X JUNIOR COLLEGE
YEAR 6 PRELIMINARY EXAMINATION
Mock Arrangement
in preparation for Candidates' Examination
Higher (than) 2

CANDIDATE NAME

MATHEMATICS 9758/01

Paper 2 **Set II**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Write your answers in the space provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need of clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

ABOUT THIS PAPER

X Junior College (XJC) is an unofficial initiative aimed at preparing pre-university and/or junior college students in Singapore for school-level and/or national-level H2 Mathematics examinations through self-prepared mock papers. It has no affiliation with any existing institution in Singapore or worldwide.

This mock paper follows closely the 9758 H2 Mathematics GCE Advanced Level syllabus, most suitable for preparation towards preliminary examinations and A–Levels. The paper intends to explore the unconventional ways and/or applications in which topics within the syllabus can be tested, which may affect the difficulty of this paper to varying degrees. **While it is ideal to attempt this paper under examination constraints, prospective candidates are reminded not to use this potentially non-conforming paper as a definitive gauge for actual performance.**

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This document consists of **26** printed pages and **2** blank page.

Section A: Pure Mathematics [40 marks]

1 A piecewise function f is given such that

$$f(x) = \begin{cases} -\sqrt{k^2 - (x-k)^2}, & 0 \le x < k, \\ x - 2k, & k \le x < 2k, \end{cases}$$

and $f(x+2k) = \frac{1}{2}f(x)$ for all real values of x, where k is a positive constant.

(i) Sketch $y = f(x)$ for $-2k \le x < 4k$. Indicate clearly all axial intercepts and minimum points. [2]

(ii) Find, in terms of k, the exact area bounded by the curve $y = f(x)$ and the x–axis for $0 \le x < 2k$. [1]

(iii) Deduce that $\displaystyle\int_{-2k}^{\infty} \mathrm{f}(x)\,\mathrm{d}x = ak^2$, for some exact real value a to be determined. [2]

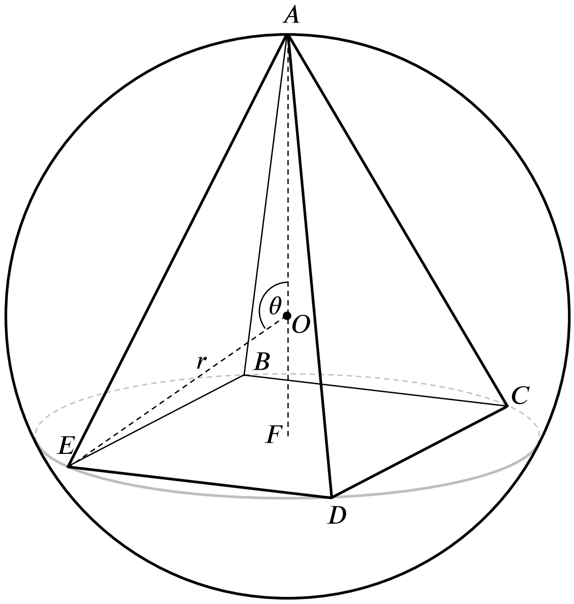
2 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 1, \quad u_2 = 2 \quad \text{and} \quad u_{n+2} = 2u_{n+1} - u_n + 2^n - 1 \text{ for } n \geq 1.$$

Using the substitution $v_n = u_{n+1} - u_n$, show that the recurrent relation above can be reduced to $v_{n+1} - v_n = 2^n - 1$. Use this result to find a general formula for u_n in terms of n. [6]

2 [Continued]

3 [The volume of a square-based pyramid is $\frac{1}{3} \times$ base area $\times$ height.]



A sphere with centre O has a fixed radius r. A right pyramid with a square base is inscribed within it, with point A as its apex and $BCDE$ as its square base. A perpendicular dropped from A passes through O and intersects $BCDE$ at point F. Point A subtends an angle θ with the corners of the base at O, where $0° < \theta < 180°$ (see diagram).

Find, in degrees, the angle θ which maximises the volume of the pyramid. Justify that the resulting volume is maximum and find its value exactly in terms of r. Show all your working clearly. [8]

3 [Continued]

4 **(a)** The complex number z is such that $\dfrac{z}{1+z^2}$ is real. If z is not real, show that $|z| = 1$. [5]

(b) A positive integer m is the smallest possible such that $\gamma = (6 + pi)\omega^2 + (-3m + qi)\omega + 2m$, where p and q are real values, is purely imaginary at two distinct ω values. Given that the two possible γ values are conjugate pairs, find these γ values in terms of q. [5]

5 With respect to an origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively such that $|\mathbf{a}| < |\mathbf{b}|$ and the angle θ between $\mathbf{a}$ and $\mathbf{b}$ is acute. Points C and D are given such that $OABC$ and $OADB$ are isosceles trapeziums, and that CD is parallel to $\mathbf{a}$ and equal in length to OB.

(i) Show that

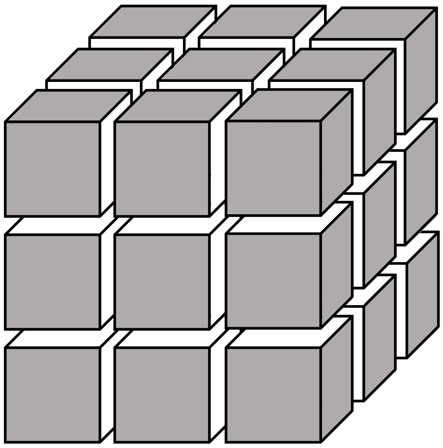
$$\overrightarrow{OC} = \left(\frac{|\mathbf{b}|}{|\mathbf{a}|} - 1\right)(\mathbf{b} - \mathbf{a}) \quad \text{and} \quad \overrightarrow{AD} = \left(\frac{|\mathbf{b}|}{|\mathbf{a}|} - 1\right)\mathbf{b}.$$ [5]

(ii) Express $|\mathbf{b}-\mathbf{a}|^2$ in terms of $\mathbf{a}$, $\mathbf{b}$ and θ. Hence, given that $\left|\overrightarrow{OC}\right| = \left|\overrightarrow{AD}\right| = |\mathbf{a}|$, find θ in degrees. [5]

(iii) Name the shape of $OADBC$. [1]

Section B: Probability and Statistics [60 marks]

6



A $3 \times 3 \times 3$ solid cube is white inside and has a black surface all around. It is then sliced into 27 pieces of $1 \times 1 \times 1$ cubes. A machine shuffles the cube slices and then blindly reassembles them into a $3 \times 3 \times 3$ cube.

Find the probability that the resulting cube will appear completely black on the surface. [6]

6 [Continued]

7 Two players A and B are contestants of a gaming competition which consists of multiple rounds of duels. To win a round, a player must be the first to win three duels. It is known that the probability of A winning a duel is p, and the outcomes of different duels are independent. Let X be the number of duels played in one round.

(i) Show that $\mathrm{P}(X = 3) = 1 - 3p + 3p^2$. [2]

(ii) Tabulate the probability distribution of X, expressing each probability in increasing powers of p. [3]

7 [Continued]

In the middle of the competition, a commentator notices that none of the rounds played have ended at the third duel. He makes a statement that the competition will continue as such in the long run.

(iii) Find the range of values of p for which the commentator's statement is true. [3]

8 A confectionery company owns a machine which produces chocolate bars such that the mass M, in grams, of a chocolate bar has an average of 250 grams with standard deviation 150 grams.

(i) Explain why the masses of the chocolate bars may not be approximated properly using a normal distribution with the given average and standard deviation. [2]

The company then decides to upgrade its machineries. To evaluate the extent of upgrade, the mass of each chocolate bars is measured using a weighing scale. It is found that the measurements are normally distributed with mean 450 grams and standard deviation σ grams such that

$$3\text{P}(437.33 < M < 462.67) = 2\text{P}(462.67 < M < 514.09) \quad \text{and} \quad \text{P}(437.33 < M < 514.09) = \text{P}(M < 450).$$

(ii) Find σ. [3]

8 [Continued]

For the rest of the question, $\sigma = 50$.

The weighing scale used is found to be faulty, and thus the recorded measurements need to be corrected. The old measurements are each scaled by a positive factor k and then moderated by m units. The new measurements now follow the normal distribution $\mathrm{N}(550, 25^2)$.

(iii) Find the value of k and m. Hence, calculate the probability that the new measurement of a randomly selected chocolate bar is now at least twice as heavy as its old measurement. [5]

9 A city is experiencing an outbreak due to a virus known to infect 2% of the population. Medical experts are examining 5000 people and their blood samples to find out the number of infected cases X among those samples.

(i) Explain why, in this context, X may not follow a binomial distribution. [1]

For the rest of the question, you may assume that X follows a binomial distribution.

(ii) Show that, for $k = 0, 1, 2, \ldots, 4999$,

$$\frac{\mathrm{P}(X = k + 1)}{\mathrm{P}(X = k)} = \frac{5000 - k}{49(k + 1)}.$$

By considering $\mathrm{P}(X = k + 1) > \mathrm{P}(X = k)$, or otherwise, find the most probable value of X. [3]

9 [Continued]

Instead of testing all 5000 blood samples individually, experts divide the samples **evenly** into m groups and make a blood mixture for each group using all samples from that group. For each blood mixture, a test is carried out using a serum which returns positive if at least 2% in that group are infected.

(iii) For $m = 100$, find the probability that the experts identify 2% of the m groups as positive just after they carried out the fifth test. [3]

(iv) By considering the **even** division of the groups, find the largest number of groups m such that, after all the groups are tested, there is more than 20% probability that exactly half of the groups are tested positive. Show your working clearly. [4]

10 In a chemical reaction involving chemicals A and B, a universal indicator (UI) is used to indicate the end of the reaction through a colour change.

A chemist sets up an automated chemical experiment to study such a reaction. In one run of the reaction, a computer records the number of acid drops x required for the UI to change colour. This run is repeated n times, maintaining the exact same reaction conditions such that the colour change is only affected by the number of acid drops. The experiment results are summarised into probabilities of occurrence of x drops in these n runs, as seen below.

x (drops)	92	93	94	95	96	97	98
Probability of occurrence in n runs	0.06	0.12	0.20	0.26	0.16	0.12	0.08

(i) By considering the probabilities of occurrence, explain why n must be a multiple of 50. [1]

(ii) Find $\mathrm{E}(x)$ and $\mathrm{Var}(x)$. [2]

Using these data, a chemist carries out a test, at 5% significance level, to determine whether the acid is stronger than the alkali, which would require at most 95 drops of the acid in each reaction on average.

(iii) Explain why there is no need for the chemist to know anything about the population distribution of the number of acid drops required for the UI to change colour. [2]

(iv) Using appropriate hypotheses and unbiased estimates, find the least n such that, according to the test, the acid is indeed stronger than the alkali. [7]

11 In computer science, the efficiency of an algorithm can be analysed using its *time complexity*, an algorithm's inherent correlation between the time taken T to solve a problem and the size of the given problem n. Time complexity analysis reveals how algorithms perform in best-case and worst-case scenarios, and it can be done on any type of algorithm, particularly sorting algorithms.

The following table shows the time complexity of the best-case and worst-case scenario for three well-known sorting algorithms: Selection Sort, Merge Sort, and Quick Sort.

Algorithm	Selection Sort	Merge Sort	Quick Sort
Best-case	$T = a + b(n^2)$	$T = a + b(n \log_2 n)$	$T = a + b(n \log_2 n)$
Worst-case	$T = a + b(n^2)$	$T = a + b(n \log_2 n)$	$T = a + b(n^2)$

A computer runs three unidentified sorting algorithms A, B and C, each of them is distinct and is either Selection Sort, Merge Sort, or Quick Sort. The time taken in milliseconds (ms) for the three algorithms A, B and C to sort n bytes of data are denoted by T_A, T_B and T_C respectively and tabulated below.

n		1000	2000	3000	4000	5000	6000	7000	8000
T_A	Best	2.3	5.3	8.4	11.5	14.9	17.6	21.4	26.1
	Worst	3.4	6.8	13.5	22.0	31.6	43.0	57.1	70.9
T_B	Best	5.0	11.0	17.3	24.0	31.2	36.9	44.5	52.1
	Worst	8.1	17.6	28.0	38.3	49.1	60.2	71.2	83.4
T_C	Best	5.1	20.2	46.5	81.0	126.9	184.1	250.0	326.0
	Worst	28.3	113.8	255.2	455.9	712.1	1030.1	1400.5	1810.8

(i) By considering product moment correlation coefficients, deduce the identity of each algorithm A, B and C. Show your working clearly. [3]

For a showcase, one of the sorting algorithms ***A***, ***B*** or ***C*** will be used to sort some size of data in exactly 3.0 ms. A student suggests using algorithm ***B*** and proposes a corresponding case and data size.

(ii) Explain why the student's proposal would be unreliable. [1]

(iii) Suggest, between ***A***, ***B*** and ***C*** and their cases, a more reliable proposal. Using the identities deduced in **(i)**, find the suitable regression line and propose a data size for the showcase, correct to the nearest byte. [3]

11 [Continued]

Further runs are observed to identify the relationship between the time taken T, in milliseconds (ms), and the memory used S, in bytes, for a given sorting algorithm. The table below is obtained from five runs.

Run	1	2	3	4	5
T	1.2	1.4	1.8	2.9	3.2
S	0.3	0.5	0.8	s	1.2

For these five runs, the regression line L of S on T is given by $S = \frac{7}{18}T - \frac{17}{300}$.

(iv) Show that $s = 1.0$ [2]

Another run of the algorithm, Run 6, takes 7.5 ms and uses 1.3 bytes of memory.

(v) Draw a scatter diagram to illustrate the values obtained from all six runs, labelling the axes clearly. Use the diagram to suggest, with justification, a better model than L for the relationship between S and T. [4]

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