Raffles Institution
H2 Mathematics (9758)
Solution for 2022 A-Level Paper 2

Section A: Pure Mathematics

Question 1

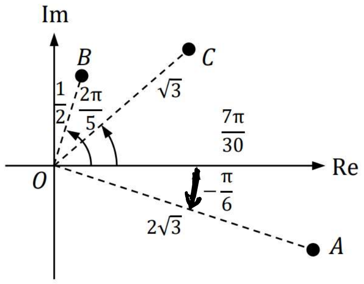
No.	Suggested Solution	Remarks for Student
	$u=\sqrt{x+2}$ $u^2=x+2$ $2u\dfrac{\mathrm{d}u}{\mathrm{d}x}=1$ $\int \dfrac{x}{\sqrt{x+2}}\ \mathrm{d}x \quad =\int \dfrac{u^2-2}{u}\ 2u\mathrm{d}u$ $=2\int\left(u^2-2\right)\ \mathrm{d}u$ $=2\left(\dfrac{1}{3}u^3-2u\right)+C$ $=\dfrac{2}{3}(x+2)^{\frac{3}{2}}-4\sqrt{x+2}+C$	

Question 2

No.	Suggested Solution	Remarks for Student
(a)	Volume $=72k=0.9h\times1000$ $\therefore k=\dfrac{900h}{72}=12.5h$	
(b)	$\dfrac{\mathrm{d}V}{\mathrm{d}t}=kt$ $V=\dfrac{1}{2}kt^2+C=\dfrac{25}{4}ht^2+C$ $V=0$ when $t=0\Rightarrow C=0$ $\therefore V=\dfrac{25}{4}ht^2$ $V=900h\Rightarrow 900h=\dfrac{25}{4}ht^2$ $t^2=144$ Since $t>0,\ t=12s$	

(c)	$\frac{\mathrm{d}V}{\mathrm{d}t}=kt+25=12.5ht+25$ $V=6.25ht^2+25t+C$ $V=0$ when $t=0\Rightarrow C=0$ $\therefore V=6.25ht^2+25t$ $t=10, V=900h\Rightarrow 900h=625h+250$ $\Rightarrow h=\frac{250}{275}=\frac{10}{11}\text{m}$	

Question 3

No.	Suggested Solution	Remarks for Student
(a)	$z_1=3-\mathrm{i}\sqrt{3}=\sqrt{12}\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$ $z_2=\frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{2\pi}{5}}$ $z_3=z_1z_2=\frac{\sqrt{12}}{2}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{5}-\frac{\pi}{6}\right)}=\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{7\pi}{30}}$ $\lvert z_3\rvert=\sqrt{3},\ \arg(z_3)=\frac{7\pi}{30}$	Need to specify $\lvert z_3\rvert$ and $\arg(z_3)$ explicitly. Answer left as $\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{7\pi}{30}}$ is not acceptable.
(b)	Let *A*, *B* and *C* represent complex numbers z_1, z_2 and z_3 respectively 	

(c)	$z_3^n=\left(\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{7\pi}{30}}\right)^n=\left(\sqrt{3}\right)^n\mathrm{e}^{\mathrm{i}\frac{7n\pi}{30}}$ $\frac{7n\pi}{30}=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\pm\frac{7\pi}{2},\pm\frac{9\pi}{2}$ $7n=15,45,75,105,\ldots$ smallest positive integer $n=15$ $\lvert z_3^{15}\rvert=\left(\sqrt{3}\right)^{15}=3^7\sqrt{3}=2187\sqrt{3}$ $\arg\left(z_3^{15}\right)=\frac{105\pi}{30}-4\pi=\frac{7\pi}{2}-4\pi=-\frac{\pi}{2}$	

Question 4

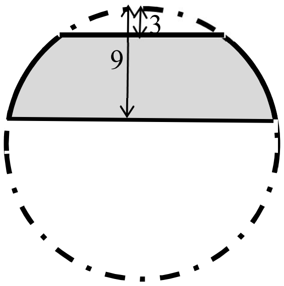
No.	Suggested Solution	Remarks for Student
(a)	$\frac{1}{9r^2+3r-2}=\frac{1}{(3r+2)(3r-1)}=\frac{1}{3(3r-1)}-\frac{1}{3(3r+2)}$	
(b)	$\sum_{r=m}^{3m}\left(\frac{1}{9r^2+3r-2}\right)=\frac{1}{3}\sum_{r=m}^{3m}\left(\frac{1}{3r-1}-\frac{1}{3r+2}\right)$ $=\frac{1}{3}\left[\frac{1}{3m-1}-\cancel{\frac{1}{3m+2}}\right.$ $+\cancel{\frac{1}{3m+2}}-\cancel{\frac{1}{3m+5}}$ $+\cancel{\frac{1}{3m+5}}-\cancel{\frac{1}{3m+8}}$ $\cdot$ $\cdot$ $\cdot$ $+\cancel{\frac{1}{9m-4}}-\cancel{\frac{1}{9m-1}}$ $\left.+\cancel{\frac{1}{9m-1}}-\frac{1}{9m+2}\right]$ $=\frac{1}{3}\left(\frac{1}{3m-1}-\frac{1}{9m+2}\right)$ $=\frac{2m+1}{(3m-1)(9m+2)}$	You are required to combine and simplify the answer as a single fraction.
(c)	$\sum_{r=1}^{n}\left(\frac{1}{9r^2+3r-2}\right)=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)=\frac{1}{6}-\frac{1}{3(3n+2)}$ $\sum_{r=1}^{\infty}\left(\frac{1}{9r^2+3r-2}\right)=\lim_{n\to\infty}\left(\frac{1}{6}-\frac{1}{3(3n+2)}\right)=\frac{1}{6}$	
(d)	$\left\|\sum_{r=1}^{n}\left(\frac{1}{9r^2+3r-2}\right)-\sum_{r=1}^{\infty}\left(\frac{1}{9r^2+3r-2}\right)\right\|<0.004$ $\frac{1}{3(3n+2)}<0.004\Rightarrow n>27.111$ Least $n=28$	Note the correct use of inequalities leading to the answer 28 which has to be an integer.

Alternatively,

n	$\left\lvert \sum_{r=1}^{n}\left(\frac{1}{9r^2+3r-2}\right)-\sum_{r=1}^{\infty}\left(\frac{1}{9r^2+3r-2}\right)\right\rvert$
27	$0.004016 > 0.004$
28	$0.003876 < 0.004$

Least n is 28.

Question 5

No.	Suggested Solution	Remarks for Student
(a)	Volume of spherical cap is $\pi\int_{r-h}^{r} x^2\ \mathrm{d}y$ $=\pi\int_{r-h}^{r}\left(r^2-y^2\right)\ \mathrm{d}y$ $=\pi\left[r^2y-\frac{1}{3}y^3\right]_{r-h}^{r}$ $=\pi\left[r^3-\frac{1}{3}r^3-r^2\left(r-h\right)+\frac{1}{3}\left(r-h\right)^3\right]$ $=\pi\left[\frac{2}{3}r^3-r^3+r^2h+\frac{1}{3}r^3-r^2h+rh^2-\frac{1}{3}h^3\right]$ $=\pi\left[rh^2-\frac{1}{3}h^3\right]$ $=\frac{1}{3}\pi h^2\left(3r-h\right)$	
(b)	$3402\pi=\frac{4}{3}\pi\left(15\right)^3-\frac{1}{3}\pi p^2\left(3\left(15\right)-p\right)-\frac{1}{3}\pi\left(3p\right)^2\left(3\left(15\right)-3p\right)$ $3402=4500-15p^2+\frac{1}{3}p^3-135p^2+9p^3$ $28p^3-450p^2+3294=0$ $p=3$ or $p=-2.5158$ (reject since $p>0$) or $p=15.587$ (reject since $p<15$) $\therefore p=3$	
(c)	Volume of 2nd ornament is $\frac{1}{3}\pi\left(3p\right)^2\left(3r-3p\right)-\frac{1}{3}\pi p^2\left(3r-p\right)$ $(p=3, r=15)$ $=846\pi$  	

Section B: Probability and Statistics

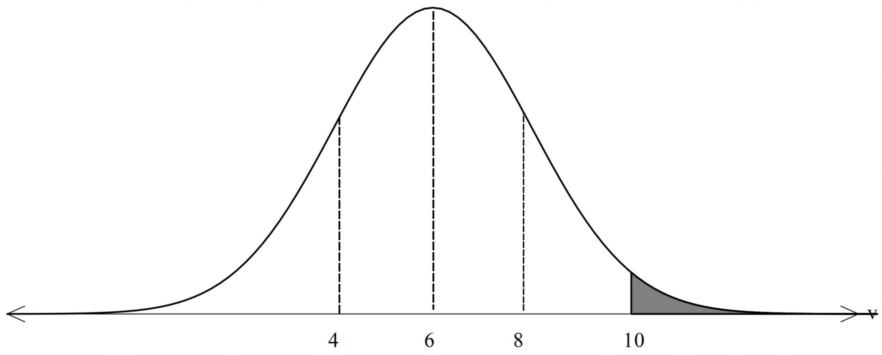
Question 6

No.	Suggested Solution	Remarks for Student
(a)	$\mathrm{P}(\text{A wins})$ $=\mathrm{P}(\text{A wins on throw 3})$ $+\mathrm{P}(\text{A wins on throw 5})$ $+\mathrm{P}(\text{A wins on throw 7})$ $+\ldots$ $=1\times\frac{5}{6}\times\frac{1}{6}+1\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+1\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+\ldots$ $=\frac{5}{36}\left(1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\left(\frac{5}{6}\right)^6+\ldots\right)$ $=\frac{5}{36}\frac{1}{1-\left(\frac{5}{6}\right)^2}$ $=\frac{5}{11}$	Exact answer is required.
(b)	$\mathrm{P}(\text{B wins on her second throw} \mid \text{B wins})$ $=\frac{\mathrm{P}(\text{B wins on throw 4})}{1-\mathrm{P}(\text{A wins})}$ $=\frac{1\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}}{1-\frac{5}{11}}$ $=\frac{275}{1296}$	Exact answer is required.

Question 7

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(a)</td><td>Sample, as not all 75 employees responded</td><td></td></tr>
<tr><td>(b)</td><td>She can randomly select a certain number of employees from each department. The number selected may be proportional to the size of the department.
Example, 10% of each department.
This way is less time consuming and fair (as there is no biasness since employees are selected randomly from each department) instead of gathering views from every employee.</td><td></td></tr>
<tr><td>(c)</td><td>
<table>
<tr><td>A
(7)</td><td>P
(6)</td><td>M
(4)</td><td>S
(3)</td><td>No.</td></tr>
<tr><td>5</td><td>1</td><td>1</td><td>1</td><td>$^7C_5 \times {}^6C_1 \times {}^4C_1 \times {}^3C_1 = 1512$</td></tr>
<tr><td>4</td><td>2</td><td>1</td><td>1</td><td>$^7C_4 \times {}^6C_2 \times {}^4C_1 \times {}^3C_1 = 6300$</td></tr>
<tr><td>4</td><td>1</td><td>2</td><td>1</td><td>$^7C_4 \times {}^6C_1 \times {}^4C_2 \times {}^3C_1 = 3780$</td></tr>
<tr><td>4</td><td>1</td><td>1</td><td>2</td><td>$^7C_4 \times {}^6C_1 \times {}^4C_1 \times {}^3C_2 = 2520$</td></tr>
<tr><td>3</td><td>2</td><td>2</td><td>1</td><td>$^7C_3 \times {}^6C_2 \times {}^4C_2 \times {}^3C_1 = 9450$</td></tr>
<tr><td>3</td><td>2</td><td>1</td><td>2</td><td>$^7C_3 \times {}^6C_2 \times {}^4C_1 \times {}^3C_2 = 6300$</td></tr>
<tr><td>3</td><td>1</td><td>2</td><td>2</td><td>$^7C_3 \times {}^6C_1 \times {}^4C_2 \times {}^3C_2 = 3780$</td></tr>
</table>
Total: 33642</td><td>Listing systematically is key in answering this question. Yes, there are 7 cases.</td></tr>
</table>

Question 8

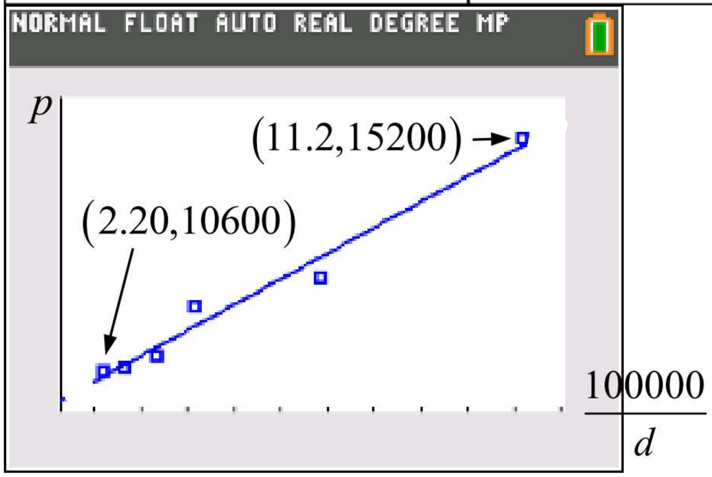
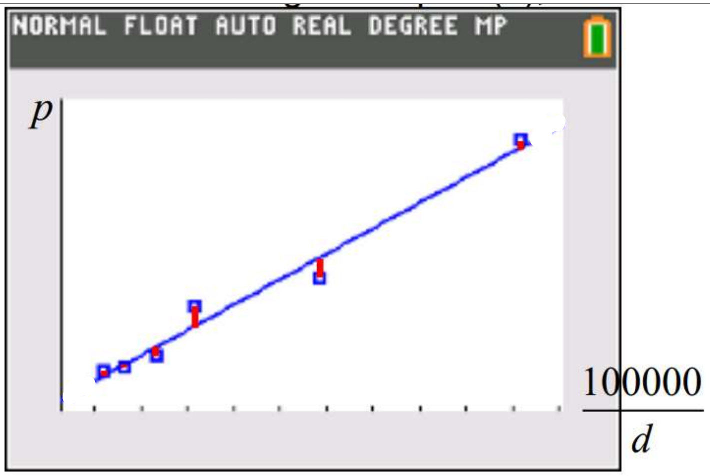
No.	**Suggested Solution**	**Remarks for Student**
(a)	$aX - bY \sim \mathrm{N}\left(ap - bs, a^2q^2 + b^2t^2\right)$	
(b) **(i)** **& (ii)**	$V \sim \mathrm{N}\left(6, 2^2\right)$  $\mathrm{P}\left(V > 10\right) = 0.0228$	Label $\mu = 6$ And also the 4 and 8
(c)	$\mathrm{E}\left(W\right) = 1.2\mathrm{Var}\left(W\right)$ $8p = 1.2\left[8p\left(1-p\right)\right]$ $p = 0$ (not meaningful) or $p = \frac{1}{6}$ $W \sim \mathrm{B}\left(8, \frac{1}{6}\right)$ $\mathrm{P}\left(W < 2\right) = \mathrm{P}\left(W \le 1\right) = 0.605$	

Question 9

No.	**Suggested Solution**	**Remarks for Student**
(a)	Note that it takes 5 moves from S to one of A,B,C,D,E or F. Let X = number of left moves after first move. $X \sim \mathrm{B}(4, p)$ (first move has probability ½ to move left or right) $\mathrm{P}(\text{counter arrives at B})$ $= \mathrm{P}(\text{first move is L and } X = 3) + \mathrm{P}(\text{first move is R and } X = 4)$ $= \frac{1}{2}\mathrm{P}(X = 3) + \frac{1}{2}\mathrm{P}(X = 4)$ $= \frac{1}{2} \times {}^4C_3 p^3 q + \frac{1}{2} \times {}^4C_4 p^4$ $= 2p^3 q + \frac{1}{2}p^4$	Note the probability is ½ for first move from S to L or R.
(b)	5 routes in total Both taking Right →Left→Left→ Left→ Left has probability $\left(\frac{1}{2}p^4\right)^2 = \frac{1}{4}p^8$ There are 4 routes where Both take Left on first move follow by another same 3 left and one right moves subsequently has probability $4 \times \left(\frac{1}{2}p^3 q\right)^2 = p^6 q^2$ Sum of above $= \frac{1}{4}p^8 + p^6 q^2$ $= \frac{1}{4}p^8 + p^6 (1-p)^2$ $= \frac{1}{4}p^8 + p^6 \left(1 - 2p + p^2\right)$ $= \frac{1}{4}p^6 \left(p^2 + 4 - 8p + 4p^2\right)$ $= \frac{1}{4}p^6 \left(5p^2 - 8p + 4\right)$ Required probability $= \dfrac{\frac{1}{4}p^6\left(5p^2 - 8p + 4\right)}{\left(2p^3 q + \frac{1}{2}p^4\right)^2} = \dfrac{\frac{1}{4}p^6\left(5p^2 - 8p + 4\right)}{\frac{1}{4}p^6 (4q + p)^2} = \dfrac{5p^2 - 8p + 4}{(4 - 3p)^2}$	Answer must be simplified and in terms of p only

(c)	$\mathrm{P}(\text{counter arrives at C})$ $=\frac{1}{2}\mathrm{P}(X=2)+\frac{1}{2}\mathrm{P}(X=3)$ $=\frac{1}{2}\times {}^4C_2p^2q^2+\frac{1}{2}\times {}^4C_3p^3q$ $=3p^2q^2+2p^3q$ $3p^2q^2+2p^3q=2p^3q+\frac{1}{2}p^4$ $3p^2q^2-\frac{1}{2}p^4=0$ $3p^2(1-p)^2-\frac{1}{2}p^4=0$ Since $p\neq 0,\ p=0.710$	Can use GC, no need to find exact value of p which is $\frac{6-\sqrt{6}}{5}$

Question 10

No.	Suggested Solution	Remarks for Student
(a)	*d* and *p* may not be linearly correlated as –0.78 is not very near to –1. The scatter diagram also suggests a non-linear negative relationship.	
(b)	No. Scaling of the values, including change of units, do not change the relationship.	
(c)	As this data becomes a outlier, since the special car has different features from the other 6 cars, it will disrupt the relationship to the rest of the data as they were taken from cars of similar version	Remember to elaborate in context
(d)		
(e)	 Squaring the distances so that the sum will not be zero or become negative, as the distances could be positive (above the line) or negative (below the line). This is referred to "method of least squares" as we are trying to fit a line such that the sum of the square of distances is the smallest.	
(f)	$p = 9398.492473 + 505.730253\left(\frac{100000}{d}\right) \approx 9400 + \frac{50600000}{d}$ $r = 0.9869962799 \approx 0.987$	
(g)	P = 19403.04743 = 19400 (3 s.f.) Not expecting it to be reliable as 5055 is not in the given range of 8954 to 45452	

Question 11

No.	**Suggested Solution**	**Remarks for Student**
(a)	Let X = time in sec taken by Zhou T = time in sec taken by Tan $X \sim \mathrm{N}\left(80, 2^2\right),\ T \sim \mathrm{N}\left(79, 3^2\right)$ $X - T \sim \mathrm{N}(1, 13)$ $\mathrm{P}(X < T) = \mathrm{P}(X - T < 0)$ $= 0.391$	Note that winning in this context means take shorter time
(b)	Respective unbiased estimates of population mean and variance are: $\bar{x} = 79.21$ $s^2 = 14.72334483 \approx 14.7$	Note that 79.21 is the exact answer
(c)	Null hypothesis H$_0$: $\mu = 80$ Alternative hypothesis H$_1$: $\mu < 80$ μ is the Zhou's population mean time Under H$_0$, $\bar{X} \sim \mathrm{N}\left(80, \frac{14.72334483}{30}\right)$ approximately by Central Limit Theorem since 30 is large $p - \text{value} = \mathrm{P}\left(\bar{X} < 79.21\right) = 0.12972836 > 0.05$ We do not reject H$_0$ and there is insufficient evidence at 5% level of significance to conclude that Zhou's times have reduced.	
(d)	He should use a 2-tail test as he is interested in testing whether his mean time has changed which could be greater or less than the original mean time.	
(e)	Assumption 1: his time still follows normal distribution Assumption 2: As he did not calculate an unbiased estimate of the population variance from the data, he has to assume that population variance remains at 3^2.	