$= \frac{0.9(1-p)}{1-(0.1+0.85p)}$$

$= 0.991$

2.6 A Mix of Probabilities

Example 14

Three events *A*, *B* and *C* are such that

$$\mathrm{P}(B)=\frac{1}{4},\quad \mathrm{P}(C)=\frac{1}{3},\quad \mathrm{P}(A\cup B)=\frac{3}{5} \text{ and } \mathrm{P}(A\cup C)=\frac{2}{3}.$$

Events *A* and *B* are independent events, while events *B* and *C* are mutually exclusive events.

(i) Find $\mathrm{P}(A)$.

(ii) Determine whether *A* and *C* are independent events.

(iii) Show that $\mathrm{P}\big((B\cup C)\mid A\big)=\frac{15}{28}$.

Solution:

(i)
$$\begin{aligned}\mathrm{P}(A\cup B)&=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)\\ &=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A)\times\mathrm{P}(B) \quad \text{as } A, B \text{ are independent}\\ \frac{3}{5}&=\mathrm{P}(A)+\frac{1}{4}-\frac{1}{4}\mathrm{P}(A)\\ \mathrm{P}(A)&=\frac{7}{15}\end{aligned}$$

(ii)
$$\begin{aligned}\mathrm{P}(A\cap C)&=\mathrm{P}(A)+\mathrm{P}(C)-\mathrm{P}(A\cup C)\\ &=\frac{7}{15}+\frac{1}{3}-\frac{2}{3}=\frac{2}{15}\\ \mathrm{P}(A)\times\mathrm{P}(C)&=\frac{7}{15}\times\frac{1}{3}=\frac{7}{45}\end{aligned}$$

Since $\mathrm{P}(A\cap C)\neq\mathrm{P}(A)\times\mathrm{P}(C)$, *A* and *C* are not independent events.

(iii)
$$\begin{aligned}\mathrm{P}\big((B\cup C)\mid A\big)&=\frac{\mathrm{P}\big((B\cup C)\cap A\big)}{\mathrm{P}(A)}\\ &=\frac{\mathrm{P}\big((B\cap A)\cup(C\cap A)\big)}{\mathrm{P}(A)}\\ &=\frac{\frac{1}{4}\times\frac{7}{15}+\frac{2}{15}}{\frac{7}{15}}=\frac{15}{28}\ \text{(shown)}\end{aligned}$$

Example 15

Three cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that

(i) all three cards are picture cards (Jack, Queen and King).
(ii) exactly one card is a club.
(iii) at least one card is a club.
(iv) all three cards are club, given that at least one is a club.

Solution:

Using permutations and combinations	**Using basic probability law (TEDIOUS)**
(i) P(all 3 are picture cards) $=\frac{^{12}C_3}{^{52}C_3}=\frac{220}{22100}=\frac{11}{1105}$	P(all 3 are picture cards) $=\frac{12}{52}\cdot\frac{11}{51}\cdot\frac{10}{50}=\frac{1320}{132600}=\frac{11}{1105}$
(ii) P(Exactly one card is a club) i.e 1 out of 13 is a club and 2 of the remaining 39 are not clubs. $=\frac{^{13}C_1\,^{39}C_2}{^{52}C_3}=0.436$	P(Exactly one card is a club) $=\frac{13}{52}\cdot\frac{39}{51}\cdot\frac{38}{50}.3=0.436$ C _ _ or _ C _ or _ _ C
(iii) P(At least one card is a club) $=1-$ P(all three cards are not clubs) $=1-\frac{^{39}C_3}{^{52}C_3}$ $=0.58647\approx0.586$	P(At least one card is a club) $=1-$ P(all three cards are not clubs) $=1-\frac{39}{52}\cdot\frac{38}{51}\cdot\frac{37}{50}$ $=0.58647$ ≈0.586
(iv) P(All three cards are club | at least one is a club) $=\frac{\text{P}\left(\text{All three cards are club}\right)}{\text{P}\left(\text{at least one of the cards is club}\right)}$ $=\frac{\frac{^{13}C_3}{^{52}C_3}}{0.58647}$ $=0.0221$	P(All three cards are club | at least one is a club) $=\frac{\text{P}\left(\text{All three cards are club}\right)}{\text{P}\left(\text{at least one of the cards is club}\right)}$ $=\frac{\frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}}{0.58647}$ $=0.0221$

Example 16

A game is played with an ordinary six-sided die. A player throws this die, and if the result is 2, 3, 4 or 5, that result is the player's score. If the result is 1 or 6, the player throws the die a second time and the sum of the two numbers resulting from both throws is the player's score. Events *A* and *B* are defined as follows:

A: the player's score is 5, 6, 7, 8 or 9

B: the player has two throws.

(i) Draw a tree diagram to represent this situation, showing all possible outcomes.

(ii) Show that P(*A*) = 1/3.

(iii) Find $\mathrm{P}(A \cap B)$.

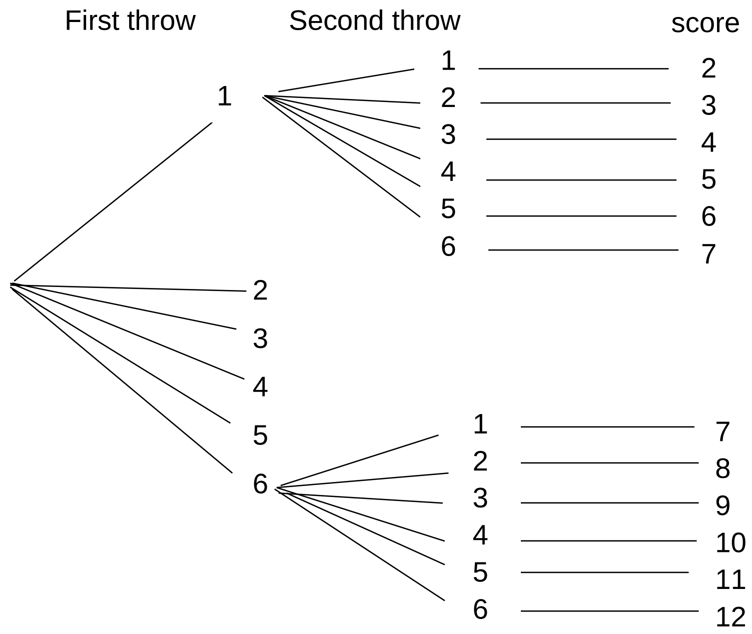
(iv) Find $\mathrm{P}(A \mid B)$ and describe what this probability meant in this context.

(v) Determine if events *A* and *B* are

(a) mutually exclusive, (b) independent.

Solution:

(i)



(ii)

$$\mathrm{P}(A) = \text{P(1st throw rolls a '5') + P(1st throw rolls a '1' or '6' and score '5,6,7,8 or 9')}$$

$$= \frac{1}{6} + \frac{1}{36} \times 6$$

$$= \frac{1}{3} \text{ (Shown)}$$

(iii) $\mathrm{P}(A \cap B) = \frac{1}{36} \times 6 = \frac{1}{6}$

(iv) $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{1}{6} \div \frac{1}{3} = \frac{1}{2}$

The probability that a player gets a score of 5,6,7,8 or 9 given that he had two throws is $\frac{1}{2}$.

(v) (a) $\mathrm{P}(A \cap B) = \frac{1}{6} \neq 0$. Therefore A & B are not mutually exclusive.

(b) $\frac{1}{2} = \mathrm{P}(A|B) \neq \mathrm{P}(A) = \frac{1}{3}$. Therefore A & B are not independent.

Or

$\mathrm{P}(A \cap B) = \frac{1}{6} \neq \mathrm{P}(A)\mathrm{P}(B) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$. Therefore A & B are not independent.

Exercise 4

1. Two events A and B are such that $\mathrm{P}(A) = \frac{1}{4}$, $\mathrm{P}(A|B) = \frac{1}{2}$ and $\mathrm{P}(B|A) = \frac{2}{3}$.

(i) Find $\mathrm{P}(A \cap B)$.
(ii) Are A and B independent events? Justify.
(iii) Are A and B mutually exclusive events? Justify.
(iv) Find $\mathrm{P}(B)$. [Ans: (i) $\frac{1}{6}$ (iv) $\frac{1}{3}$]

Solution:

i) $\mathrm{P}(A \cap B)$ $= \mathrm{P}(B|A) \times \mathrm{P}(A)$ $= \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$	ii) $P(A|B) = \frac{1}{2} \neq P(A)$ $\therefore$ A & B are not independent	iii) Since $\mathrm{P}(A \cap B) = \frac{1}{6} \neq 0$ $\therefore$ A & B are not mutually exclusive.	iv) $\mathrm{P}(B)$ $= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A|B)}$ $= \frac{1}{6} \div \frac{1}{2} = \frac{1}{3}$

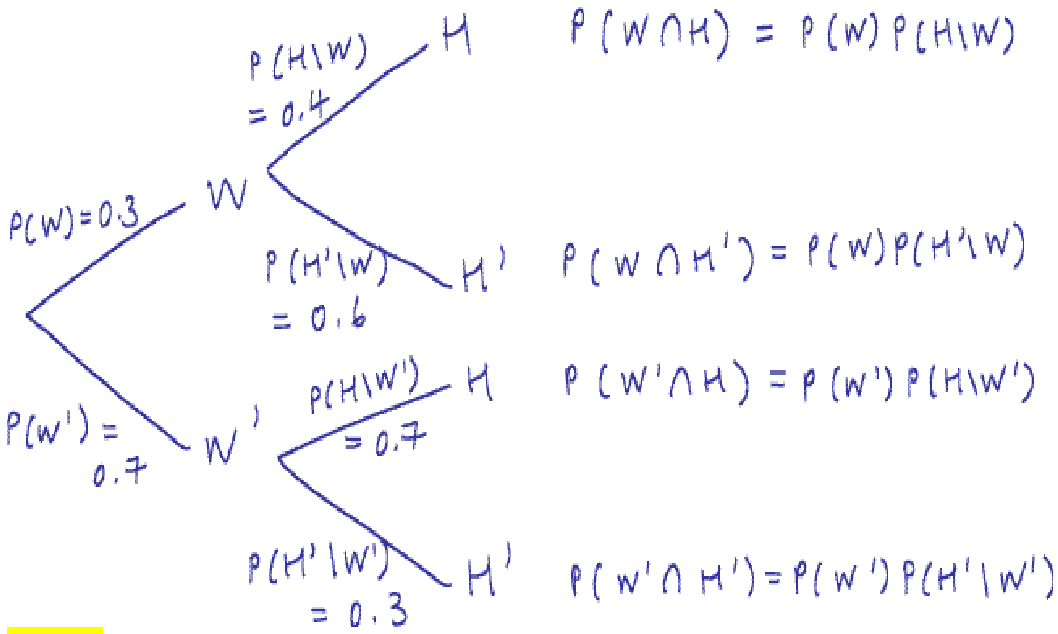
2. The probability that a golfer hits the ball if it is windy is 0.4 and the probability that he hits the ball if it is not windy is 0.7. The probability that it is windy is 0.3. Draw a tree diagram and hence find the probability that

(i) he hits the ball;
(ii) it was not windy given that he does not hit the ball.

[Ans: (i) 0.61; (ii) 7/13]

Solution:

Events :
W : it is windy
H : golfer hits the ball



(i) P(H) = (0.3)(0.4) + (0.7)(0.7)= 0.61

(ii) $\mathrm{P}(W^{'} \mid H^{'}) = \dfrac{\mathrm{P}(W^{'} \cap H^{'})}{\mathrm{P}(H^{'})} = \dfrac{0.7(0.3)}{1-\mathrm{P}(H)} = \dfrac{7}{13}$

3. Seven cards, labeled A, B, C, D, E, F, G, are thoroughly shuffled and dealt out face upwards on a table. Find the probabilities that
(a) the first three cards to appear are the cards labeled A, B, C, in that order,
(b) the first three cards to appear are the cards labeled A, B, C, but in any order,
(c) the seven cards appear in that original order : A, B, C, D, E, F, G.

[Ans: (a) 1/210; (b) 1/35; (c) 1/5040]

Solution

(a) Required prob = $\dfrac{1}{7} \times \dfrac{1}{6} \times \dfrac{1}{5} = \dfrac{1}{210}$

(b) Required prob = $\dfrac{1}{7} \times \dfrac{1}{6} \times \dfrac{1}{5} \times 3! = \dfrac{1}{35}$

(c) Required prob = $\dfrac{1}{7} \times \dfrac{1}{6} \times \dfrac{1}{5} \times \dfrac{1}{4} \times \dfrac{1}{3} \times \dfrac{1}{2} \times \dfrac{1}{1} = \dfrac{1}{5040}$

4. The events A and B are such that $\mathrm{P}(A) = 0.45$, $\mathrm{P}(B) = 0.35$ and $\mathrm{P}(A \cup B) = 0.7$.

(i) Find the value of $\mathrm{P}(A \cap B)$.

(ii) Are the events A and B independent? Explain why.

(iii) Find $\mathrm{P}(A \mid B)$, leaving your answer as a fraction.

[Ans: (i) 0.1;(iii) $\dfrac{2}{7}$]

Solution:

(i) $\mathrm{P}(A \cap B)$=0.45 + 0.35 – 0.7 = 0.1

(ii) $\mathrm{P}(A)\times\mathrm{P}(B)=0.45\times0.35$
$=0.1575$

Since $\mathrm{P}(A \cap B) \neq \mathrm{P}(A)\times\mathrm{P}(B)$,

A & *B* are not independent events.

(iii) $\mathrm{P}(A|B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=\dfrac{0.1}{0.35}=\dfrac{2}{7}$

<u>Question</u>

Is $\mathrm{P}(A|B)=\mathrm{P}(B|A)$?

No. They are the same only when P(*A*) = P(*B*)

Practice Questions

1. The probabilities that a person travelling in the eastern region of Singapore will visit Tampines or Changi or both places are 0.45, 0.36 and 0.18 respectively. Find the probability that such a person

(i) who is visiting Tampines will also visit Changi;

(ii) who is visiting Changi will also visit Tampines.

[Ans: (i) 0.4; (ii) 0.5]

Solution:

(i) $\mathrm{P}(T)=0.45;\mathrm{P}(C)=0.36;\mathrm{P}(T\cap C)=0.18$

$$\mathrm{P}(C\,|\,T)=\frac{\mathrm{P}(C\cap T)}{\mathrm{P}(T)}=\frac{0.18}{0.45}=0.4$$

(ii) $$\mathrm{P}(T\,|\,C)=\frac{\mathrm{P}(C\cap T)}{\mathrm{P}(C)}=\frac{0.18}{0.36}=0.5$$

2. A and B are two events such that $\mathrm{P}(A)=0.7$, $\mathrm{P}(B)=0.2$ and $\mathrm{P}(A\cup B)=0.8$. Find

(i) $\mathrm{P}(A\,|\,B)$ (ii) $\mathrm{P}(B\,|\,A)$

(iii) $\mathrm{P}(A'\cap B')$ (iv) $\mathrm{P}(A'\,|\,B)$

(v) $\mathrm{P}(B'\,|\,A')$

[Ans: (i) $\frac{1}{2}$ (ii) $\frac{1}{7}$ (iii) 0.2 (iv) $\frac{1}{2}$ (v) $\frac{2}{3}$]

Solution:

(i)

$$\begin{aligned}\mathrm{P}(A\cap B)&=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cup B)\\&=0.7+0.2-0.8\\&=0.1\end{aligned}$$

$$\mathrm{P}(A\,|\,B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=\frac{0.1}{0.2}=\frac{1}{2}$$

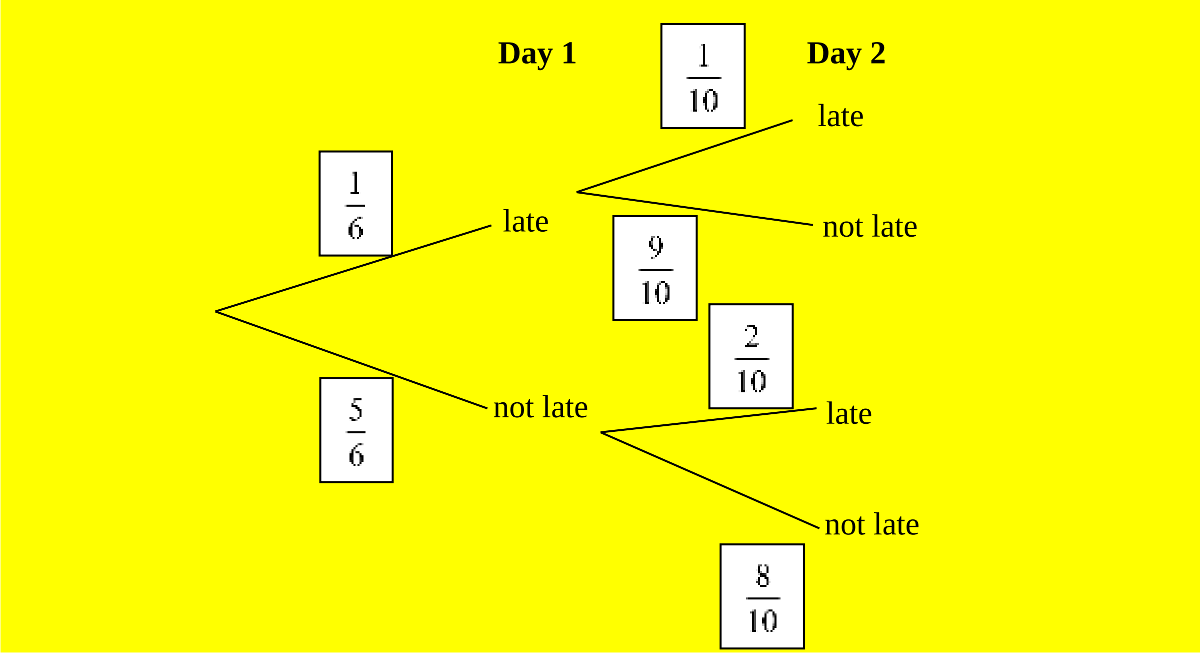
(ii) $$\mathrm{P}(B\,|\,A)=\frac{\mathrm{P}(B\cap A)}{\mathrm{P}(A)}=\frac{0.1}{0.7}=\frac{1}{7}$$

(iii) $\mathrm{P}(A'\cap B')=1-\mathrm{P}(A\cup B)=1-0.8=0.2$

(iv) $$\mathrm{P}(A'\,|\,B)=\frac{\mathrm{P}(A'\cap B)}{\mathrm{P}(B)}=\frac{\mathrm{P}(B)-\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=\frac{0.2-0.1}{0.2}=\frac{1}{2}$$

(v) $$\mathrm{P}(B'\,|\,A')=\frac{\mathrm{P}(B'\cap A')}{\mathrm{P}(A')}=\frac{\mathrm{P}(B'\cap A')}{1-\mathrm{P}(A)}=\frac{0.2}{0.3}=\frac{2}{3}$$

3. The probability that a new delivery boy will deliver the newspaper late on his first day of work is $\frac{1}{6}$. The probability that he will deliver the newspaper late again if he had delivered late the day before is $\frac{1}{10}$. There is 20% chance that he will deliver the newspaper late if was not late the day before.



(a) Fill in the boxes in the tree diagram above.

(b) Find the probability that

(i) the newspaper will not be delivered late on his first day of work,

(ii) the newspaper will be delivered late on his first two days of work,

(iii) the newspaper will be delivered late on just one of his first two days of work.

[Ans: (bi) $\frac{5}{6}$ (bii) $\frac{1}{60}$ (biii) $\frac{19}{60}$]

Solution:

(b) (i) P(newspaper will not be delivered late on his first day of work)

$= 1-\frac{1}{6}=\frac{5}{6}$

(ii) P(delivered late on his first two days of work)

$=\frac{1}{6}\times\frac{1}{10}=\frac{1}{60}$

(iii) P(delivered late on just one of his first two days of work)

$=\frac{1}{6}\times\frac{9}{10}+\frac{5}{6}\times\frac{2}{10}=\frac{19}{60}$

4. A seed merchant sells three types of flower seed, F_1, F_2 and F_3. They are sold as a mixture where the proportions are 4 : 4 : 2 respectively. The germination rates of the three types are 45%, 60% and 35% respectively.

Draw a tree diagram to illustrate this data. Hence calculate the probabilities that

(a) a randomly chosen seed will germinate;

(b) given that the seed is of type F_3, it will not germinate;

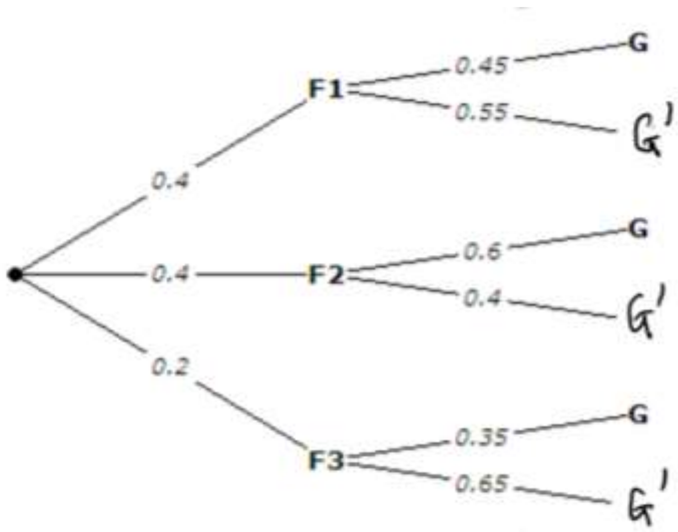
(c) the seed is of type F_3 if it does not germinate;

(d) given that a randomly chosen seed does germinate, it is of type F_2.

[Ans: (a) 0.49; (b) 0.65; (c) $\frac{13}{51}$; (d) $\frac{24}{49}$]

Solution:

Events are G : germinate, G' : does not germinate



(a) P(seed will germinate) = P(G) = 0.4×0.45 + 0.4 ×0.6 + 0.2×0.35 = 0.49

(b) P(given that the seed is of type F_3, it will not germinate) = P($G' \mid F_3$) = 0.65

(c) P(the seed is of type F_3 given that it does not germinate) = P($F_3 \mid G'$)

$$= \frac{P(F_3 \cap G')}{P(G')} = \frac{0.2 \times 0.65}{(1-0.49)} = \frac{13}{51}$$

(d) $P(F_2 \mid G) = \frac{P(F_2 \cap G)}{P(G)} = \frac{0.4 \times 0.6}{(0.49)} = \frac{24}{49}$

5. [2010 H2 A Level Exam P2 Q8]
The digits 1, 2, 3, 4 and 5 are arranged randomly to form a five-digit number. No digit is repeated. Find the probability that
(i) the number is greater than 30 000, [1]
(ii) the last two digits are both even, [2]
(iii) the number is greater than 30 000 and odd. [4]

[Ans: 0.6; 0.1; 0.35]

Solution

(i) P(number > 30000) = $\frac{3\times 4!}{5!}=\frac{3}{5}$ or 0.6

(ii) P(last 2 digits both even) = P(last 2 digits are 2,4 or 4,2) = $\frac{3!\times 2}{5!}=\frac{1}{10}$ or 0.1

(iii) Case 1 : 1st digit is 4 and last digit is 1/3/5
No. of such numbers = $3!\times 3$
Case 2 : 1st digit is 3/5 and last digit is 1/3/5
No. of such numbers = $2\times 3!\times 2$
P(number > 30000 and odd) = $\frac{3!\times 3+2\times 3!\times 2}{5!}=\frac{7}{20}$ or 0.35

6. When a person needs a minicab, it is hired from one of three firms X, Y and Z. Of the hirings, 40% are from X, 50% are from Y and 10% are from Z. For cabs hired from X, 9% arrive late, with the corresponding percentages for cabs hired from firms Y and Z being 6% and 20% respectively.

Calculate the probability that the next cab hired

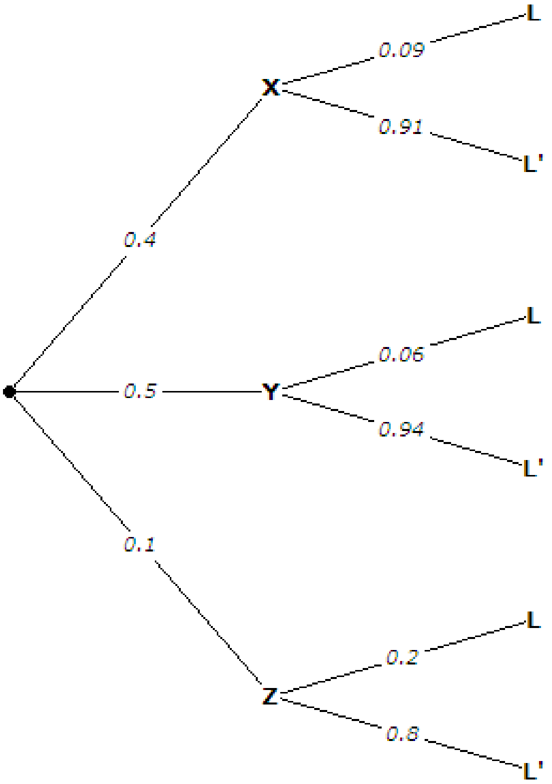
(i) will be from X and will not arrive late,

(ii) will arrive late.

[Ans: (i) 0.364 (ii) 0.086]

Solution:
X : hired from firm X
Y : hired from firm Y
Z: hired from firm Z

L : arrive late



(i) P(will be from X and will not arrive late) = $0.4 \times 0.91 = 0.364$

(ii) P(will arrive late) = $0.4 \times 0.09 + 0.5 \times 0.06 + 0.1 \times 0.2 = 0.086$

7 [RJC/2009/H2/P2/Q6]

(a) Two events A and B are such that $\mathrm{P}(A)=0.2$ and $\mathrm{P}(A\cup B)=0.6$. Find $\mathrm{P}(B)$ if

(i) A and B are mutually exclusive,

(ii) A and B are independent.

(b) In 2008, an insurance company classifies 10% of their car policy holders as "good risks", 60% as "average risks" and 30% as "bad risks". Their statistical database in 2008 has shown that of those classified as "good risks", only 1% were involved in at least one accident whereas of those classified as "bad risks", 25% were involved in at least one accident. As for those classified as "average risks", 15% were involved in at least one accident. Find the probability that

(i) a randomly chosen car policy holder was involved in at least one accident in 2008.

(ii) a randomly chosen car policy holder was classified as "good risks" if the car policy holder was not involved in any accidents in 2008.

[Ans (ai) 0.4; (aii) 0.5; (bi) 0.166, (bii) 0.119]

Solution:

(a) (i) Since A and B are mutually exclusive,

$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)$

$\Rightarrow \mathrm{P}(B)=\mathrm{P}(A\cup B)-\mathrm{P}(A) = 0.6-0.2 = 0.4$

(a) (ii) Since A and B are independent,

$\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B) = 0.2\,\mathrm{P}(B) \ \ldots (1)$

Also, $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$

$\Rightarrow \mathrm{P}(B)=\mathrm{P}(A\cup B)-\mathrm{P}(A)+\mathrm{P}(A\cap B)$

$\Rightarrow \mathrm{P}(B)=0.6-0.2+0.2\mathrm{P}(B)$ [from (1)]

$\Rightarrow 0.8\mathrm{P}(B)=0.4 \Rightarrow \mathrm{P}(B)=0.5$

(b) (i) Required probability

$= (0.1\times 0.01)+(0.6\times 0.15)+(0.3\times 0.25)$

$= 0.166$

(b) (ii)

$$\mathrm{P}(\text{good risks} \mid \text{no accidents}) = \frac{\mathrm{P}(\text{good risks and no accidents})}{\mathrm{P}(\text{no accidents})}$$

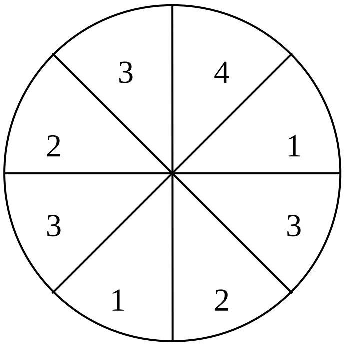
$$= \frac{0.1\times(1-0.01)}{1-0.166}$$

$$= \frac{0.099}{0.834} = 0.119$$

8. [2016/PJC/Promo/Q9]

In a carnival, a stall owner designed a game making use of a wheel which is equally divided into eight sectors, with numbers on the sectors as shown in the diagram. To play the game, a player spins the wheel once.

If the number obtained is '1' or '2', the player will throw an unbiased die and the score is the number shown on the die.
If the number obtained is '3', the player will throw two unbiased dice and the score is the sum of the numbers shown on the dice.
If the number obtained is '4', the score is recorded as 4.



(i) Find the probability that a player obtains the highest score. [2]

(ii) Find the probability that a player has a score which is at least 4. [3]

(iii) Find the probability that a player spins a '2', given that his score is at least 4. [3]

Two players are randomly chosen. Find the probability that one player obtains the highest score and the other player has a score which is at least 4. [3]

[Ans (i) $\frac{1}{96}$; (ii) $\frac{23}{32}$; (iii) $\frac{4}{23}$; $\frac{23}{1536}$]

Solution:

(i) P(highest score) = P (spin '3') P (sum = 12)

$$=\left(\frac{3}{8}\right)\left(\frac{1}{36}\right)$$

$$=\frac{1}{96}$$

(ii) P(score ≥ 4)

$= $ P (spin '1' or '2') P(die score ≥ 4) + P (spin '3') P (sum ≥ 4) + P (spin '4')

$$=\left(\frac{4}{8}\right)\left(\frac{3}{6}\right)+\left(\frac{3}{8}\right)\left(\frac{33}{36}\right)+\frac{1}{8}$$

$$=\frac{23}{32}$$

(iii) P(spin '2' | score ≥ 4) = $\dfrac{P(\text{spin}'2' \cap \text{score} \geq 4)}{P(\text{score} \geq 4)}$

$$= \frac{\left(\frac{2}{8}\right)\left(\frac{3}{6}\right)}{\frac{23}{32}}$$

$$= \frac{4}{23}$$

Required Probability = P(highest score) P(score ≥ 4) $\times$ 2

$$= \left(\frac{1}{96}\right)\left(\frac{23}{32}\right)\times 2$$

$$= \frac{23}{1536}$$

Summary

Classical/ Theoretical Definition of Probability

If the sample space S consists of a finite number of <u>equally likely</u> outcomes, then the probability of an event A is defined as

$$\mathrm{P}(A) = \frac{\text{number of outcomes in event } A}{\text{total number of possible outcomes}} = \frac{\mathrm{n}(A)}{\mathrm{n}(S)}$$

Some Important Results of Probability

1. $0 \le \mathrm{P}(A) \le 1$
2. $\mathrm{P}(A') = 1 - \mathrm{P}(A)$
3. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

Conditional Probability (finding the probability of event A, given that event B has already occurred)

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \text{ or } \mathrm{P}(A \mid B) = \frac{\mathrm{n}(A \cap B)}{\mathrm{n}(B)}$$

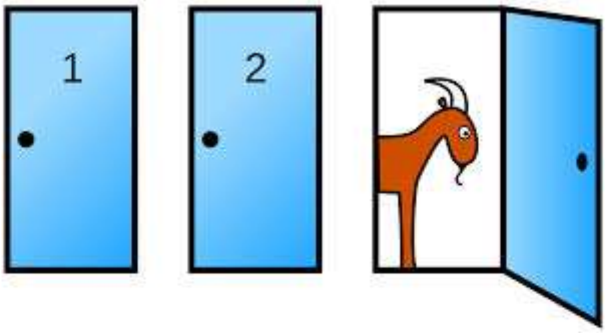
Mutually Exclusive and Independent Events

Events	**Definition**	**Test**
Mutually Exclusive	Event A and Event B are said to be mutually exclusive if they **cannot occur at the same time**.	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ or $\mathrm{P}(A \cap B) = 0$
Independent	Two events A and B are independent events if the occurrence of event A **does not affect** the occurrence of event B and vice versa.	$\mathrm{P}(A \mid B) = \mathrm{P}(A)$ or $\mathrm{P}(B \mid A) = \mathrm{P}(B)$ or $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

Checklist

- ☐ I understand that probability of an event measures how likely the event will occur;
- ☐ I can use table of outcomes, Venn Diagrams or tree diagram to calculate the probabilities;
- ☐ I can use a Venn diagram to draw and interpret probabilities such as $\mathrm{P}(A')$, $\mathrm{P}(A \cup B)$, $\mathrm{P}(A \cap B)$, $\mathrm{P}(A \mid B)$;
- ☐ I understand that when 2 events are mutually exclusive, it means the probability of these 2 events happening at the same time is 0;
- ☐ I understand that when 2 events are independent, it means $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.

Learning Experience Worksheet: Let's Make a Deal!



The Monty Hall Problem

There are 3 doors – behind one door is a new car (the grand prize) & behind the other two are goats.
The contestant of the game show chooses one door and then the host opens one of the other two revealing a goat. Note that the host knows behind which door lies the grand prize. The contestant will be invited to choose to take the prize behind the door he picked initially or he may switch door and take the prize behind the other remaining closed door.

The question is: To maximise their chances of winning, should they switch doors or stay with their initial pick or does it make no difference?

Introduction

Switching doors in Let's Make a Deal is in fact the better strategy. The following simple experiment will allow students to get hands-on experience with what is known as the Monty Hall problem and allow them to prove that switching doors is the preferred strategy. It also allows them see how through amassing trials, experimentally determined probabilities will tend to approach the theoretical value.

Learning Objectives

By the end of this activity students should:

- Understand the difference between theoretical and experimental probabilities
- Observe that as the number of experimental trials increases, experimental probabilities grow closer to the theoretical probability•
- Use a tree diagram to enumerate outcomes and determine the theoretical probability of events

Skills

Students should develop:
• Data collection skills
• The ability to analyse data and calculate probability
• Collaboration and communication skills

Materials needed

A piece of cardboard with three doors, 1 plastic toy car and 2 goat figurines.

Lesson Planning

Teacher starts off the lesson by explaining the Monty Hall Problem (as above). You may also invite a student to play a few rounds of the game using a simulation http://math.ucsd.edu/~crypto/Monty/monty.html.

Teacher is to explain to the students the two simple strategies a player could follow:

1) Switch door after the first pick to the one that wasn't opened OR
2) Stay with the first pick even after you are shown that there was a goat behind one of the remaining doors.

Teacher then ask the students what they think the best strategy is and why before they start performing the experiment.

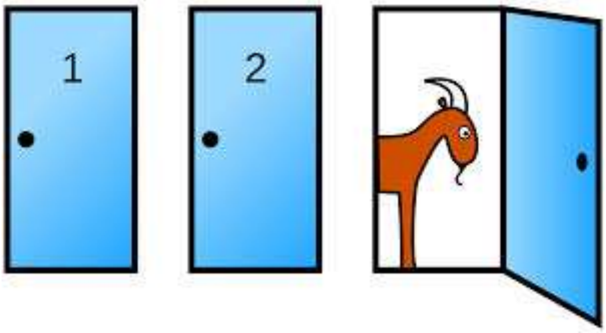
Divide the class into 4 – 5 groups of 4 to 5 students each. Within each group, one of students plays host and another student plays the role of the contestant. The other students will be the observer and recording of the outcomes. Each team should play the game 50 times and they are required to record in a table the number of wins and losses when the contestant switches or does not switch door.

After all teams complete their experiment, teacher may compile the class data and compute the winning probability for the two different strategies. At the same time, the students may continue to complete the worksheet in their assigned teams.

Estimated Time Required

1. **Introduction and instructions – 5 mins**
2. **Students to conduct experiments – 15 mins**
3. **Students to complete worksheet – 15 mins**
4. **Teacher led class in discussion & conclusion – 20 mins**

Worksheet
Learning Experience Worksheet: Let’s Make a Deal!



The Monty Hall Game
Suppose you're on a game show, and you're given the choice of three doors: behind one door is the Grand Prize and behind the other two doors are goats. You pick a door, say Door 1, and the host (who knows what is behind each door) opens another door, say Door 2, revealing a goat. The host then offers you the opportunity to change your selection to Door 3. You know that the Grand Prize is either behind Door 1 or Door 3. Should you stick with your original choice or switch? Does it make any difference?

Instructions for Students
You will experiment with the problem described in the introduction. The problem is known as "The Monty Hall Problem," named for the game show host of Let's Make a Deal.

1. You will play the Monty Hall Game as an experiment to determine whether or not you win more often when you switch doors. **What is your prediction of the outcome? Why?**

2. Play the Monty Hall Game. Record your results in the table below.
Be sure to play 50 times WITH switching doors and 50 times WITHOUT switching doors.

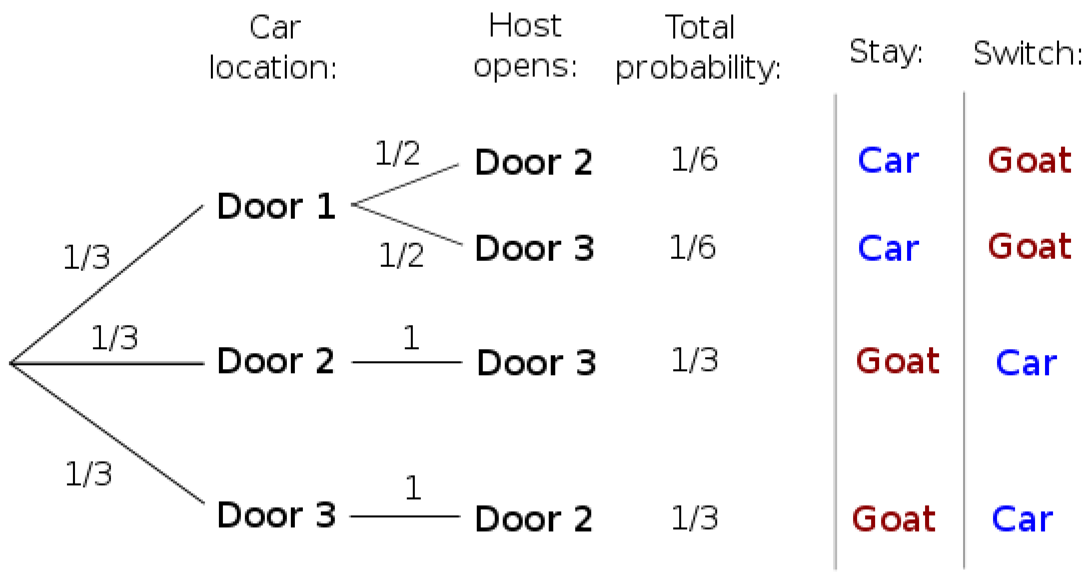
STRATEGY	**Switch Doors**	**Don’t Switch Doors**
WINS		
LOSSES		
Probability of Winning the Grand Prize		

3. Do your results show a difference in your probability of winning based on whether or not you decided to switch doors ? Explain.

4. How do your results compared with your prediction in (1)?

5. As you & your team are collecting data to find the experimental probability, try to think about what the theoretical probability of winning would be if you switched doors and what it would be if you didn't switch. (Hint: You may construct a Tree Diagram).

Assume that the contestant chose Door 1 initially.



Therefore the theoretical probability of winning the grand prize if he chooses to switch door is $\frac{2}{3}$.

Extension of Monty Hall Problem

Suppose we have n doors, with a car behind 1 of them. The probability of choosing the door with the car behind it on your first pick, is $\frac{1}{n}$

Monty then opens k doors, where $0 \le k \le n-2$ (he has to leave your original door and at least one other door closed).

The probability of picking the car if you choose a different door, is the chance of not having picked the car previously, which is $\frac{n-1}{n}$, times the probability of picking it *now*, which is $\frac{1}{n-k-1}$.

This gives us a total probability of $(\frac{n-1}{n})(\frac{1}{n-k-1}) = \frac{1}{n}\frac{n-1}{n-k-1}$

If Monty opens no doors, $k=0$ and that reduces to $\frac{1}{n}$.

For all $k > 0$, $\frac{n-1}{n-k-1} > 1$ and so the probability of picking the car on your second guess is greater than $\frac{1}{n}$.

Hence the decision is the contestant should switch door!

Now if k is at its maximum value of $n-2$, the probability of picking a car after switching becomes $(\frac{n-1}{n})(\frac{1}{n-(n-2)-1}) = \frac{n-1}{n}$.

Note that for $n = 3$, this is the solution to the original Monty Hall problem.