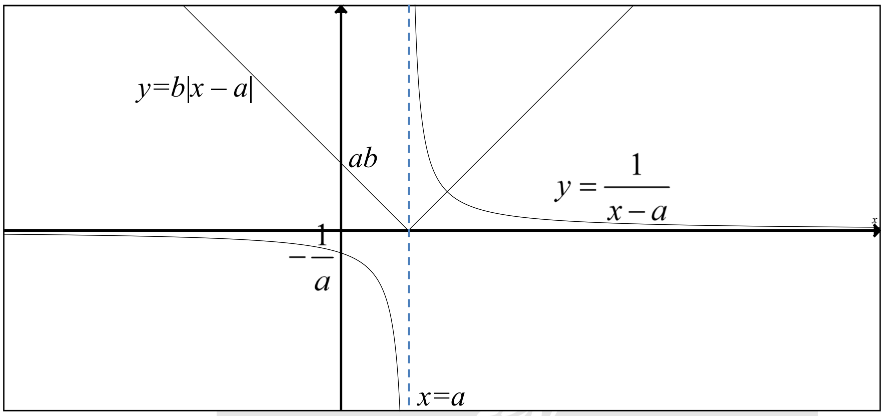
Raffles Institution
H2 Mathematics (9758)
Solution for 2017 A-Level Paper 1

Question 1

No.	Suggested Solution	Remarks for Student
	$e^{2x}\ln(1+ax)$ $=\left(1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\ldots\right)\left(ax-\frac{(ax)^2}{2}+\frac{(ax)^3}{3}-\ldots\right), \quad -1<ax\le 1$ $=ax-\frac{(ax)^2}{2}+2ax^2+\frac{(ax)^3}{3}-x(ax)^2+\frac{ax(2x)^2}{2}+\ldots$ $=ax+\left(\frac{4a-a^2}{2}\right)x^2+\left(\frac{a^3-3a^2+6a}{3}\right)x^3+\ldots$ No term in x^2, we have $4a-a^2=0 \Rightarrow a=0$ (rejected since $a\neq 0$) or $a=4$.	

Question 2

No.	**Suggested Solution**	**Remarks for Student**
(i)		
(ii)	To solve $\frac{1}{x-a} < b\lvert x-a\rvert$, we consider the graphs $y=\frac{1}{x-a}$ and $y=b\lvert x-a\rvert$ drawn in (i), noting they intersect at a point where $x>a$. $\frac{1}{x-a}=b(x-a),\ x>a.$ $(x-a)^2=\frac{1}{b} \Rightarrow x=a+\sqrt{\frac{1}{b}},\ \left(\text{reject } x=a-\sqrt{\frac{1}{b}} \text{ since } x>a\right)$ $\frac{1}{x-a}<b\lvert x-a\rvert$ $x>a+\sqrt{\frac{1}{b}}$ or $x<a$	You need to refer to (i) if you are using “Hence”.

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Question 3

No.	Suggested Solution	Remarks for Student
(i)	$y^2 - 2xy + 5x^2 - 10 = 0 \ldots (1)$ Differentiate (1) with respect to x: $2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y + 10x = 0$ $y\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x} - y + 5x = 0 \ \ldots (2)$ When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $y = 5x$ Sub into (1): $25x^2 - 10x^2 + 5x^2 - 10 = 0 \Rightarrow 20x^2 = 10$. $\therefore x = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$	
(ii)	Differentiate (2) with respect to x: $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} + 5 = 0 \ldots (3)$ When $x = \frac{1}{\sqrt{2}}$, $y = \frac{5}{\sqrt{2}}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Sub into (3): $\frac{5}{\sqrt{2}}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{1}{\sqrt{2}}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5 = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{5\sqrt{2}}{4} < 0$ Thus, $\left(\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$ is a maximum turning point.	Use 2nd Derivative Test here. 1st derivative test cannot be used as $\frac{\mathrm{d}y}{\mathrm{d}x}$ depends on both x and y. Note that exact values with working are required as no calculator is allowed for this question.

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Question 4

No.	Suggested Solution	Remarks for Student
(i)	$y=\dfrac{4x+9}{x+2}=\dfrac{4(x+2)+1}{x+2}=4+\dfrac{1}{x+2}$ Note that C is defined on all values of x except –2. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1}{(x+2)^2}<0$ for all $x\neq-2$ Thus, gradient of C is negative for all points on C.	
(ii)	$y=4+\dfrac{1}{x+2}$ Asymptotes are $x=-2,\ y=4$	
(iii)	A translation of 2 units in the positive x direction follow by a translation of 4 units in the negative y direction.	Replace x by $(x-2)$ follow by replacing y by $(y+4)$ is not acceptable.

Question 5

No.	Suggested Solution	Remarks for Student
(i)	$\mathrm{f}(x)=x^3+ax^2+bx+c$ $\mathrm{f}(1)=8 \Rightarrow 1+a+b+c=8 \Rightarrow a+b+c=7$ ...(1) $\mathrm{f}(2)=12 \Rightarrow 8+4a+2b+c=12 \Rightarrow 4a+2b+c=4$ ...(2) $\mathrm{f}(3)=25 \Rightarrow 27+9a+3b+c=25 \Rightarrow 9a+3b+c=-2$ ...(3) Solving (1), (2) and (3), $a=-\frac{3}{2},\ b=\frac{3}{2},\ c=7$	
(ii)	$\mathrm{f}(x)=x^3-\frac{3}{2}x^2+\frac{3}{2}x+7$ We know that in general a cubic function has 1, 2 or 3 x-intercepts. $\mathrm{f}'(x)=3x^2-3x+\frac{3}{2}$ $=3(x^2-x)+\frac{3}{2}$ $=3\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)+\frac{3}{2}$ $=3\left(x-\frac{1}{2}\right)^2+\frac{3}{4}>0$ for all x since $\left(x-\frac{1}{2}\right)^2\geq 0$ for all x $\mathrm{f}(x)\to-\infty$ as $x\to-\infty$ and $\mathrm{f}(x)\to\infty$ as $x\to\infty$. Since gradient of curve is always positive, curve is always increasing. (in particular, f is 1-1) So, $\mathrm{f}(x)=0$ has only one root. Using GC, the root is –1.33 (3 s.f.)	
(iii)	$\mathrm{f}'(x)=2 \Rightarrow 3\left(x-\frac{1}{2}\right)^2+\frac{3}{4}=2$ $\Rightarrow \left(x-\frac{1}{2}\right)^2=\frac{5}{12}$ $x=\frac{1}{2}\pm\sqrt{\frac{5}{12}}$	

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Question 6

No.	Suggested Solution	Remarks for Student
(i)	It is a line passing through the point A with position vector $\underset{\sim}{a}$ and is parallel to the vector $\underset{\sim}{b}$.	
(ii)	It is a plane with normal vector $\underset{\sim}{n}$ such that the dot product of the position vector of any point on the plane with the normal has the value d. Since $\underset{\sim}{n}$ is a unit vector, the magnitude of d is the distance from the origin to the plane.	
(iii)	$(\underset{\sim}{a}+t\underset{\sim}{b}).\underset{\sim}{n}=d \quad \Rightarrow \underset{\sim}{a}.\underset{\sim}{n}+t\underset{\sim}{b}.\underset{\sim}{n}=d$ $\because \underset{\sim}{b}.\underset{\sim}{n}\neq 0,\ t=\dfrac{d-\underset{\sim}{a}.\underset{\sim}{n}}{\underset{\sim}{b}.\underset{\sim}{n}}$ $\therefore \underset{\sim}{r}=\underset{\sim}{a}+\left(\dfrac{d-\underset{\sim}{a}.\underset{\sim}{n}}{\underset{\sim}{b}.\underset{\sim}{n}}\right)\underset{\sim}{b}$ which is the position vector of the point of intersection of the line in (i) and the plane in (ii).	

Question 7

No.	Suggested Solution	Remarks for Student
(i)	$\int \sin 2mx \sin 2nx \ \mathrm{d}x$ $= -\frac{1}{2}\int -2\sin 2mx \sin 2nx \ \mathrm{d}x$ $= -\frac{1}{2}\int \cos(2m+2n)x - \cos(2m-2n)x \ \mathrm{d}x$ $= -\frac{1}{4m+4n}\sin(2m+2n)x + \frac{1}{4m-4n}\sin(2m-2n)x + C$	
(ii)	$\int_0^{\pi} (\mathrm{f}(x))^2 \ \mathrm{d}x$ $= \int_0^{\pi} (\sin 2mx + \sin 2nx)^2 \ \mathrm{d}x$ $= \int_0^{\pi} \left(\sin^2 2mx + \sin^2 2nx + 2\sin 2mx \sin 2nx\right) \mathrm{d}x$ $= \int_0^{\pi} \left(\frac{1-\cos 4mx}{2} + \frac{1-\cos 4nx}{2} + 2\sin 2mx \sin 2nx\right) \mathrm{d}x$ $= \left[\frac{1}{2}x - \frac{1}{8m}\sin 4mx\right]_0^{\pi} + \left[\frac{1}{2}x - \frac{1}{8n}\sin 4nx\right]_0^{\pi}$ $+ \left[-\frac{1}{2m+2n}\sin(2m+2n)x + \frac{1}{2m-2n}\sin(2m-2n)x\right]_0^{\pi}$ $= \pi \qquad (\because \sin k\pi = 0 \text{ for } k \in \mathbb{Z})$	

Question 8

No.	Suggested Solution	Remarks for Student
(a)	$z^2(1-\mathrm{i})-2z+(5+5\mathrm{i})=0$ $z=\dfrac{2\pm\sqrt{2^2-4(1-\mathrm{i})5(1+\mathrm{i})}}{2(1-\mathrm{i})}$ $z=\dfrac{2\pm\sqrt{4-40}}{2(1-\mathrm{i})}$ $z=\dfrac{2\pm 6\mathrm{i}}{2(1-\mathrm{i})}$ $z=\dfrac{2+6\mathrm{i}}{2(1-\mathrm{i})}\times\dfrac{1+\mathrm{i}}{1+\mathrm{i}}$ or $z=\dfrac{2-6\mathrm{i}}{2(1-\mathrm{i})}\times\dfrac{1+\mathrm{i}}{1+\mathrm{i}}$ $z=\dfrac{2+2\mathrm{i}+6\mathrm{i}-6}{4}$ or $z=\dfrac{2+2\mathrm{i}-6\mathrm{i}+6}{4}$ $z=-1+2\mathrm{i}$ or $z=2-\mathrm{i}$	You can also divide through by $(1-\mathrm{i})$ before applying the quadratic formula Note that detailed working is required as calculator is not allowed.
(b) **(i)**	$w=1-\mathrm{i}$ $w^2=(1-\mathrm{i})^2=-2\mathrm{i}$ $w^3=-2\mathrm{i}(1-\mathrm{i})=-2-2\mathrm{i}$ $w^4=(w^2)^2=(-2\mathrm{i})^2=-4$ $w^4+pw^3+39w^2+qw+58=0$ $-4+p(-2-2\mathrm{i})+39(-2\mathrm{i})+q(1-\mathrm{i})+58=0$ Compare Real and Imaginary parts: $-4-2p+q+58=0$ ...(1) $-2p-78-q=0$ ...(2) $(1)+(2)$: $-4p-24=0\Rightarrow p=-6$ Sub into (2): $12-78-q=0\Rightarrow q=-66$	
(ii)	$w^4-6w^3+39w^2-66w+58=0$ Note that $(1-\mathrm{i})$ is a root and coefficients are real, thus $(1+\mathrm{i})$ is also a root One quaratic factor is $(w-(1-\mathrm{i}))(w-(1+\mathrm{i}))=w^2-2w+2$ $\therefore w^4-6w^3+39w^2-66w+58=(w^2-2w+2)(w^2+pw+q)$ By comparing constant terms, $58=2q\Rightarrow q=29$ By comparing linear term, $-66=-2(29)+2p\Rightarrow p=-4$ $\therefore w^4-6w^3+39w^2-66w+58=(w^2-2w+2)(w^2-4w+29)$	

	Alternatively $\begin{array}{r} w^2-4w+29 \\ w^2-2w+2\overline{\big) w^4-6w^3+39w^2-66w+58} \\ -\left(w^4-2w^3+2w^2\right) \\ \hline -4w^3+37w^2-66w+58 \\ -\left(-4w^3+8w^2-8w\right) \\ \hline 29w^2-58w+58 \\ -\left(29w^2-58w+58\right) \\ \hline 0 \end{array}$	

Question 9

No.	Suggested Solution	Remarks for Student
(a)	$S_n = \sum_{r=1}^{n} u_r = An^2 + Bn$	
(i)	$u_n = S_n - S_{n-1}$ $= An^2 + Bn - A(n-1)^2 - B(n-1)$ $= A(2n-1) + B$	
(ii)	$u_{10} = 48 \Rightarrow 19A + B = 48$ $u_{17} = 90 \Rightarrow 33A + B = 90$ Solving, $A = 3,\ B = -9$	
(b)	$r^2(r+1)^2 - (r-1)^2 r^2 = r^2\left[(r+1)^2 - (r-1)^2\right]$ $= r^2\left[(r+1)-(r-1)\right]\left[(r+1)+(r-1)\right]$ $= r^2(2)(2r)$ $= 4r^3$ $\sum_{r=1}^{n} r^3 = \frac{1}{4}\sum_{r=1}^{n}\left(r^2(r+1)^2 - (r-1)^2 r^2\right)$ $= \frac{1}{4}\Big[(1^2)(2^2) - (0^2)(1^2)$ $+(2^2)(3^2) - (1^2)(2^2)$ $+(3^2)(4^2) - (2^2)(3^2)$ $+ \ldots$ $+(n-1)^2 n^2 - (n-2)^2(n-1)^2$ $+n^2(n+1)^2 - (n-1)^2 n^2\Big]$ $= \frac{1}{4}n^2(n+1)^2$	
(c)	Let $a_n = \frac{x^n}{n!}$. $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} = \frac{x}{n+1}$ $\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = \lim_{n\to\infty}\frac{\|x\|}{n+1} = 0 < 1$ for each x. Therefore $\sum_{r=0}^{\infty}\frac{x^r}{r!}$ converges. $\sum_{r=0}^{\infty}\frac{x^r}{r!} = \mathrm{e}^x$ from MF26.	

Question 10

No.	Suggested Solution	Remarks for Student
	$l_C : \underset{\sim}{r} = \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix},\ \lambda \in \mathbb{R}$ and $l_N : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 5 \\ a+1 \end{pmatrix},\ \mu \in \mathbb{R}$	
(i)	$\lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 5 \\ a+1 \end{pmatrix}$ $3\lambda = 1 + 4\mu \Rightarrow 3\lambda - 4\mu = 1 \quad ...(1)$ $\lambda = 2 + 5\mu \Rightarrow \lambda - 5\mu = 2 \quad ...(2)$ $-2\lambda = -1 + \mu(a+1) ...(3)$ Solving (1) and (2): By GC, $\mu = -\frac{5}{11}$ and $\lambda = -\frac{3}{11}$ $\therefore \frac{6}{11} = -1 - \frac{5}{11}(a+1) \Rightarrow a = -4.4$	
(ii)	$\overrightarrow{OR} = \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$, for some $\lambda \in \mathbb{R}$, since R lies on C. If angle PRQ is 90º, then $\overrightarrow{RP}.\overrightarrow{RQ} = 0 \quad \Rightarrow \begin{pmatrix} 1-3\lambda \\ 2-\lambda \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 5-3\lambda \\ 7-\lambda \\ -3+2\lambda \end{pmatrix} = 0$ $(5-15\lambda-3\lambda+9\lambda^2)+(14-7\lambda-2\lambda+\lambda^2)+(3-6\lambda-2\lambda+4\lambda^2)=0$ $14\lambda^2 - 35\lambda + 22 = 0$ Discriminant $= 35^2 - 4(14)(22) = -7 < 0 \Rightarrow$ no solution. Therefore not possible for angle PQR to be 90º.	
(iii)	Length PR is as small as possible implies PR is perpendicular to C. $\overrightarrow{RP}.\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 0 \quad \Rightarrow \begin{pmatrix} 1-3\lambda \\ 2-\lambda \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 0$ $3 - 9\lambda + 2 - \lambda + 2 - 4\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$ Coordinates of R is $\left(\frac{3}{2}, \frac{1}{2}, -1\right)$ Exact minimum length $\lvert\overrightarrow{RP}\rvert = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 0^2} = \frac{1}{2}\sqrt{10}$	

Question 11

No.	Suggested Solution	Remarks for Student
(i) (a)	$\dfrac{\mathrm{d}v}{\mathrm{d}t}=c$	
(b)	$v=ct+d$ $v=4$ when $t=0\Rightarrow d=4$ $v=29$ when $t=2.5\Rightarrow 29=2.5c+4\Rightarrow c=10$ $\therefore v=10t+4$	
(ii)	$\dfrac{\mathrm{d}v}{\mathrm{d}t}=10-kv$ $\int\dfrac{1}{10-kv}\mathrm{d}v=\int\mathrm{d}t$ $-\dfrac{1}{k}\ln\left(10-kv\right)=t+b$ $10-kv=\mathrm{e}^{-kt-kb}=A\mathrm{e}^{-kt}$, where $A=\mathrm{e}^{-kb}$ is a positive constant $v=\dfrac{10-A\mathrm{e}^{-kt}}{k}$ $v=0$ when $t=0$: $A=10$ $\therefore v=\dfrac{10}{k}\left(1-\mathrm{e}^{-kt}\right)$	
(iii)	$v=\dfrac{10}{k}\left(1-\mathrm{e}^{-kt}\right)$ $v\to 40$ as $t\to\infty\Rightarrow 40=\dfrac{10}{k}\Rightarrow k=\dfrac{1}{4}$ $\therefore v=40\left(1-\mathrm{e}^{-\frac{1}{4}t}\right)$ When $v=0.9\times 40=36$, $36=40\left(1-\mathrm{e}^{-\frac{1}{4}t}\right)\Rightarrow \mathrm{e}^{-\frac{1}{4}t}=0.1\Rightarrow t=9.21\ s.$	