1 Let $\mathbf{A}$ be an $n \times n$ matrix such that $\mathbf{A}^2 = \mathbf{O}$. If $\mathbf{u}$ is a vector in $\mathbb{R}^n$ such that $\mathbf{Au} \neq \mathbf{0}$, show that $\mathbf{u}$ and $\mathbf{Au}$ are linearly independent. [3]

2 The Lemniscate of Bernoulli is a curve with polar equation $r^2 = a^2 \cos 2\theta$, where a is a positive constant.

(i) Find the total area of the regions bounded by the lemniscate. [3]

(ii) Find an expression for the total length of the lemniscate, leaving your answer in the form $4a\int_0^{\frac{\pi}{4}} \sqrt{\mathrm{f}(\theta)}\, \mathrm{d}\theta$, where $\mathrm{f}(\theta)$ is a function of θ to be determined. Simplify your answer. [3]

3 A sequence of real numbers $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation

$$u_{n+2} = u_{n+1} + \left(c^2 - \frac{1}{4}\right)u_n$$

where c is a real number.

(i) Describe the behaviour of the sequence in the case where

(a) $c = \frac{1}{2}$, [1]

(b) $c = 0$ as n becomes very large. [3]

(ii) Determine the range of values of c for which the sequence is convergent for any values of u_1 and u_2, justifying your answer. [4]

4 The set $\mathbf{M}_{n\times n}(\mathbb{R})$ denotes the vector space consisting of all real $n \times n$ matrices and with the usual operations of addition and scalar multiplication.

Let $\mathbf{v}$ be a nonzero constant vector in $\mathbb{R}^n$ and

$$S_n = \{\mathbf{X} \in \mathbf{M}_{n\times n}(\mathbb{R}) \mid \mathbf{v} \in \text{null space of } \mathbf{X}\}.$$

(i) Show that S_n is a subspace of $\mathbf{M}_{n\times n}(\mathbb{R})$. [3]

(ii) In the case where $n = 2$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find a basis for S_2. [3]

(iii) Explain why the dimension of S_n is $n^2 - n$. [2]

(iv) Show that the set $\{\mathbf{X} \in \mathbf{M}_{n\times n}(\mathbb{R}) \mid \mathbf{v} \in \text{column space of } \mathbf{X}\}$ is not a subspace of $\mathbf{M}_{n\times n}(\mathbb{R})$. [2]

5 A patient undergoing medical treatment is required to take a dose of 60 mg of a particular drug once a day at the start of each day. It has been found that the amount of drug in the patient's bloodstream at the end of a day is $\frac{1}{16}$ of the amount present at the start of the day.

Let a_n denote the amount of drug, in mg, in the patient's bloodstream at the end of the n^{th} day of taking the drug. There is no drug present in the patient's bloodstream before the first day of taking the drug.

(i) By formulating a recurrence relation for a_n, find an expression for a_n in terms of n for $n \geq 0$. [4]

(ii) State the value of $\lim_{n\to\infty} a_n$. [1]

After a long period of time, the patient visits his doctor for a follow up session. During the session, the doctor tells the patient that he needs to increase the daily dosage of the drug intake from 60 mg to *d* mg starting from the next day. An overdose is said to have occurred once the amount of the drug present in the bloodstream exceeds 80 mg.

(iii) Find the maximum value of *d* such that an overdose will never occur. [5]

6 A curve *C* is defined parametrically by

$$x = \sqrt{t}\,,\ y = t\cos 2t \ \text{ for } 0 \leq t \leq \pi.$$

The region below the *x*-axis bounded by *C* and the *x*-axis is denoted by *R*. A solid is obtained by rotating *R* completely about the *y*-axis. Find

(i) the surface area of this solid, and [3]

(ii) the exact volume of this solid. [4]

Use Simpson's Rule for four strips to estimate the volume of the solid, correct to 5 decimal places. Comment on the accuracy of your estimate with reference to your answer from part **(ii)**. [3]

7 **Do not use a calculator in answering this question.**

The ancient Greek mathematician, Theon of Smyrna, discovered a method to approximate the value of $\sqrt{2}$. He found that if $\frac{a}{b}$ is a fraction that approximates $\sqrt{2}$, then the fraction $\frac{a+2b}{a+b}$ provides an even better approximation to $\sqrt{2}$. Starting with an initial approximation $\frac{a_0}{b_0}$, successive approximations $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ can be generated by the relations

$$a_n = a_{n-1} + 2b_{n-1},$$
$$b_n = a_{n-1} + b_{n-1}.$$

(i) Let $\mathbf{v}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$. State the 2×2 matrix $\mathbf{A}$ such that $\mathbf{v}_n = \mathbf{A}\mathbf{v}_{n-1}$.

Express $\mathbf{v}_n$ in the form $\mathbf{B}\mathbf{v}_0$, where $\mathbf{B}$ is a matrix to be determined in terms of $\mathbf{A}$. [2]

(ii) Express $\mathbf{A}^n$ in the form $\mathbf{PDP}^{-1}$ where $\mathbf{P}$ is an invertible matrix and $\mathbf{D}$ is a diagonal matrix. [5]

(iii) Hence, using $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, show that $\frac{a_n}{b_n}$ tends to $\sqrt{2}$ as n tends to infinity. [4]

8 Let $\mathbf{P}_2 = \left\{ ax^2 + bx + c \,\middle|\, a, b, c \in \mathbb{R} \right\}$ denote the vector space of polynomials in x with degree at most 2, with real coefficients. The function $\mathrm{T} : \mathbf{P}_2 \to \mathbf{P}_2$ is defined by

$$\mathrm{T}\left(\mathrm{f}(x)\right) = \mathrm{f}(x) + \mathrm{f}'(x).$$

(i) Show that T is a linear transformation. [2]

(ii) Find a basis for the null space of T. Hence, state the rank of T. [3]

(iii) Show that $\mathrm{T}^2\left(ax^2+bx+c\right) = ax^2 + (4a+b)x + 2a + 2b + c$ and find $\mathrm{T}^3\left(ax^2+bx+c\right)$.

Hence, formulate a conjecture for $\mathrm{T}^n\left(ax^2+bx+c\right)$, where n is a positive integer, and prove your conjecture by induction. [7]

(iv) Show that if $a \neq 0$ and the equation $ax^2+bx+c=0$ has real roots, then the equation

$$\mathrm{T}^n\left(ax^2+bx+c\right) = 0$$

has real and distinct roots for any positive integer n. [2]

9 It is given that the equation $\mathrm{f}(x) = 0$, where $\mathrm{f}(x) = \mathrm{e}^{-x^2} + x^2 - 2x$, has exactly two real roots α and β with $\alpha < \beta$.

(i) Without finding the values of α and β, show that there exists an integer N such that $N < \alpha < N + 1 < \beta < N + 2$, where the value of N is to be determined. [2]

(ii) A student attempts to use the iterative formula $x_{n+1} = \mathrm{e}^{-x_n^{\,2}} + x_n^{\,2} - x_n$ to find the values of α and β. With the aid of a sketch, explain why this iterative formula is suitable for finding an approximation only for α, but not for β. [4]

(iii) Use the iterative formula in part **(ii)** with a suitable initial approximation to obtain an approximation to α, giving your answer to 2 decimal places. [3]

(iv) Use one iteration of the Newton-Raphson method with initial approximation $x_0 = 1.5$ to obtain an approximation to β, giving your answer to 4 decimal places. [2]

(v) Prove that $\mathrm{f}''(x) > 0$ for all non-zero real values of x.

Hence, determine the range of values of k such that one iteration of the Newton-Raphson with initial approximation $x_0 = k$ will produce an overestimate of β. [4]

10 Figure 1 shows a photograph of a hollow, cylindrical wall light at an MRT station in Singapore. The axis of the cylinder is vertical and parallel to the wall.

Figure 1

(i) Give a reason why the outline of the light on the wall is expected to be a hyperbola. [1]

For this question, it may be assumed that:

- the light source is a single point at the centre of the cylinder,
- the outline of light on the wall above and below the wall light are from the same hyperbola,
- any light ray that is reflected inside the cylinder at least once and reaches the wall will not produce any noticeable effect on the wall,
- the thickness of the cylinder is negligible.

The radius and height of the cylindrical wall light are 4 cm and 12 cm respectively.

Figures 2 and 3 below show the front view and side view of the wall light respectively. The cartesian plane is superimposed onto the front view with the light source at the origin, while the cylinder is touching the wall as seen in the side view.

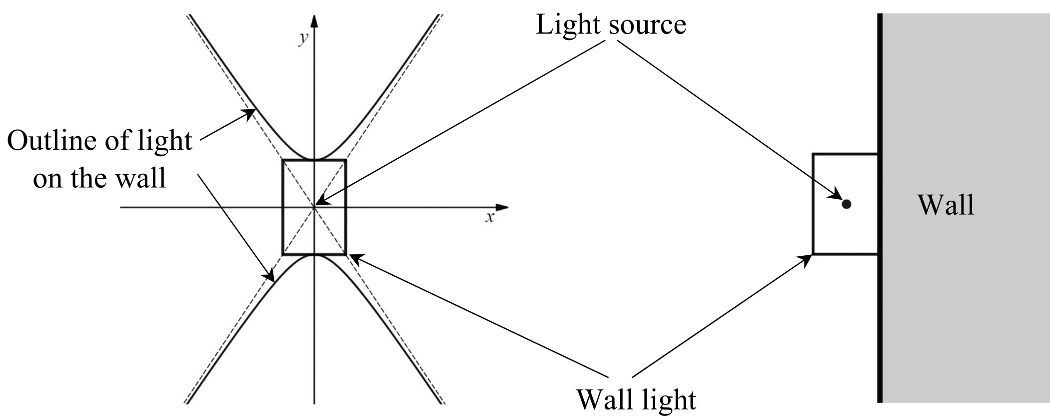

Figure 2: Front view **Figure 3: Side view**

Take 1 unit on the cartesian plane to represent 1 cm.

(ii) Show that the equation of the outline of the light on the wall is $\frac{y^2}{36} - \frac{x^2}{16} = 1$. [4]

An MRT station staff member decided to install a supporting rod of length 2 cm at the same horizontal level as the light source as shown in Figure 4 below. As a result, a new outline of light on the wall is obtained, giving rise to a new hyperbola on the wall.

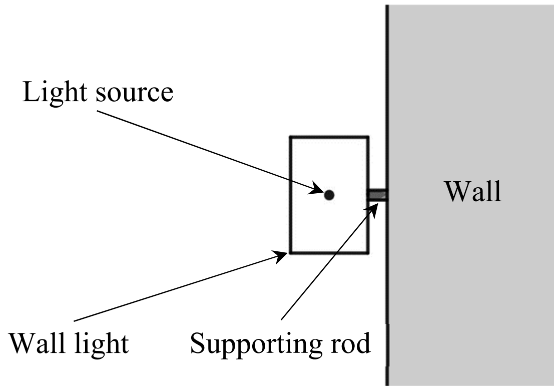

Figure 4

(iii) Did the installation of the supporting rod affect the eccentricity of the hyperbola? Justify your answer. [2]

(iv) Find the equation of the new outline of the light on the wall. [2]

The photographer observed that in the photograph in Figure 1, the bottom of the wall light has the shape of an ellipse. After some measurements were taken, it was found that the length of the major axis is three times the length of the minor axis.

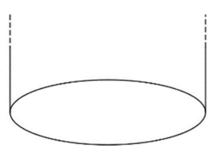

Figure 5: Bottom of wall light as seen in photograph

(v) Find the eccentricity of this ellipse. [2]

(vi) Assuming that the camera was sufficiently far away from the wall light and aimed at the bottom of the wall light when the photograph in Figure 1 was taken, estimate the angle of elevation of the bottom of the wall light from the camera. [2]