

H2 MATHEMATICS 9758

Topic 11: DIFFERENTIATION APPLICATIONS

TUTORIAL WORKSHEET



1 Given that $y = 3x^4 - 16x^3 + 24x^2 - 9$. Without the use of a GC,

(a) find the stationary point(s) of its graph and determine their nature.

$(0, -9)$ min pt, $(2, 7)$ stat pt of inflex

(b) find the range of x , if any, for which the curve is

(i) strictly increasing,

$0 < x < 2$ or $x > 2$

(ii) concave downwards.

$\frac{2}{3} < x < 2$

(c) sketch the curve showing all the important features.

2 N10/I/Q4

Given that $x^2 - y^2 + 2xy + 4 = 0$, find $\frac{dy}{dx}$ in terms of x and y .

For the curve with equation $x^2 - y^2 + 2xy + 4 = 0$, find the coordinates of each point at which the tangent is parallel to x -axis.

$(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

3 N2012/I/8

The curve C has equation

$$x - y = (x + y)^2$$

It is given that C has only one turning point.

(i) Show that $1 + \frac{dy}{dx} = \frac{2}{2x + 2y + 1}$.

(ii) Hence, or otherwise, show that $\frac{d^2y}{dx^2} = -\left(1 + \frac{dy}{dx}\right)^3$

(iii) Hence state, with a reason, whether the turning point is a maximum or a minimum.

- 4 The parametric equations of a curve are $x = ct, y = \frac{c}{t^2}$, where c is a constant.

- (i) Find the equation of the tangent to the curve at the point $P\left(cp, \frac{c}{p^2}\right)$.

$$p^3y + 2x = 3cp$$

- (ii) Hence find the coordinates of the points Q and R where the tangent meets the x - and y -axes respectively.

$$Q\left(\frac{3cp}{2}, 0\right), R\left(0, \frac{3c}{p^2}\right)$$

- (iii) Find the Cartesian equation of the curve.

$$x^2y = c^3$$

- (iv) Find a Cartesian equation of the locus of the mid-point of QR as p varies.

$$x^2y = \frac{27c^3}{32}$$

- 5 The line l is tangent to the curve $y = 1 - \frac{1}{x}$, where $x \neq 0$, at the point where $x = a$.

- (i) Show that the equation of l may be expressed in the form $a^2y - x = a^2 - 2a$.

- (ii) The tangent line l passes through a fixed point (X, Y) . Give a brief argument to explain why there cannot be more than 2 tangents passing through (X, Y) .

- (iii) Find the value of a for which the line l passes through the origin, and find the equation of l in this case.

$$a = 2, y = \frac{x}{4}$$

- 6 Find the equations of the tangents to the curve with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$, which makes an angle of 45° with the positive x -axis, measured in an anticlockwise sense.

$$y = x - \sqrt{a^2 + b^2} \text{ or } y = x + \sqrt{a^2 + b^2}$$

- 7 NJC Prelim 09/I/7 (Anchor Q – Connected rate of change)

In a triangle ABC , $AB = 3$ cm and $AC = 2$ cm. If angle BAC is increasing at a constant rate of 0.1 radians per second, find the rate of increase of the length BC at the instant

where $BAC = \frac{\pi}{3}$ radians.

$$0.196 \text{ cm/s} [4]$$

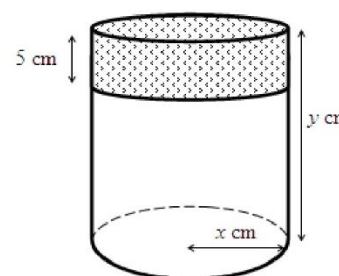
8 HCI Prelim 08/I/3

A right circular cone with radius 3 cm and height 9 cm is initially full of water. Water is leaking from the circular base of the cone at a constant rate of $2 \text{ cm}^3 \text{ s}^{-1}$, find the exact rate of change of the depth of water when the depth of water is 6 cm.

$$\boxed{-\frac{2}{\pi} \text{ cm/s}} [5]$$

9 HCI Prelim 10/II/1 (Anchor Q – Maxima/ Minima)

A tin has a fitting cylindrical lid which overlaps its cylindrical body by 5 cm. When completely closed, it has base radius x cm and height y cm, as shown in the diagram. The body and the lid are made from thin metal sheet such that the difference in their radii is negligible. The total area of metal sheet used to make the tin and its lid is $400\pi \text{ cm}^2$.



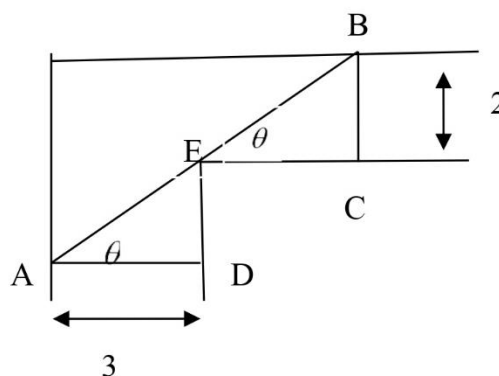
Show that the volume $V \text{ cm}^3$ of the tin is given by

$$V = \pi x(200 - x^2 - 5x). \quad [2]$$

If x varies, find using differentiation the values of x and y for which V has its maximum value.

$$\boxed{x = \frac{20}{3}, y = \frac{55}{3}} [4]$$

- 10 The figure shows two corridors meet at right angles and are 2 m and 3 m wide respectively. θ is the angle marked on the given figure and AB is a thin metal tube which must be kept horizontal and cannot be bent as it moves around the corner from one corridor to the other.



- (a) Show that the length AB is given by

$$L = 3 \sec \theta + 2 \csc \theta.$$

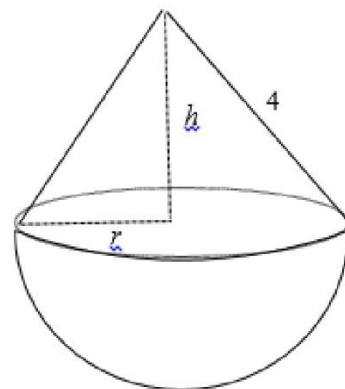
- (b) Show that $\frac{dL}{d\theta} = 0$ when $\theta = \tan^{-1} \sqrt[3]{\frac{2}{3}} \approx 41.14^\circ$

- (c) Find L when $\theta = \tan^{-1} \sqrt[3]{\frac{2}{3}}$ and comment on the significance of this value.

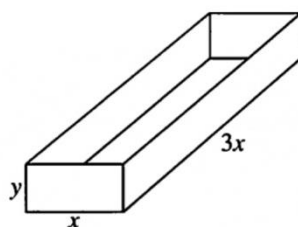
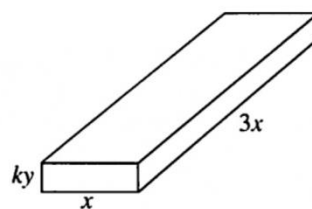
$$\boxed{7.023}$$

11 N14/I/11

A toy manufacturer makes a toy which consists of a hemisphere of radius r cm joined to a circular cone of base radius r cm and height h cm. The manufacturer determines that the length of the slant edge of the cone must be 4 cm and that the total volume of the toy, V cm³, should be as large as possible.



- (i) Find a formula for V in terms of r . Given that $r = r_1$ is the value of r which gives the maximum value of V , show that r_1 satisfies the equation $45r^4 - 768r^2 + 1024 = 0$.
- (ii) Find the two solutions to the equation in part (i) for which $r > 0$, giving your answers correct to 3 decimal places. $3.951, 1.207$
- (iii) Show that one of the solutions found in part (ii) does not give a stationary value of V . Hence write down the value of r_1 and find the corresponding value of h . $3.951, 0.625$
- (iv) Sketch the graph showing the volume of the toy as the radius of the hemisphere varies.

12 N10/I/9**Box****Lid**

A company requires a box made of cardboard of negligible thickness to hold 300 cm^3 of powder when full. The length of the box is $3x \text{ cm}$, the width is $x \text{ cm}$ and the height is $y \text{ cm}$. The lid has depth $ky \text{ cm}$, where $0 < k \leq 1$ (see diagram).

- (i) Use differentiation to find, in terms of k , the value of x which gives a minimum total external surface area of the box and the lid.

$$x = \left[\frac{200}{3}(k+1) \right]^{\frac{1}{3}}$$

- (ii) Find also the ratio of the height to the width, $\frac{y}{x}$, in this case, simplifying your answer.

$$\frac{3}{2(k+1)}$$

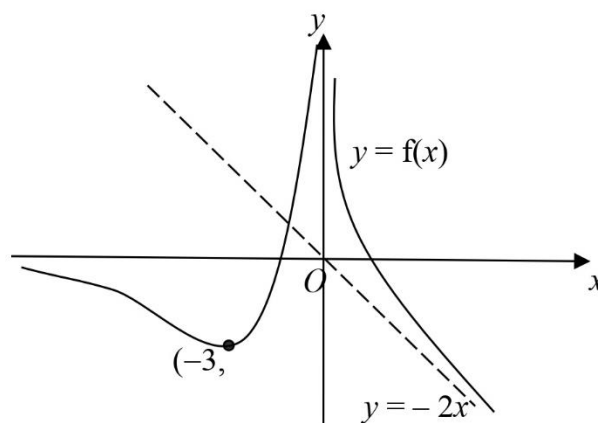
- (iii) Find the values between which $\frac{y}{x}$ must lie.

$$\left[\frac{3}{4}, \frac{3}{2} \right]$$

- (iv) Find the value of k for which the box has square ends.

$$\frac{1}{2}$$

- 13 The diagram shows the graph of $y = f(x)$, which has a minimum point at $(-3, -3)$.

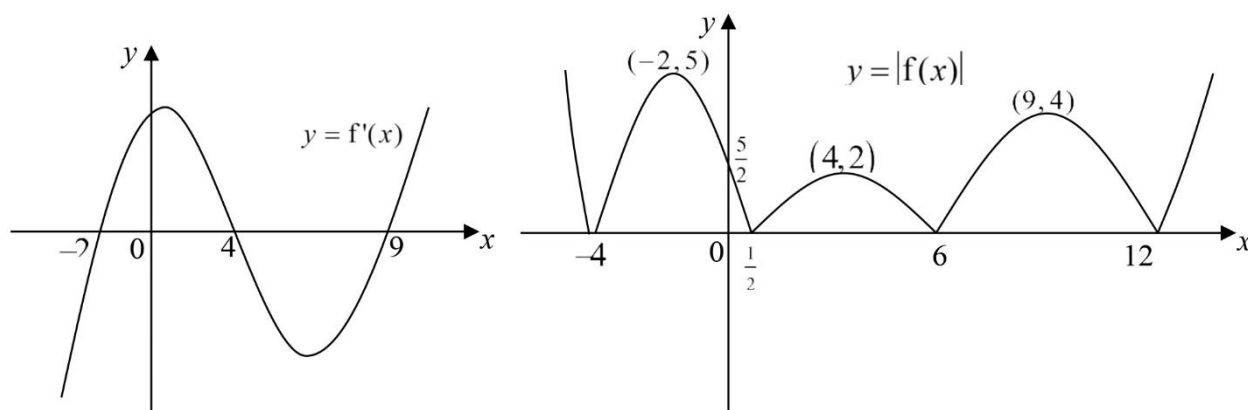


The curve is such that both the x and y -axis are asymptotes and $y = -2x$ is an oblique asymptote.

- (a) State the range of values of x for which f is strictly decreasing.
 (b) Sketch the graph of $y = f'(x)$.

14 PJC Prelim 08/II/1

The given sketches show the graphs of $y = f'(x)$ and $y = |f(x)|$.



Sketch the graph of $y = f(x)$, showing clearly the stationary points and intercepts. [3]

Supplementary Questions

- 15** Find the x -coordinates of all the stationary points on the curve $y = \frac{x^3}{(x+2)^2}$, stating, with reasons, the nature of each point.

$$x = -6 \text{ max pt, } x = 0 \text{ stat pt of inflex}$$

- 16** Find the coordinates and determine the nature of the turning points on the curve $y = 3x^4 - 16x^3 + 18x^2$. Sketch the graph of $y = 3x^4 - 16x^3 + 18x^2$ and state the range of values of k for which the equation $3x^4 - 16x^3 + 18x^2 = k$ has precisely two real roots for x .

$$k > 5 \text{ or } -27 < k < 0$$

- 17** Find by differentiation the x -coordinate of the stationary point of the curve $y = x^2 - k^2 \ln\left(\frac{x}{a}\right)$ where k and a are positive constants, and determine the nature of the stationary point.

$$\frac{k}{\sqrt{2}} \text{ min pt}$$

18 A curve C has parametric equations

$$x = \sin^3 \theta, \quad y = 3 \sin^2 \theta \cos \theta, \quad \text{for } 0 \leq \theta \leq \frac{1}{2} \pi.$$

(i) Show that $\frac{dy}{dx} = 2 \cot \theta - \tan \theta$.

(ii) Show that C has a turning point when $\tan \theta = \sqrt{k}$, where k is an integer to be determined. Find in non-trigonometric form, the exact coordinates of the turning point and explain why it is a maximum.

$$\left(\frac{2\sqrt{6}}{9}, \frac{2\sqrt{3}}{3} \right)$$

19 N14/II/1 (Anchor Q – Tangent/ Normal)

A curve C has parametric equations

$$x = 3t^2, \quad y = 6t.$$

(i) Find the value of t at the point on C where the tangent has gradient 0.4. $2.5[3]$

(ii) The tangent at the point $P(3p^2, 6p)$ on C meets the y -axis at the point D . Find the Cartesian equation of the locus of the mid-point of PD as p varies.

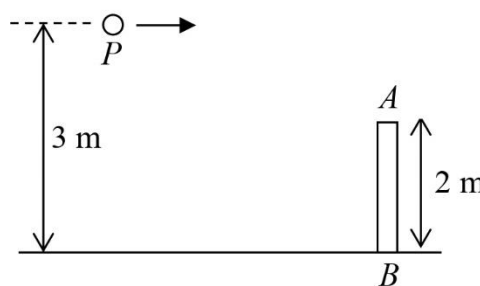
$$y^2 = \frac{27}{2}x \quad [4]$$

20 ACJC Prelim 11/I/7

An isosceles triangle has fixed base of length b cm. The other 2 equal sides of the triangle are each decreasing at the constant rate of 3 cm per second. How fast is the area changing when the triangle is equilateral? Leave your answer in terms of b .

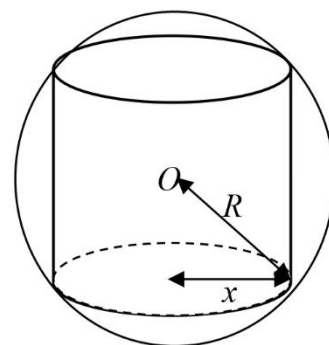
$$-\sqrt{3}b$$

21 The given diagram shows a vertical pole AB of length 2 m on a horizontal ground, with points A and B representing the top and bottom ends respectively. A moving point P approaches the pole on a horizontal path of 3 m above the ground and at a constant speed of 2 m/s. If θ is the angle APB , find the rate of change of θ with respect to time when P is at a horizontal distance of 8 m from the pole.



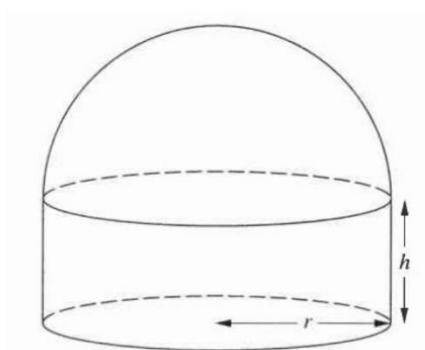
$$0.0514 \text{ rad s}^{-1}$$

- 22** A cylinder of radius x is inscribed in a fixed sphere of internal radius R , the circumferences of the circular ends being in contact with the inner surface of the sphere. Show that $A^2 = 16\pi^2 x^2 (R^2 - x^2)$, where A is the curved surface area of the cylinder. Prove that, if x is made to vary, the maximum value of A is obtained when the height of the cylinder is equal to its diameter.



23 N12/I/10

[It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



A model of a concert hall is made up of three parts.

- The roof is modelled by the curved surface of a hemisphere of radius r cm.
- The walls are modelled by the curved surface of a cylinder of radius r cm and height h cm.
- The floor is modelled by a circular disc of radius r cm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

- (i) It is given that the volume of the model is a fixed value $k \text{ cm}^3$, and the external surface area is a minimum. Use differentiation to find the values of r and h in terms

of k . Simplify your answers.

$$r = h = \left(\frac{3k}{5\pi} \right)^{\frac{1}{3}} [7]$$

- (ii) It is given instead that the volume of the model is 200 cm^3 and its external surface area is 180 cm^2 . Show that there are two possible values of r . Given also that $r < h$, find the value of r and the value of h .

$$r = 3.04, h = 4.88 [5]$$