H2 Mathematics
The Minimalist Problem Set
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This minimalist problem set of 12 pure mathematics topics is designed to support every H2 Mathematics student in:

- Developing critical thinking, problem-solving, and the practical application of acquired knowledge.

- Realising the best results within a short, intensive period of focused practice before the A-level examination.

A bonus topic *permutations & combinations* has also been included.

30 August 2023

Topic 1a: Inequalities and equations

Q1

(i) Show algebraically that $x^2 + x + 4 > 0$ for all real values of x.

(ii) Hence, without using a calculator, solve the inequality $\dfrac{1-x}{2+x} \le \dfrac{2}{x}$.

Answer(s):

(ii) $x < -2$ or $x > 0$

Q2

(i) Sketch in a single diagram, the graphs of $y = \left|\dfrac{x+1}{x-1}\right|$ and $y = -(x+3)^3$.

(ii) Hence solve the inequality $\left|\dfrac{x+1}{x-1}\right| > -(x+3)^3$.

(iii) Using the answer obtained in part (ii), deduce the range of values of x that satisfies $\left|\dfrac{e^x+1}{e^x-1}\right| > -(e^x+3)^3$.

Answer(s):

(ii) $x > -3.84, x \ne 1$

(iii) $x \in \mathbb{R}, x \ne 0$

Q3

Find the exact solutions of the equation $\left|x^2-1\right|-1=3x-2$.

Answer(s):

$x=3, \frac{1}{2}\left(\sqrt{17}-3\right)$

Q4

Solve the inequality $\sin 3x>\frac{1}{2}$, where $-\frac{\pi}{2}\le x<\frac{\pi}{2}$.

Answer(s):

$-\frac{\pi}{2}\le x<-\frac{7\pi}{18}$ or $\frac{\pi}{18}<x<\frac{5\pi}{18}$

Topic 1b: Discriminants

Q1

The curve C has equation $y = \dfrac{x^2 + 2x + q}{x - 1}$, $x \neq 1$, where q is a constant. Given that C has two stationary points, show that $q > -3$.

Q2

Use an algebraic method to show that the curve $y = \dfrac{x^2 + 2x + 1}{x - 1}$, $x \neq 1$ does not lie between 0 and 3, not including 0 and 3.

Topic 2: System of linear equations

Q1

Let $\mathrm{f}(x)$ be a cubic polynomial. It is given that the graph of $y=\mathrm{f}(x)$ passes through the origin and has a stationary point at the point $(-6,-3)$. The tangent at the point where $x=-4$ is parallel to the line $6y=x+1$. Find $\mathrm{f}(x)$.

Answer(s):

$$\mathrm{f}(x)=\frac{1}{72}x^3+\frac{1}{4}x^2+\frac{3}{2}x$$

Q2

A collection of 41 shapes consisting of identical triangles, squares and pentagons have a total of 169 sides.

The base and height of each triangle have the same length as the side of each square. Given that the total area of the triangles is twice the total area of the squares, find the number of triangles, squares and pentagons respectively.

Answer(s):

16 triangles, 4 squares, 21 pentagons

Q3
Find positive integers x, y, s, t such that the given chemical reaction is balanced, in terms of the number of molecules of the elements Na, P, Ca and F.
$x\mathrm{Na_3P} + y\mathrm{CaF_2} \rightarrow s\mathrm{NaF} + t\mathrm{Ca_3P_2}$

Answer(s):
$x = 2, y = 3, s = 6, t = 1$

Topic 3: Graphs

Q1

The curve C has equation $y=\dfrac{ax+3}{x+b}$ where a, b are positive integers such that $ab<3$.

(i) State the equations of the asymptotes of C, in terms of a and b.

(ii) Sketch the graph of C, indicating all asymptotes and axial intercepts.

Answer(s):

(i) $y=a, x=-b$

(ii) $\left(0,\dfrac{3}{b}\right),\left(-\dfrac{3}{a},0\right)$

Q2

The curve C has equation $\dfrac{(x-1)^2}{9}-y^2=1$.

(i) Sketch the curve C, showing clearly the equations of asymptotes, axial intercepts and coordinates of turning points, if any.

(ii) Given that k is a positive constant and C intersects the curve with equation $y^2+\dfrac{x^2}{k^2}=1$ at exactly two distinct points, state the range of values of k.

Answer(s):

(ii) $2<k<4$

Q3

The curve C has equation $y = \dfrac{x^2 - 2x - 3}{x + 2}$.

(i) Find the equations of the asymptotes of C.

(ii) Draw a sketch of C, which should include the asymptotes, and state the coordinates of the points of intersection of C with the x-axis.

(iii) On the same diagram draw a sketch of the curve $y = \dfrac{4}{(x+4)^2}$, showing clearly the asymptotes.

(iv) Hence show that the equation $x^4 + 6x^3 - 3x^2 - 60x - 56 = 0$ has exactly two real roots.

Answer(s):

(i) $x = -2,\ y = x - 4$

(ii) $(-1, 0), (3, 0)$

Q4

A curve C has parametric equations $x = t + t^2,\ y = t^2 + t^3$ where $-2 \le t \le 2$.

(i) Sketch C, indicating the end-points and the behaviour near the origin.

(ii) Explain why the curve does not exist when $x < -\dfrac{1}{4}$.

(iii) Find the gradient of the tangent to C at (2, 2) and the angle that this tangent makes with respect to the horizontal.

Answer(s):

(iii) 1.67; 59.0°

Q5

Determine, with working, whether the curves $y = \dfrac{6x+25}{8x+12}$ and $x^2 + y^2 = \dfrac{125}{16}$ are tangent to each other at $x = -2.5$.

Answer(s):

Yes,

the curves touch each other at $(-2.5, -1.25)$ and

both curves give the same gradient of -2 at $(-2.5, -1.25)$.

Topic 4: Transformations

Q1

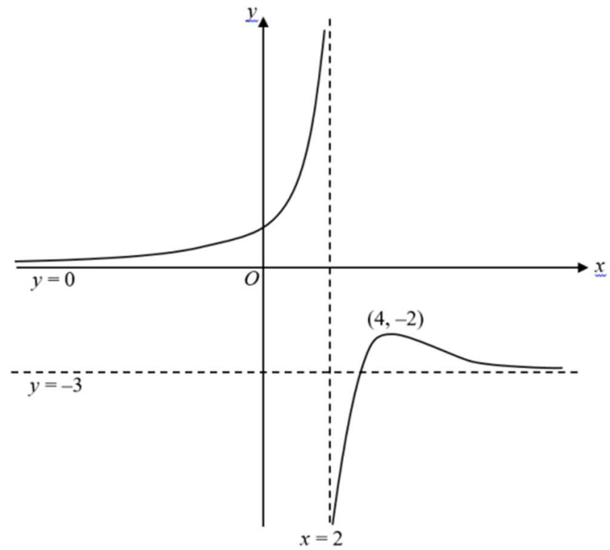
A curve with equation $4x^2 + y^2 - 2y - 24 = 0$ is transformed by a translation of 6 units in the positive y-direction, followed by a stretch parallel to the x-axis with scale factor 2. Find the equation of the new curve in the form $(x-h)^2 + (y-k)^2 = r^2$.

Answer(s):

$x^2 + (y-7)^2 = 25$

Q2

The diagram below shows the graph of $y = \mathrm{f}(x)$.



Sketch, separately, the graphs of

(i) $y = |\mathrm{f}(x)|$,

(ii) $y = \mathrm{f}(|x|)$,

(iii) $y = 3\mathrm{f}(2x+1)$,

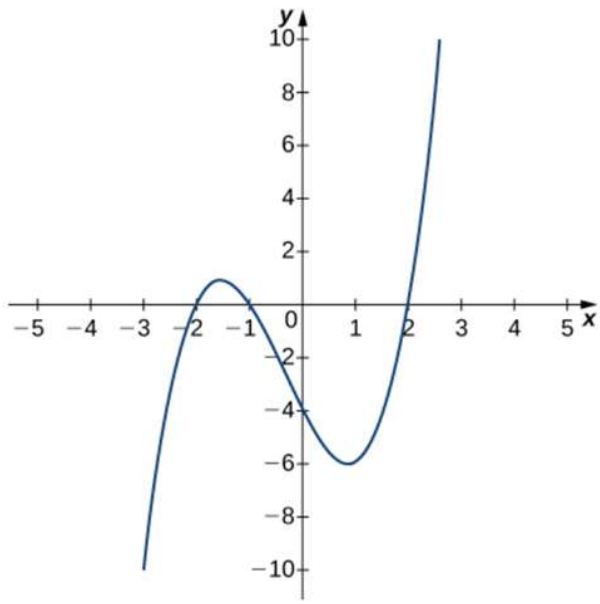
(iv) $y = \dfrac{1}{\mathrm{f}(x)}$,

(v) $y = \mathrm{f}'(x)$.

For part (iii), describe a sequence of transformations on $y = \mathrm{f}(x)$ to obtain $y = 3\mathrm{f}(2x+1)$.

Q3

Below is the graph of $y = f'(x)$.



(i) List all intervals where f is increasing or decreasing.

(ii) Indicate the locations of all stationary points of f and determine their nature.

(iii) What can you say about the concavity of f for the interval between the turning points of f'?

Answer(s):

(i) f is increasing for $x \in (-2, -1) \cup (2, \infty)$;

f is decreasing for $x \in (-\infty, -2) \cup (-1, 2)$

(ii) $x = -2$ (minimum);

$x = -1$ (maximum);

$x = 2$ (minimum)

(iii) f is concave down since $f'(x)$ is decreasing over the interval.

Topic 5: Functions

Q1

The function f is defined by $\mathrm{f}: x \mapsto \left|\frac{1}{2x+1}\right|,\ x < -\frac{1}{2}$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ and explain clearly why the inverse function f^{-1} exists. On the same diagram, sketch the graph of $y = \mathrm{f}^{-1}(x)$, indicating clearly the geometrical relationship with the graph of $y = \mathrm{f}(x)$.

(ii) Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}.

Another function g is defined by $\mathrm{g}: x \mapsto -2-\sqrt{x-5},\ x > 5$.

(iii) Sketch the graph of $y = \mathrm{g}(x)$, showing the main relevant features.

(iv) Show that fg exists, and find its rule, domain and range.

(v) Determine the value of $(\mathrm{fg})^{-1}(0.2)$.

Answer(s):

(ii) $\mathrm{f}^{-1}(x) = \frac{1}{2}\left(-1-\frac{1}{x}\right), x > 0$

(iv) $\mathrm{fg}(x) = \frac{1}{3+2\sqrt{x-5}}, x > 5$; $\mathrm{R}_{\mathrm{fg}} = \left(0, \frac{1}{3}\right)$

(v) $(\mathrm{fg})^{-1}(0.2) = 6$

Q2
It is given that

$$\mathrm{h}(x)=\begin{cases}-4x-12, & \text{for } -4\le x\le -2,\\ x|x|, & \text{for } -2<x\le 2.\end{cases}$$

and that $\mathrm{h}(x)=\mathrm{h}(x+6)$ for all real values of *x*.

(i) Evaluate $\mathrm{h}(-4)$ and $\mathrm{h}(12)$.

(ii) Sketch the graph of $y=\mathrm{h}(x)$ for $-4\le x\le 6$ and explain why h^{-1} does not exist.

Answer(s):
(i) $\mathrm{h}(-4)=4, \mathrm{h}(12)=0$

Q3
A function is said to be a self-inverse if $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ for all *x* in the domain. Find the value of the constant *k* so that $\mathrm{g}(x)=\dfrac{3x-5}{x+k}$ is a self-inverse function.

Answer(s):
$k=-3$

Q4

The function f is defined by

$$\mathrm{f}(x)=\begin{cases}3x^2+2, & \text{for } x\in\mathbb{R}, 0<x\le 1,\\ \dfrac{1}{x}, & \text{for } x\in\mathbb{R}, x>1.\end{cases}$$

(i) Sketch the graph of $y=\mathrm{f}(x)$.

(ii) Show that f^2 exists and define f^2 in a similar form to f.

(iii) Define f^{-1} in a similar form to f.

Answer(s):

(ii) $$\mathrm{f}^2(x)=\begin{cases}\dfrac{1}{3x^2+2}, & \text{for } x\in\mathbb{R}, 0<x\le 1,\\ \dfrac{3}{x^2}+2, & \text{for } x\in\mathbb{R}, x>1.\end{cases}$$

(iii) $$\mathrm{f}^{-1}(x)=\begin{cases}\dfrac{1}{x}, & \text{for } x\in\mathbb{R}, 0<x<1,\\ \sqrt{\dfrac{x-2}{3}}, & \text{for } x\in\mathbb{R}, 2<x\le 5.\end{cases}$$

Topic 6a: Arithmetic and geometric progressions

Q1

An arithmetic series A has first term a and common difference d, where a and d are non-zero. A convergent geometric series G has common ratio r.

The first three terms of G are equal to the first, eleventh and seventeenth terms of A, respectively.

(i) Find r.

(ii) Hence find the exact ratio of the sum to infinity of G to the sum of the first four terms of G.

Answer(s):

(i) $r = \frac{3}{5}$

(ii) 625:544

Q2

A patient is administered a 500 mg dose of Drug A on the first day. Each subsequent day, the dosage of Drug A is reduced by 20 mg.

(i) Find the total amount of Drug A (in mg) administered at the end of two weeks.

Another patient is administered a 500 mg dose of Drug B every six hours. At the end of each six-hour period, 32% of the amount of drug present at the start of the six-hour period remains.

(ii) Find the amount of Drug B (in mg) present immediately after the patient takes the third dose.

(iii) An overdose occurs when the amount of Drug B exceeds 735 mg. Calculate the maximum number of doses the patient can take to avoid overdosing.

Answer(s):

(i) 5180

(ii) 711

(iii) 6

Q3

(a) Evaluate $\sum_{r=3}^{2n}\left(3^r - 2r\right)$.

(b) It is given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

Find a formula for $(-5)(2)+(-3)(3)+(-1)(4)+\ldots+n$ th term, in factorised form.

Answer(s):

(a) $\frac{27}{2}\left(3^{2n-2}-1\right)-(2n-2)(2n+3)$

(b) $\frac{1}{6}n(n-5)(4n+11)$

Q4

A couple decided to purchase a flat and took a mortgage loan of \$100000 from a bank on 1 January 2023 to finance their flat. The bank charges an interest rate of 2% per month, so that on the last day of each month, the amount that they owe the bank increases by 2%. As part of the mortgage loan policy, they are required to pay \$3000 on the first day of each month, starting from February 2023.

(i) Taking January 2023 as the first month, show that the amount of money the couple owed the bank on the last day of the n th month is given by $153000-50000(1.02)^n$.

(ii) It is given that the amount of money that the couple owed the bank will first become less than \$50000 in the N th month. Find the value of N.

Answer(s):

(ii) $N = 37$

Q5

The sum of the first n terms of a series is given by the expression $6-\frac{2^{n+1}}{3^{n-1}}$. By finding an expression for the n th term of the series, show that this is a geometric series, and state the values of the first term and the common ratio.

Answer(s):

$a=2; r=\frac{2}{3}$

Topic 6b: General sequences and series

Q1

A sequence is defined by the relation $u_{n+1} = 2u_n + An$.

(i) Given that $u_1 = 4$ and $u_2 = 13$, find A and u_3.

Given that the n th term is given by $u_n = a\left(2^n\right) + bn + c$.

(ii) Find the constants a, b and c.

(iii) Find $\sum_{r=1}^{n} u_r$ in terms of n.

Answer(s):

(i) $A = 5, u_3 = 36$

(ii) $a = 7, b = -5, c = -5$

(iii) $-\frac{5}{2}n^2 - \frac{15}{2}n + 14\left(2^n\right) - 14$

Q2

A sequence is defined by $u_1 = 1$ and $u_{r+1} = 3u_r - 1$ for $r > 0$.

(i) Write down the first 5 terms of the sequence and find $\sum_{r=7}^{15} u_r$.

(ii) If $u_1 = 5$ instead, what can you say about the behaviour of the sequence?

(iii) If $u_{r+1} = \frac{1}{3u_r - 1}$ instead, estimate the value of the limit that the sequence converges to.

Answer(s):

(i) 1, 2, 5, 14, 41; 3587049

(ii) A constant sequence with all terms being 0.5.

(iii) -0.434

Topic 7: Method of differences

Q1

(i) Use the method of differences to show that $\sum_{r=2}^{n}\frac{1}{r^3-r}=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{n}+\frac{1}{n+1}\right]$.

(ii) Give a reason why the series $\sum_{r=2}^{\infty}\frac{1}{r^3-r}$ is convergent and write down its value.

(iii) Find $\sum_{r=n+1}^{2n}\frac{1}{r^3-r}$.

[There is no need to express your answer as a single algebraic fraction.]

Answer(s):

(ii) $\frac{1}{4}$

(iii) $\frac{1}{4n}+\frac{1}{4n+2}-\frac{1}{2n+2}$

Q2

Given that $u_r=\frac{1}{r!}$. Show that $u_r-u_{r+1}=\frac{1}{r!+(r-1)!}$.

Hence evaluate the series $\frac{1}{2!+3!}+\frac{1}{3!+4!}+...+\frac{1}{(N-1)!+N!}$ in terms of *N*.

Deduce that $\sum_{r=3}^{N}\frac{1}{r!+(r+2)!}<\frac{1}{6}$.

Answer(s):

$\frac{1}{6}-\frac{1}{(N+1)!}$

Q3

Show that $\tan^{-1}(2n+1)-\tan^{-1}(2n-1)=\tan^{-1}\left(\frac{1}{2n^2}\right)$.

Hence determine the exact value $\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{2n^2}\right)$.

Answer(s):

$\frac{\pi}{4}$

Topic 8: Differentiation and its applications

Q1

Steel sheets of negligible thickness are used to make cans in the shape of a right-circular cylinder of radius r cm and height h cm. It is given that a can is to hold 300 cm^3 of liquid.

(i) Find, in terms of r, an expression for the external surface area, A cm^2, of a closed can.

(ii) Find, using differentiation, the exact value of r which produces a can with a minimum value of A. Hence, find the minimum value of A.

(iii) Deduce that $\dfrac{h}{r}=2$ when A is a minimum.

(iv) Liquid is dispensed into an empty can at a rate of 100 cm^3/s. Find the rate of increase of the depth of the liquid, H, when A is a minimum.

Answer(s):

(i) $A=2\pi r^2+\dfrac{600}{r}$

(ii) $r=\sqrt[3]{\dfrac{150}{\pi}}$; minimum $A=248\ \text{cm}^3$

(iv) 0.806 cm/s

Q2

The equation of a curve C is $x^2-2xy+2y^2=k$, where k is a constant.

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

Given that C has two points for which the tangents are parallel to the line $y=x$, find the range of values of k.

Given that $k=4$, find the exact coordinates of each point on C at which the tangent is parallel to the y-axis.

Answer(s):

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y-x}{2y-x}$; $k>0$; $\left(-2\sqrt{2},-\sqrt{2}\right),\left(2\sqrt{2},\sqrt{2}\right)$

Q3

A curve is defined by the parametric equations $x = 3t + 2$ and $y = t + \frac{3}{t}$, where t is a non-zero parameter.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the y-coordinate of the point P where $x = 11$.

(ii) Find the equation of the tangent at P.

(iii) Determine, with working, whether the tangent at P will meet the curve again.

Answer(s):

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2 - 3}{3t^2}$; $y = 4$

(ii) $y = \frac{2}{3}x - \frac{10}{3}$

(iii) Yes, at the point where $t = -1$.

Q4

Find the turning points of the curve with parametric equations $x = t^2 - 4t$, $y = 2t^3 - 6t$ and distinguish between them.

Answer(s):

$(-3, -4)$ is a minimum point, $(5, 4)$ is a maximum point

Q5

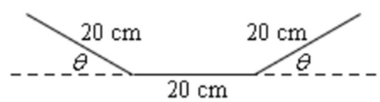
The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show, using parametric differentiation, that the normal to the curve at P has equation $t^3x - ty = c\left(t^4 - 1\right)$.

(ii) The tangent at P cuts the x-axis at X, and the normal at P cuts the y-axis at Y. Show that as t varies, the locus of the midpoint of XY, has cartesian equation $y = \dfrac{c^4 - x^4}{2c^2x}$.

Q6

A trough for holding water is formed by taking a piece of sheet metal 60 cm wide and folding the 20 cm on either end up as shown below. Use differentiation to determine the angle θ that will maximise the amount of water that the trough can hold.



Answer(s):

$\theta = 0.936$ radians

Q7

A curve C has equation $y = \dfrac{2x+1}{xy+3}$.

(a) Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, in terms of x and y.

(b) Show that there is no point on C for which the tangent is parallel to the y-axis.

Answer(s):

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2-y^2}{2xy+3}$

Q8

Find the exact values of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for the curve $y^4 + 5y^2 = x^4 - 5x^2$ at the point $(3, 2)$.

Answer(s):

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{2}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\dfrac{5}{8}$

Topic 9: Maclaurin series

Q1

Given that $y = \mathrm{e}^x \sin x$, express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form $k\mathrm{e}^x \sin(x+\alpha)$ where k and α are constants to be found. Express the next two derivatives in similar form and hence, obtain the expansion of y in ascending powers of x up to and including the term in x^3. Deduce an approximate value of $\int_0^{0.2} \mathrm{e}^x \sin x \, \mathrm{d}x$ and comment of its accuracy.

Answer(s):

$k = \sqrt{2}, \alpha = \frac{\pi}{4}$; $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$; $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 2\sqrt{2}\mathrm{e}^x \sin\left(x + \frac{3\pi}{4}\right)$; $y = x + x^2 + \frac{1}{3}x^3 + \ldots$;

$\frac{57}{2500}$ is very accurate because $x = 0.2$ is sufficiently small and close to $x = 0$.

Q2

It is given that $y = \left(1+\tan^{-1} x\right)^2$, show that $\left(1+x^2\right)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$. Find the Maclaurin series for y, up to and including the term in x^3. Hence write down the equation of the tangent to the curve $y = \left(1+\tan^{-1} x\right)^2$ at the point where $x = 0$.

Answer(s):

$1 + 2x + x^2 - \frac{2}{3}x^3 + \ldots$; $y = 1 + 2x$

Q3

(i) Use results of standard series in MF26 to find the first four terms in the series expansion of $\cos\left[\ln(1+x)\right]$.

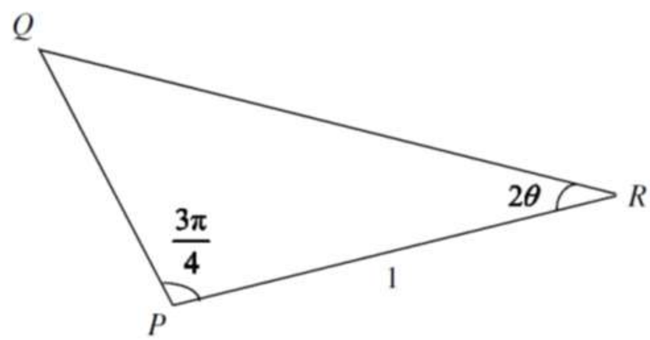
(ii) Hence obtain the Maclaurin expansion of $\sin\left[\ln(1+x)\right]$, up to and including the term in x^3.

Answer(s):

$1-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{12}x^4+\ldots$; $x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$

Q4

Refer to the triangle PQR below with the given information.



(i) Show that $QR=\dfrac{1}{\cos 2\theta-\sin 2\theta}$.

(ii) Given that θ is sufficiently small, show that $QR\approx 1+a\theta+b\theta^2$, for constants a and b to be determined.

Answer(s):

(ii) $a=2, b=6$

Topic 10a: Integration techniques

Q1

Using partial fractions, find $\int \frac{x^2 - x + 3}{2 - 2x + x^2 - x^3}\,\mathrm{d}x$.

Answer(s):

$-\ln|x-1| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$

Q2

Find the exact value of k such that $\int_0^{\frac{1}{2k}} \frac{1}{\sqrt{1-k^2x^2}}\,\mathrm{d}x = \int_0^1 \ln(2x+1)\,\mathrm{d}x$.

Answer(s):

$k = \frac{\pi}{9\ln 3 - 6}$

Q3

Find $\int \cos x \cos \frac{x}{2}\,\mathrm{d}x$ and hence find the exact value of $\int_0^{\pi} |\cos x| \cos \frac{x}{2}\,\mathrm{d}x$.

Answer(s):

$\frac{1}{3}\sin\frac{3x}{2}+\sin\frac{x}{2}+c$; $\frac{4\sqrt{2}}{3}-\frac{2}{3}$

Q4

Use integration by parts twice to find $\int x^2 \cos 2x\,\mathrm{d}x$.

Answer(s):

$\frac{1}{2}x^2\sin 2x+\frac{1}{2}x\cos 2x-\frac{1}{4}\sin 2x+c$

Q5

Use the substitution $x = a\sin\theta$ to find $\int \frac{x^2}{\left(a^2 - x^2\right)^{\frac{3}{2}}} \mathrm{d}x$.

Answer(s):

$\frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a} + c$

Q6

(i) Prove the identity $\tan 2\theta - \tan\theta = \tan\theta \sec 2\theta$.

(ii) Hence find the exact value of $\int_0^{\frac{\pi}{6}} \tan\theta \sec 2\theta \,\mathrm{d}\theta$.

Answer(s):

$\frac{1}{2}\ln\left(\frac{3}{2}\right)$

Topic 10b: Applications of integration

Q1

Explain, with a diagram, how the area under the curve $y = x^3 + 2$ for $0 \le x \le 1$ may be approximated by n rectangles of equal width. Show that this area is approximately $\frac{9}{4} - \frac{1}{2n} + \frac{1}{4n^2}$ and deduce the exact value of $\int_0^1 \left(x^3 + 2\right) \mathrm{d}x$. What is the least value of n such that the approximate area exceeds 99.9% of the actual area?

Answer(s):

$\frac{9}{4}$; least $n = 222$

Q2

(i) Use the substitution $u = 2x + 1$ to find $\int x\sqrt{2x+1}\,\mathrm{d}x$.

The region R is bounded by the curve $y = x\sqrt{2x+1}$, the y-axis and the line $y = \frac{1}{\sqrt{2}}$.

(ii) Find the exact area of R using your result in part (i).

(iii) Find the exact volume of the solid generated when R is rotated through four right angles about the x-axis.

Answer(s):

(i) $\frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + c$

(ii) $\left(\frac{11}{60}\sqrt{2} - \frac{1}{15}\right)$ units ^{2}

(iii) $\frac{17}{96}\pi$ units ^{3}

Q3

A curve is defined parametrically by the equations $x = \sin t$ and $y = \cos^3 t$, $-\pi \le t \le \pi$.

(i) Show that the area enclosed by the curve is given by $k\int_0^{\frac{\pi}{2}} \cos^4 t \, \mathrm{d}t$, where k is a constant to be found.

(ii) Hence find the exact area enclosed by the curve.

Answer(s):

(i) $k = 4$

(ii) $\dfrac{3\pi}{4}$ units^2

Q4

A curve C has parametric equations $x = \tan\theta$ and $y = 2\sin 2\theta$, $0 \le \theta < \dfrac{\pi}{2}$. The finite region S is bounded by C, the line $x = \dfrac{1}{\sqrt{3}}$ and the x-axis. This region is rotated through 2π radians about the x-axis to form a solid of revolution. By considering the cartesian equation of C, find the volume of the solid of revolution, correct to 4 significant figures.

Answer(s):

2.277 units^3

Topic 11: Differential equations

Q1

Verify that $y=(x+1)\mathrm{e}^{-x}$ is a solution to the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x}+y=\mathrm{e}^{-x}$.

Q2

An ecologist is investigating the change in the population of a particular species of insects of size x (in thousands) at time t (in months). She suggests that x and t are related by the differential equation $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}=9-5t$.

(i) Find the general solution of this differential equation.

(ii) Given that there are 1000 insects initially and that there is no change in population on the third month, find the particular solution of this differential equation.

Answer(s):

(i) $x=\dfrac{9}{2}t^2-\dfrac{5}{6}t^3+ct+d$

(ii) $x=\dfrac{9}{2}t^2-\dfrac{5}{6}t^3-\dfrac{9}{2}t+1$

Q3

When a constant voltage is applied to an electrical circuit with resistance R and inductance L, the current i satisfies the differential equation $L\frac{\mathrm{d}i}{\mathrm{d}t}+iR=V$ according to Kirchhoff's voltage law. Find the value of the current after 2 seconds if $R=10$ ohms, $L=75$ henrys and $V=220$ volts, and it is given that $i=0$ when $t=0$.

Answer(s):

5.15

Q4

A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at $3°\mathrm{C}$ and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is $\theta°\mathrm{C}$. The rate of change of the temperature of the water in the bottle follows Newton's law of cooling and is given by the differential equation $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{3-\theta}{125}$.

(i) By solving the differential equation, show that $\theta = A\mathrm{e}^{-0.008t}+3$, where A is a constant.

(ii) The water in the bottle was $16°\mathrm{C}$ when it was put in the refrigerator. Find the time taken for the water in the bottle to fall to $10°\mathrm{C}$.

Answer(s):

77.4 minutes

Q5

Using the substitution $z = xy$, convert the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x\sqrt{1 - x^2 y^2}$ to a separable equation. Hence find the general solution of the original differential equation.

Answer(s):

$$y = \frac{\sin\left(x^2 + c\right)}{x}$$

Q6

The mass, x grams of a certain substance present in a chemical reaction at time t minutes satisfies the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(1 + x - x^2\right)$, where $0 \le x \le \frac{1}{2}$ and k is a constant. It is given that $x = \frac{1}{2}$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{4}$ when $t = 0$.

(i) Show that $k = -\frac{1}{5}$.

(ii) By first expressing $1 + x - x^2$ in completed square form, find t in terms of x.

(iii) Hence find

(a) the exact time taken for the mass of the substance present in the chemical reaction to become half of its initial value,

(b) the time taken for there to be none of the substance present in the chemical reaction, giving your answer correct to 3 decimal places.

(iv) Express the solution of the differential equation in the form $x = \mathrm{f}\left(t\right)$ and sketch the part of the curve with this equation which is relevant in this context.

Answer(s):

(ii) $\frac{5}{4} - \left(x - \frac{1}{2}\right)^2$; $t = -\sqrt{5}\ln\left(\frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x}\right)$

(iii)(a) $t = \sqrt{5}\ln\left(\frac{2\sqrt{5} + 1}{2\sqrt{5} - 1}\right)$ minutes (b) 2.152 minutes

(iv) $x = \frac{\left(\sqrt{5} + 1\right) + \left(1 - \sqrt{5}\right)e^{\frac{t}{\sqrt{5}}}}{2\left(1 + e^{\frac{t}{\sqrt{5}}}\right)}$

Topic 11: Vectors

Q1

If $\mathbf{a}=\mathbf{i}+3\mathbf{j}$, $\mathbf{b}=2\mathbf{i}-5\mathbf{j}$ and $\mathbf{c}=-2\mathbf{j}+4\mathbf{k}$, find unit vectors in the direction of $\mathbf{a}+\mathbf{b}-\mathbf{c}$ and $3\mathbf{a}+2\mathbf{c}$. Give the geometrical meanings of $|\mathbf{a}\times\mathbf{d}|$ and $|\mathbf{a}.\mathbf{d}|$ if $\mathbf{d}$ is a unit vector.

Answer(s):

$$\frac{1}{5}\begin{pmatrix}3\\0\\-4\end{pmatrix};\ \frac{1}{\sqrt{98}}\begin{pmatrix}3\\5\\8\end{pmatrix}$$

$|\mathbf{a}\times\mathbf{d}|$ is the perpendicular distance of the point A to the line OD.

$|\mathbf{a}.\mathbf{d}|$ is the length of projection of **a** onto **d**.

Q2

The points A, B and C have position vectors **a**, **b**, **c** respectively referred to an origin O.

(a) Given that the point X lies on AB produced such that $AB:BX=2:1$, find the position vector of X, in terms of **a** and **b**.

(b) If Y lies on BC, between B and C such that $BY:YC=1:3$, find the position vector of Y, in terms of **b** and **c**.

(c) Given that Z is the midpoint of AC, show that X, Y and Z are collinear.

(d) Calculate $XY:YZ$.

Answer(s):

(a) $\overrightarrow{OX}=-\frac{1}{2}\mathbf{a}+\frac{3}{2}\mathbf{b}$

(b) $\overrightarrow{OY}=\frac{3}{4}\mathbf{b}+\frac{1}{4}\mathbf{c}$

(d) 1:1

Q3

The vectors **a** and **b** are such that $\mathbf{a}.\mathbf{b} = 21$, $|\mathbf{a}| = 5\sqrt{2}$ and $|\mathbf{b}| = 3$. Determine the exact value of $|\mathbf{a} \times \mathbf{b}|$ and show that the length of $(\mathbf{a} - \mathbf{b})$ is $\sqrt{17}$.

Answer(s):

$|\mathbf{a} \times \mathbf{b}| = 3$

Q4

(i) Expand and simplify the vector expression $(\mathbf{a} - 4\mathbf{b}) \times (\mathbf{a} + 3\mathbf{b})$.

(ii) If **a** and **b** are perpendicular, deduce that $|(\mathbf{a} - 4\mathbf{b}) \times (\mathbf{a} + 3\mathbf{b})| = \lambda |\mathbf{a}||\mathbf{b}|$, where λ is an integer.

Answer(s):

(i) $7\,\mathbf{a} \times \mathbf{b}$

(ii) $\lambda = 7$

Q5

The lines l_1 and l_2 have equations $\mathbf{r} = \begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -7 \\ 19 \\ 14 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix}$ respectively, where $\lambda, \mu \in \mathbb{R}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. The point H on the line segment PQ is such that $3PH = HQ$.

(i) Find the acute angle between the lines l_1 and l_2.

(ii) By considering $\overrightarrow{PQ}$, find the position vectors of P and Q.

(iii) Show that the coordinates of H are $(4, 6, 1)$.

(iv) Find the length of projection of $\overrightarrow{AH}$ onto the line l_1, where point A has coordinates $(6, 5, -6)$.

Answer(s):

(i) $65.9°$

(ii) $\overrightarrow{OP} = \begin{pmatrix} 8 \\ 4 \\ -4 \end{pmatrix}; \overrightarrow{OQ} = \begin{pmatrix} -8 \\ 12 \\ 16 \end{pmatrix}$

(iv) 3 units

Q6

A line l has equation $\mathbf{r}=\begin{pmatrix}-4\\0\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\1\end{pmatrix}$, where $\lambda\in\mathbb{R}$.

(i) Find the position vector of P, the foot of the perpendicular from the origin O to l.

(ii) Find a cartesian equation of the plane Π_1 containing O and l.

(iii) It is given that l also lies in a plane Π_2 with equation $\mathbf{r}\cdot\begin{pmatrix}2\\k\\1\end{pmatrix}=-3$. Show that $k=\frac{7}{2}$.

(iv) Find the angle between Π_1 and Π_2.

(v) Find the length of projection of the vector $-4\mathbf{i}+5\mathbf{k}$ onto Π_2.

(vi) The point Q has coordinates $(\lambda, 0, 5)$ and its distance from Π_2 is 4 units. Find the possible coordinates of Q.

Answer(s):

(i) $\overrightarrow{OP}=\begin{pmatrix}-2.5\\-1\\5.5\end{pmatrix}$

(ii) $10x+19y+8z=0$

(iv) $6.8°$

(v) 6.36 units

(vi) $\left(\sqrt{69}-1,0,5\right)$ or $\left(-\sqrt{69}-1,0,5\right)$

Q7

(a) The point $A(2,4,-5)$ is reflected in the line with equation $\mathbf{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+t\begin{pmatrix}4\\2\\1\end{pmatrix}, t\in\mathbb{R}$ to give the point A'. Determine the coordinates of A'.

(b) The point $B(1,-1,0)$ is reflected in the plane $x+y-z=6$ to give the point B'. Determine the coordinates of B'.

(c) L is the line of intersection of the two planes $2x-2y+5z=-4$ and $x+2y-2z=-8$. Find the cartesian equation of L and the angle between L and the $x-z$ plane.

Answer(s):

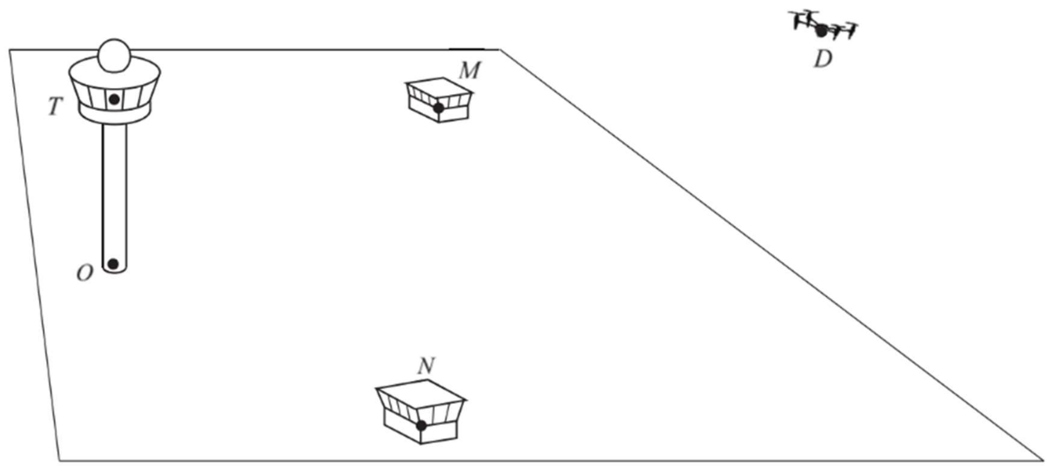
(a) $\left(\frac{38}{21},-\frac{44}{21},\frac{167}{21}\right)$

(b) $(5,3,-4)$

(c) $\frac{x+4}{-1}=\frac{y+2}{1.5}=z$; $46.7°$

Q8 Application in a real-life context

At an airport, an air traffic control room T is located in a vertical air traffic control tower, 70 metres above the ground level. Let $O(0, 0, 0)$ be the foot of the air traffic control tower and all points (x, y, z) are defined relative to O where the units are in kilometres. Two observation posts at the points $M(0.8, 0.6, 0)$ and $N(0.4, -0.9, 0)$ are located within the perimeters of the airport as shown.



(i) State the coordinates of T and find the distances OM and ON.

An air traffic controller on duty at T spots an intruding drone in the vicinity of the airport. The two observation posts at M and N are alerted immediately. A laser rangefinder at M directs a laser beam in the direction $2\mathbf{i}+7\mathbf{j}-\mathbf{k}$ at the drone to determine its position, D. The position D is confirmed using another laser bean from N, which passes through the point $(0.8, 0.75, 0.3)$, directed at the drone.

(ii) Find the coordinates of D.

A Drone Catcher is deployed instantly from O and flies in a straight line directly to D to intercept the intruding drone.

(iii)Find the acute angle between the flight path of the Drone Catcher and the horizontal ground.

At the same time, a Jammer Gun is fired at the intruding drone to disrupt its control signals. The Jammer Gun is located at a point G on the plane p containing the points T, M and N.

(iv) Find the cartesian equation of p.

It is also known that the Jammer Gun is at the foot of the perpendicular from the intruding drone to plane p.

(v) Find the coordinates of G and the distance GD in metres.

Answer(s):

(i) $T(0,0,70)$; $OM = 1$ km , $ON = 0.985$ km

(ii) $D(0.56, -0.24, 0.12)$

(iii) $11.1°$

(iv) $10.5x - 2.8y + 96z = 6.72$

(v) $G(0.547, -0.237, 0.00325)$; $GD = 117$ metres

Topic 12: Complex numbers

Q1

Without the use of a calculator, find the roots of the equation $z^2 = 33 + 56\text{i}$ in cartesian form.

Answer(s):

$z = -7 - 4\text{i}, z = 7 + 4\text{i}$

Q2

Find the complex numbers z and w satisfying the equations $z + 2\text{i}w = 7 - 3\text{i}$ and $2z - (3 + \text{i})w = -1 + \text{i}$.

Answer(s):

$w = \frac{5}{17} - \frac{48}{17}\text{i}, z = \frac{23}{17} - \frac{61}{17}\text{i}$

Q3

Do not use a graphic calculator in answering this question.

The equation $z^3 + az^2 + 4z - 4\mathrm{i} = 0$ has a root $z = \mathrm{i}$. Find the value of a and hence solve the equation.

Answer(s):

$a = -\mathrm{i}; z = \mathrm{i}, z = -2\mathrm{i}, z = 2\mathrm{i}$

Q4

The complex number z has modulus r and argument θ, where $r > 0$ and $0 < \theta < \pi$. Given that the complex number w is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}$ and $(wz)^6$ is real and negative, find the possible values of θ.

Answer(s):

$\theta = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$

Q5

Let z be a complex number such that $\left|\frac{z-\mathrm{i}}{z+2\mathrm{i}}\right|$ and $|z|=\frac{5}{2}$, find the exact value of $|z+3\mathrm{i}|$.

Answer(s):

$|z+3\mathrm{i}|=\frac{7}{2}$

Q6

The complex number z is given by $z=1+\cos\theta+\mathrm{i}\sin\theta$ where $-\pi<\theta<\pi$.

(i) Show that $z=2\cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}$.

(ii) Find the modulus and argument of $\frac{1}{\mathrm{i}z^2}$ in terms of θ.

Answer(s):

(ii) $\left|\frac{1}{\mathrm{i}z^2}\right|=\frac{1}{4}\sec^2\frac{\theta}{2}$, $\arg\left(\frac{1}{\mathrm{i}z^2}\right)=-\frac{\pi}{2}-\theta$

Q7

Illustrate on an Argand diagram points representing $z, \frac{1}{z}, z^2$ and $z^* - z^2$ when $z = 2 + \mathrm{i}$. Find the area of the shape formed by the points representing the origin, $z^* - z^2$, $\frac{1}{z}$, z and z^2.

Answer(s):

4 units ^{2}

Q8

Find the first three positive integer/real values of n such that

(i) $\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^n$ is always real,

(ii) $\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)^n$ is always imaginary.

Answer(s):

(i) $n = 6, 12, 18$

(ii) $n = 1.5, 4.5, 7.5$

Q9

Consider the complex numbers $z=-1+\mathrm{i}$ and $w=\frac{1}{\sqrt{2}}\left[\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)\right]$.

(i) Express z in the form $r\left(\cos\theta+\mathrm{i}\sin\theta\right)$.

(ii) Calculate zw, giving your answer in the form $a+b\mathrm{i}$ where $a,b\in\mathbb{R}$.

(iii) Hence find the value of $\tan\left(\frac{\pi}{12}\right)$ in the form $c+d\sqrt{3}$ where $c,d\in\mathbb{Z}$.

(iv) Find the smallest $p\in\mathbb{Q}^{+}$ such that $\left(zw\right)^{p}\in\mathbb{R}^{+}$.

Answer(s):

(i) $\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right)+\mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right]$

(ii) $\frac{-\sqrt{2}-\sqrt{6}}{4}+\frac{\sqrt{2}-\sqrt{6}}{4}\mathrm{i}$

(iii) $2-\sqrt{3}$

(iv) Smallest $p=\frac{24}{13}$

Bonus topic 1: Permutations & combinations

Q1

(a) A team of 7 people is to be chosen from 5 women and 7 men. Calculate the number of different ways in which this can be done if

(i) there are no restrictions,

(ii) the team is to contain more women than men.

(b) (i) How many different 4-digit numbers, less than 5000, can be form using 4 of the 6 digits 1, 2, 3, 4, 5 and 6 if no digit can be used more than once?

(ii) How many of these 4-digit numbers are divisible by 5?

Answer(s): (a)(i) 792 (ii) 196 (b)(i) 240 (ii) 48

Q2

Four boys and three girls are to be seated in a row. Calculate the number of different ways that this can be done if

(i) the boys and girls sit alternately,

(ii) the boys sit together and the girls sit together,

(iii) a boy sits at each end of the row.

Answer(s): (i) 144 (ii) 288 (iii) 1440

Q3
Four boys and three girls are to be seated in a circle. Find the **probability** that
(i) the girls are seated together,

(ii) the girls are seated separately from one another.

Answer(s): (i) $\frac{1}{5}$ (ii) $\frac{1}{5}$

Q4
Find the number of ways in which the letters of the word CHOCOLATES can be arranged if there are no restrictions.

How many 4-letter words may be formed from the letters of the word CHOCOLATES?

Answer(s): 907200, 2190

Q5

Different coloured pegs, each of which is painted in one and only one of the six colours red, white, black, green, blue and yellow, are to be placed in four holes, with one peg in each hole. Pegs of the same colour are indistinguishable. Calculate how many different arrangements of pegs placed in the four holes so that they are all occupied which can be made from

(a) six pegs, all of different colours,

(b) two red and two white pegs,

(c) two red, one white and one black peg,

(d) two pegs of one colour, one peg of another colour and one peg of a third colour,

(e) twelve pegs, two of each colour.

Answer(s): (a) 360   (b) 6   (c) 12   (d) 720   (e) 1170

Q6

(a) Find the number of different ways that the 13 letters of the word ACCOMMODATION can be arranged in a line if all the vowels (A, I, O) are next to each other.

(b) There are 7 Chinese, 6 European and 4 American students at an international conference. Four of the students are to be chosen to take part in a television broadcast. Find the number of different ways the students can be chosen if at least one Chinese and at least one European student are included.

Answer(s): (a) 604800   (b) 1841

Q7

Find the number of ways in which the letters of the word CONFERENCE can be arranged such that the two C's are separated and the two N's are separated.

Answer(s): 97440

Q8

Three straws are to be drawn at random without replacement from a pack of 10 straws. The straws, indistinguishable apart from the colours, are placed in an opaque box. It is given that four of the straws are purple, two are blue and the remaining straws are pink, orange, yellow and green. Let X denote the number of colours of the straws drawn. Find the probability distribution of X.

Answer(s):

x	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{30}$	$\frac{11}{30}$	$\frac{3}{5}$