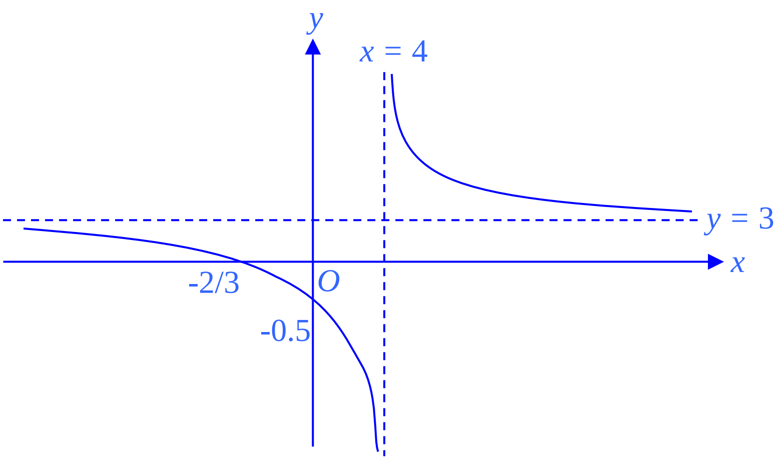
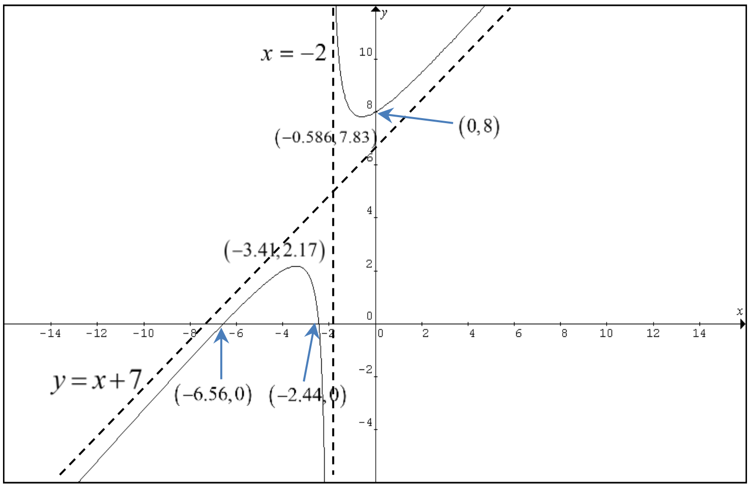
1.1 Graphs and Transformations 1 (Suggested Solutions)

Curve Sketching

1(i)	Asymptotes : $y = 3$, $x = 4$
(ii)	
(iii)	$kx = \dfrac{3x+2}{x-4}$ $kx^2 - 4kx - 3x - 2 = 0$ $kx^2 + (-4k-3)x + (-2) = 0$ ---(*) For $k = 0$, (*) is a linear equatio and not possible to have two roots. For $k \neq 0$, (*) is a quadratic equation and we wand discriminant > 0 $(-4k-3)^2 - 4(k)(-2) > 0$ $16k^2 + 24k + 9 + 8k > 0$ $16k^2 + 32k + 9 > 0$ $k < -1.66$ *or* $k > -0.339$ (3 s.f.) $\therefore$ set of values = $\{k \in \mathbb{R} : k < -1.66 \text{ or } k > -0.339,\ k \neq 0\}$
2(i)	Asymptotes : $y = x + 7$ and $x = -2$
(ii)	$k = \dfrac{x^2 + 9x + 16}{x+2}$ $k(x+2) = x^2 + 9x + 16$ $x^2 + (9-k)x + (16-2k) = 0$ Discriminant < 0 $(9-k)^2 - 4(16-2k) < 0$ $k^2 - 10k + 17 < 0$ $(k-5)^2 - 8 < 0$ $(k - 5 - \sqrt{8})(k - 5 + \sqrt{8}) < 0$ $5 - \sqrt{8} < k < 5 + \sqrt{8}$

(iii)



3(i)

Asymptotes: $y=\dfrac{x}{2}$ and $x=1$

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}-\frac{A}{(x-1)^2}=0$$

$$(x-1)^2=2A$$

Therefore, for C not to have stationary points, $A<0$.

4(i)

$c=1$

When $x=0, y=5, c=1$,

$5=\dfrac{b}{1}\Rightarrow b=5$

$y=\dfrac{2x^2+ax+b}{x+c}=2x+(a-2)+\dfrac{7-a}{x+1}$

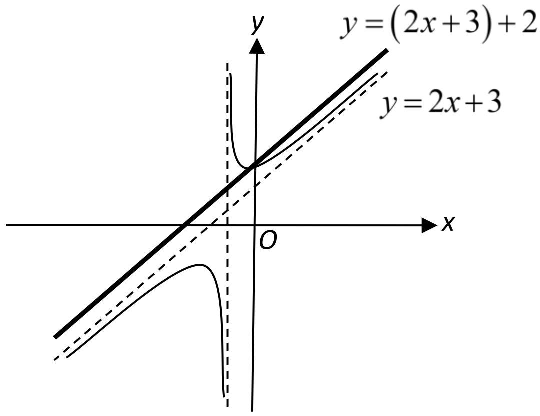
When $x=-1, y=1$,

$y=2x+a-2$

$1=2(-1)+a-2$

$\Rightarrow a=5$

$\therefore a=5,\ b=5$

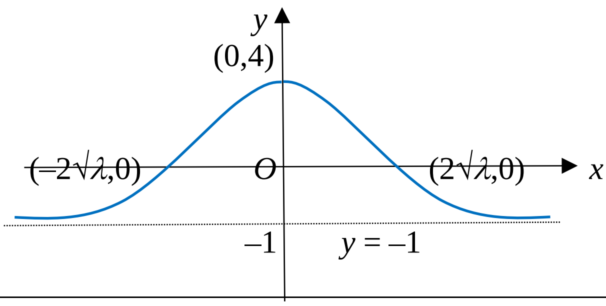


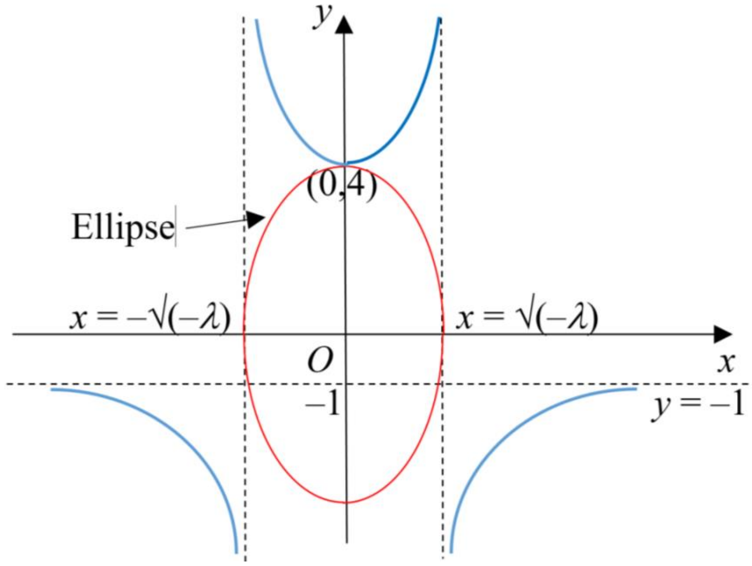
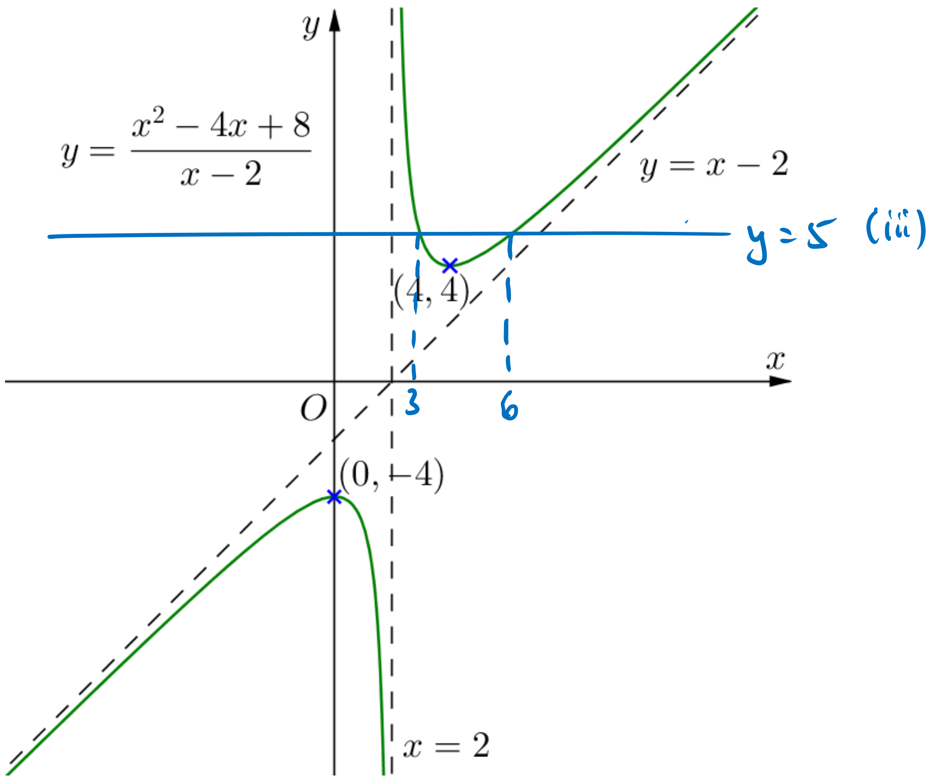
(ii)

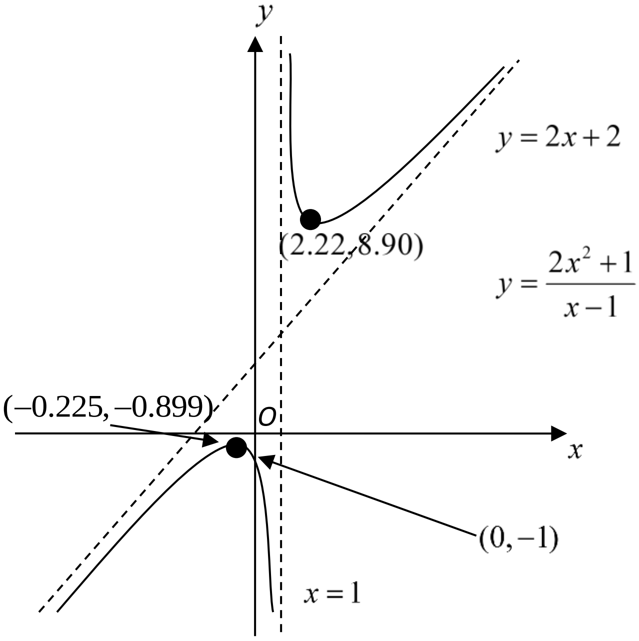
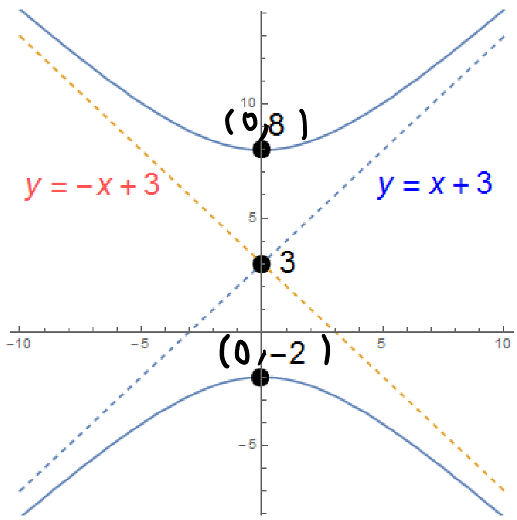
$2x+5=\dfrac{2x^2+ax+b}{x+c}$

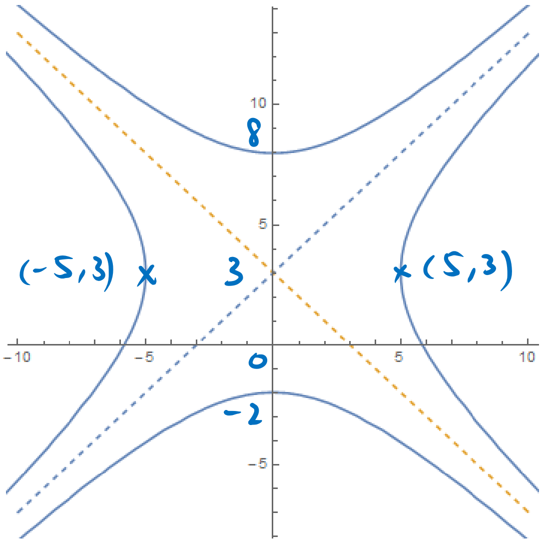
<u>One</u> root.

5(i)



(ii)	
	$\left(\frac{4\lambda - x^2}{hx^2 + h\lambda}\right)^2 = 1 + \frac{x^2}{\lambda} \Rightarrow \frac{x^2}{\left(\sqrt{-\lambda}\right)^2} + \frac{1}{h^2}\left(\frac{4\lambda - x^2}{x^2 + \lambda}\right)^2 = 1$ So insert the graph $\frac{x^2}{\left(\sqrt{-\lambda}\right)^2} + \frac{y^2}{h^2} = 1$ which is an Ellipse with centre O. For only one real root, the ellipse must meet the graph in (ii) at exactly one point. Hence h = 4 and the corresponding root is x = 0.
6(i)	
(ii)	$x > 2$
(iii)	From GC : $\frac{x^2 - 4x + 8}{x - 2} = 5 \Rightarrow x = 3$ or 6

	$\therefore$ the range of values of x for which $\dfrac{x^2-4x+8}{x-2} \geq 5$ is $x \geq 6$ or $2 < x \leq 3$.
(iv)	$m > 1$
7	 For no real solutions to the equation $2x^2+1 = a(x-1)\cos(bx+c)$, $x \neq 1$ range of values for a is $-0.899 < a < 0.899$
8(i)	Note that $x^2 - y^2 + 6y + 16 = 0$ $x^2 - \left((y-3)^2 - 9\right) + 16 = 0$ $(y-3)^2 - x^2 = 25$ $\dfrac{(y-3)^2}{5^2} - \dfrac{(x-0)^2}{5^2} = 1$ The equations of the asymptotes are $y - 3 = \pm\frac{5}{5}x \Rightarrow y = x+3$ or $y = -x+3$. 

(ii)	From the graph, it follows that $m<-1$ or $m>1$.
(iii)	The other hyperbola with the same asymptotes are $$x^2-(y-3)^2=25$$ $$\frac{(x-0)^2}{5^2}-\frac{(y-3)^2}{5^2}=1$$ Therefore, $p=0,\ q=3$.
(iv)	
(v)	The circle $x^2+(y-3)^2=r^2$ will intersect C_2 either 0, 2 or 4 times. So, $n=0,2,4$.
9(i)	Since $x=1$ is a vertical asymptote, we have $b=-1$. Also, $y=x-2$ is an oblique asymptote, so $y=x-2+\frac{k}{x-1}$, where k is a constant. <u>Method 1:</u> $y=\frac{x^2+ax-4}{x-1}$ $=\frac{x(x-1)+x+ax-4}{x-1}$ $=x+\frac{(a+1)(x-1)+a+1-4}{x-1}$ $=x+(a+1)+\frac{a-3}{x-1}$ Comparing the form we have earlier, we have $a+1=-2\Rightarrow a=-3$ <u>Method 2:</u> $y=x-2+\frac{k}{x-1}$ $=\frac{x^2-3x+2+k}{x-1}$ Comparing coefficient of x in given equation, $a=-3$. **Intercepts:** Let $x=0\Rightarrow y=4$

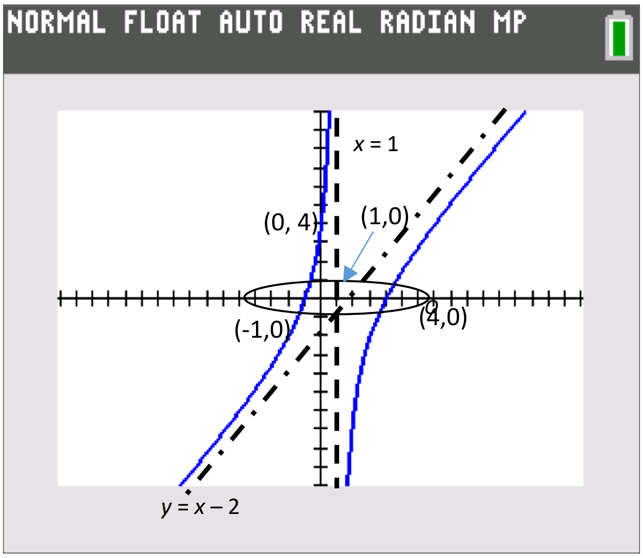
Let $y=0 \Rightarrow \dfrac{x^2-3x-4}{x-1}=0$

$\Rightarrow x^2-3x-4=0$

$\Rightarrow (x-4)(x+1)=0$

$\Rightarrow x=4$ or $x=-1$

(*Note to students*: you can just use GC for this.)



(ii)

<u>Method 1</u>

$$x^2-2x-20+21\left(\frac{x^2-3x-4}{x-1}\right)^2=0$$

$$(x-1)^2-1-20+21\left(\frac{x^2-3x-4}{x-1}\right)^2=0$$

$$(x-1)^2+21\left(\frac{x^2-3x-4}{x-1}\right)^2=21$$

$$\frac{(x-1)^2}{21}+\left(\frac{x^2-3x-4}{x-1}\right)^2=1 \Rightarrow \frac{(x-1)^2}{21}+(y)^2=1$$

The graph to add on is an ellipse centred at (1, 0) with horizontal axis length $\sqrt{21}$ and vertical axis length 1.

From graph, the ellipse intersects curve C at 4 distinct points, therefore it has 4 real distinct roots.

<u>Method 2</u>

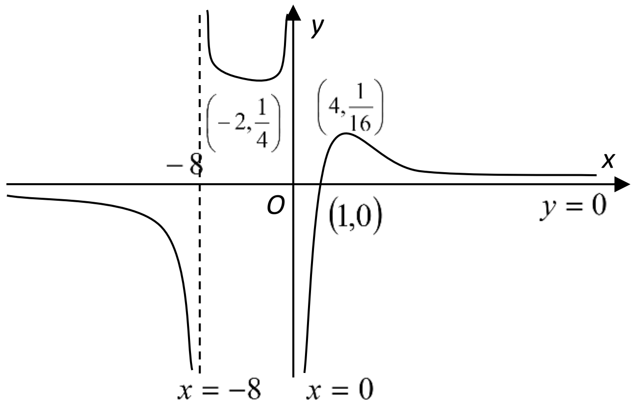
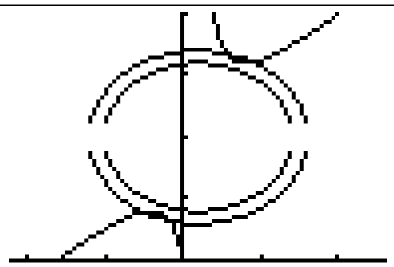
$$x^2-2x-20+21\left(\frac{x^2-3x-4}{x-1}\right)^2=0$$

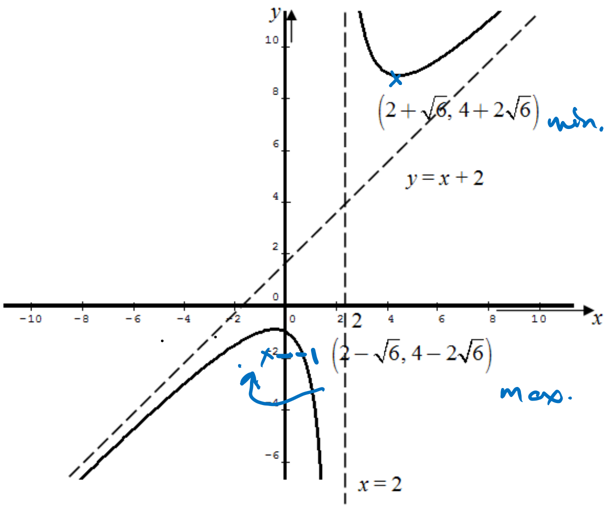
$$x^2-2x-20+21(y)^2=0$$

$$21y^2=-x^2+2x+20 \Rightarrow y=\pm\sqrt{\frac{-x^2+2x+20}{21}}$$

The graph to add on is $y=\pm\sqrt{\dfrac{-x^2+2x+20}{21}}$ (*Students can use the GC for this*)

From the graph, $y=\pm\sqrt{\dfrac{-x^2+2x+20}{21}}$ intersects curve *C* at 4 distinct points, therefore it has 4 real distinct roots.

10 (a)(i)	$y=\dfrac{x^2+8x}{x+k}$. Since vertical asymptote is $x = 1$, $\therefore\ k = -1$ $\therefore\ y=\dfrac{x^2+8x}{x+k}=\dfrac{x^2+8x}{x-1}=x+9+\dfrac{9}{x-1}$ $\Rightarrow$ Oblique Asymptote is $y = x + 9$
(a)(ii)	
(b)	 By guess and check, $r = 7$
11(i)	$y=\dfrac{x^2+a}{x-a}=x+a+\dfrac{a^2+a}{x-a}$ Given the oblique asymptote of C is $y = x + 2$, therefore $a = 2$.
(ii)	$y=\dfrac{x^2+a}{x-a}=x+a+\dfrac{a^2+a}{x-a}$ $\dfrac{dy}{dx}=\dfrac{2x(x-a)-(x^2+a)}{(x-a)^2}=\dfrac{x^2-2ax-a}{(x-a)^2}$ OR $\dfrac{dy}{dx}=1-\dfrac{a^2+a}{(x-a)^2}$ For turning points, let $\dfrac{dy}{dx}=0$ $\dfrac{x^2-2ax-a}{(x-a)^2}=0$ $x^2-2ax-a=0$ .... 1 If curve C has two turning points, discriminant >0 $(-2a)^2-4(1)(-a)>0$ $4a^2+4a>0$ $a(a+1)>0$ $a<-1$ or $a>0$ $\therefore$ set of values = $\{a\in\mathbb{R}: a<-1 \text{ or } a>0\}$

(iii)	(ii) When $a=2$, $y=\dfrac{x^2+2}{x-2}$, $x\neq 2$ $y(x-2)=x^2+2$ $x^2-yx+2y+2=0$ If curve C cannot exists for $y_1<y<y_2$, discriminant <0 $(-y)^2-4(1)(2y+2)<0$ $y^2-8y-8<0$ Let $y^2-8y-8=0$ $y=\dfrac{8\pm\sqrt{(-8)^2-4(1)(-8)}}{2(1)}=\dfrac{8\pm\sqrt{96}}{2}$ $=\dfrac{8\pm 4\sqrt{6}}{2}=4\pm 2\sqrt{6}$ Hence C cannot exists for $4-2\sqrt{6}<y<4+2\sqrt{6}$ where $y_1=4-2\sqrt{6}$ and $y_2=4+2\sqrt{6}$
(iv)	$y=\dfrac{x^2+2}{x-2}=x+2+\dfrac{6}{x-2}$ 
12(i)	C has a vertical asymptote at $x=0$. $\therefore r=0$ (ans) $y=\dfrac{(px+q)^2}{x}=\dfrac{p^2x^2+2pqx+q^2}{x}=p^2x+2pq+\dfrac{q^2}{x}$ As $x\to\infty$, $\dfrac{q^2}{x}\to 0$. $y\to p^2x+2pq$. Oblique asymptote: $y=p^2x+2pq$ Comparing coefficient of x: $p^2=9\Rightarrow p=3$ or -3 (rejected $\because$ p is non-negative constant) constant: $2pq=\lambda\Rightarrow q=\dfrac{\lambda}{2p}=\dfrac{\lambda}{6}$ (shown)
(ii)	$y=9x+\lambda+\dfrac{\lambda^2}{36x}\Rightarrow\dfrac{dy}{dx}=9-\dfrac{\lambda^2}{36x^2}$

To find stationary point(s): $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$9 - \frac{\lambda^2}{36x^2} = 0 \Rightarrow 324x^2 = \lambda^2$

$\therefore x = \pm\frac{\lambda}{18}$

For $x = \frac{\lambda}{18}$, $y = 9\left(\frac{\lambda}{18}\right) + \lambda + \frac{\lambda^2}{36\left(\frac{\lambda}{18}\right)} = 2\lambda$.

For $x = \frac{\lambda}{18}$, $y = 9\left(-\frac{\lambda}{18}\right) + \lambda + \frac{\lambda^2}{36\left(-\frac{\lambda}{18}\right)} = 0$.

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\lambda^2}{18x^3}$

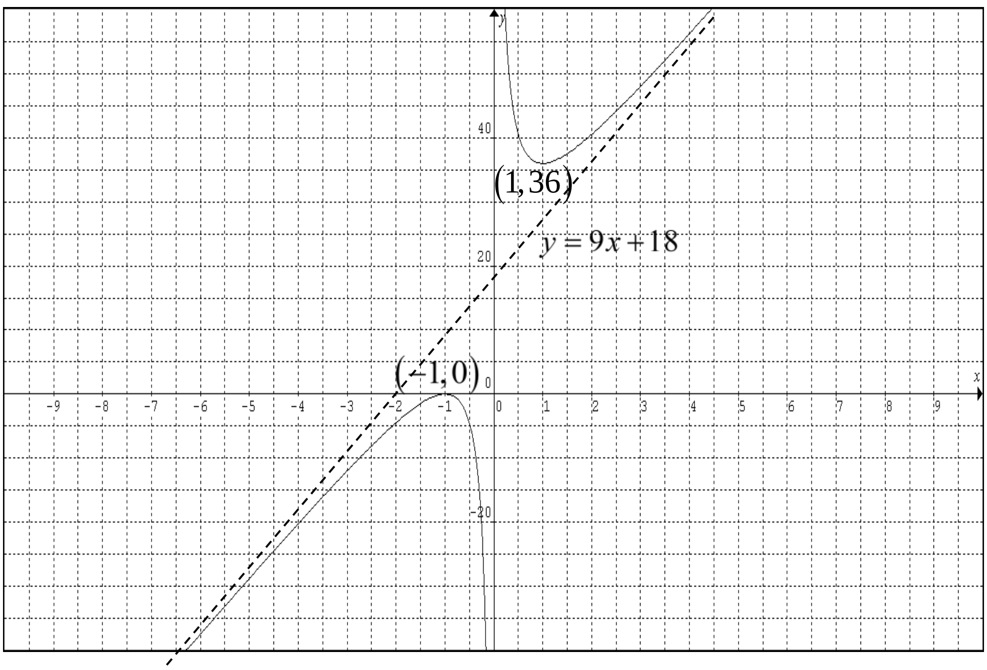
For $x = \frac{\lambda}{18}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\lambda^2}{18\left(\frac{\lambda}{18}\right)^3} = \frac{18^2}{\lambda}(>0)$.

$\therefore \left(\frac{\lambda}{18}, 2\lambda\right)$ is minimum point.

For $x = -\frac{\lambda}{18}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\lambda^2}{18\left(\frac{\lambda}{18}\right)^3} = -\frac{18^2}{\lambda}(<0)$.

$\therefore \left(-\frac{\lambda}{18}, 0\right)$ is maximum point.

(iii) For $\lambda = 18$, vertical asymptote at $x = 0$,
Oblique asymptote: $y = 9x + 18$,
Stationary points at $(-1,\ 0)$ and $(1,\ 36)$.
No y-intercept. x-intercept at $(-1,\ 0)$.



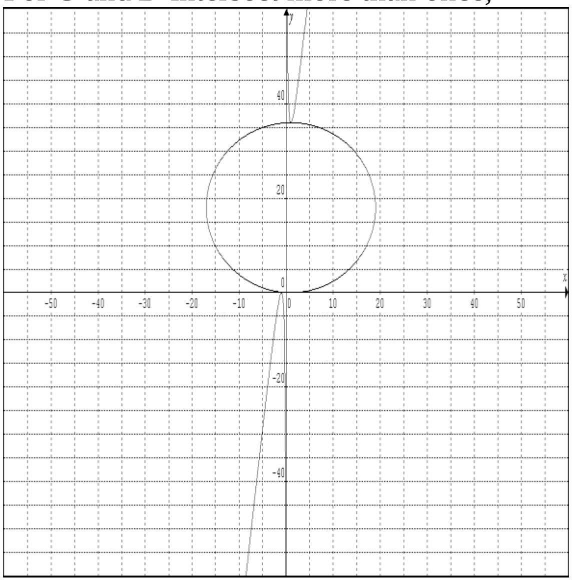
$x = a\sin\theta + 1 \Rightarrow \sin\theta = \dfrac{x-1}{a}$, $\qquad y = a\cos\theta + 18 \Rightarrow \cos\theta = \dfrac{y-18}{a}$

Using trigonometric identity,

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \left(\frac{x-1}{a}\right)^2 + \left(\frac{y-18}{a}\right)^2 = 1$$

$$\Rightarrow (x-1)^2 + (y-18)^2 = a^2 \text{ (ans)}$$

For C and D intersect more than once,



Note that D is a circle with centre at (1, 18) and radius a unit.

$\therefore$ by guess and check,
least integer value of $a = 19$

13(i)

For C_2, when $x = 0$,

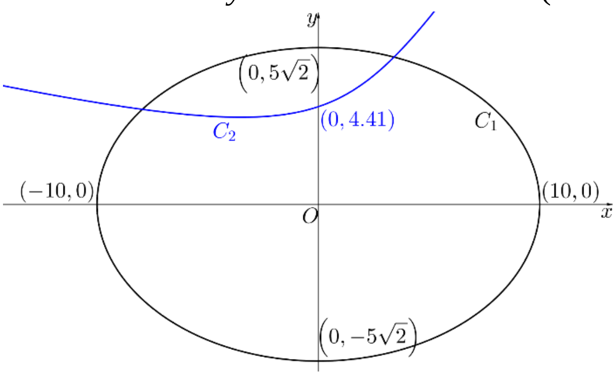
$$2e^{-t} - 4e^{2t} = 0$$

$$4e^{2t} = 2e^{-t}$$

$$e^{3t} = \frac{1}{2}$$

$$t = \frac{1}{3}\ln\frac{1}{2} = -\frac{1}{3}\ln 2$$

Therefore, $y = 3e^{\frac{1}{3}\ln 2} + e^{-\frac{2}{3}\ln 2} = 4.41$ (to 3 s.f.)

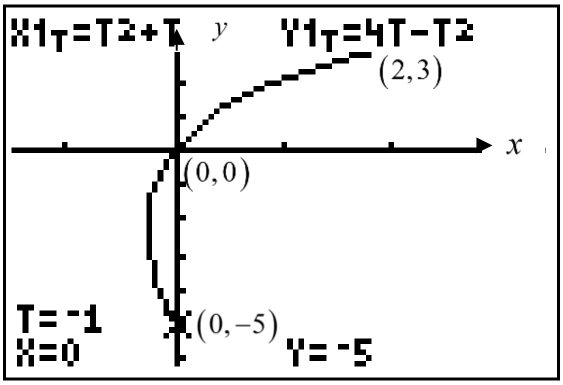


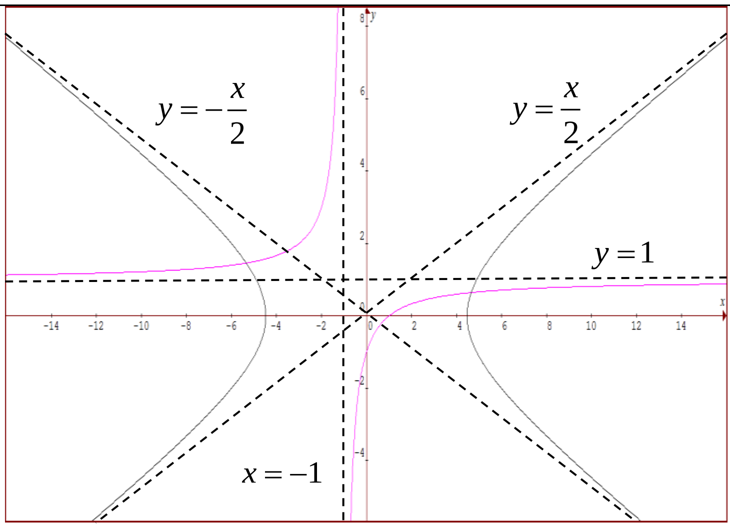
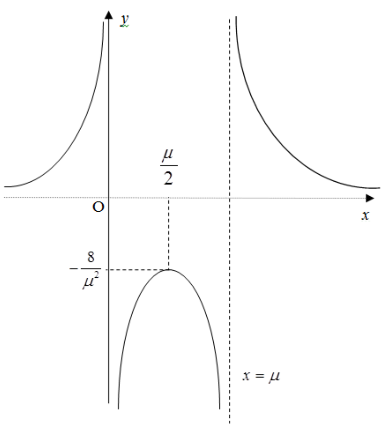
(ii)

$x = 2e^{-t} - 4e^{2t}$ (1)

$y = 3e^{-t} + e^{2t}$ (2)

$2\times(2) - 3\times(1): 2y - 3x = 14e^{2t}$ (3)

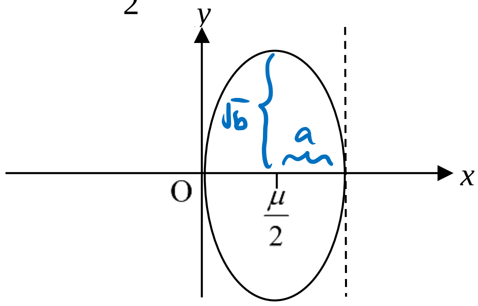
	$4\times(2)+1\times(1): 4y+x=14e^{-t}$ $e^{t}=\dfrac{14}{x+4y} \quad (4)$ Substituting (4) into (3), $2y-3x=14\left(\dfrac{14}{x+4y}\right)^2$ $(x+4y)^2(2y-3x)=2744$
14(a) (i)	
(ii)	When $x=\dfrac{5}{16}$, $\dfrac{5}{16}=t^2+t \Rightarrow t^2+t-\dfrac{5}{16}=0$ $\Rightarrow t=-\dfrac{5}{4}(NA \because -1\le t\le 1)$, or $t=\dfrac{1}{4}$ $\dfrac{dx}{dt}=2t+1,\ \dfrac{dy}{dt}=4-2t$ $\dfrac{dy}{dx}=\dfrac{\left[\dfrac{dy}{dt}\right]}{\left[\dfrac{dx}{dt}\right]}=\dfrac{4-2t}{2t+1}$ When $t=\dfrac{1}{4}$, $\dfrac{dy}{dx}=\dfrac{4-2\left(\dfrac{1}{4}\right)}{2\left(\dfrac{1}{4}\right)+1}=\dfrac{7}{3}$
(iii)	$x+y=5t \Rightarrow t=\dfrac{x+y}{5}$ Substitute $t=\dfrac{x+y}{5}$ into $x=t^2+t$ $\Rightarrow x=\left(\dfrac{x+y}{5}\right)^2+\left(\dfrac{x+y}{5}\right)$ Alternate form: $y=\pm5\sqrt{x+\dfrac{1}{4}}-\dfrac{5}{2}-x$; $x=10\pm5\sqrt{4-y}-y$; $(x+y)^2=5(4x-y)$

(b)	 Axial intercepts (should be shown on graph): $(1,0);(0,-1)$ for $y=\frac{x-1}{x+1}$. $(\sqrt{20},0);(-\sqrt{20},0)$ for $\frac{x^2}{20}-\frac{y^2}{5}=1$.
	Substitute $y=\frac{x-1}{x+1}$ into $\frac{x^2}{20}-\frac{y^2}{5}=1$. $\frac{x^2}{20}-\frac{\left(\frac{x-1}{x+1}\right)^2}{5}=1$ $\Rightarrow x^2-4\left(\frac{x-1}{x+1}\right)^2=20$ y Number of solutions = 2
15	$y=\frac{2}{x(x-\mu)}$ Equations of asymptotes: $y=0, \quad x=0, \quad x=\mu$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2(2x-\mu)}{(x^2-\mu x)^2}=0$ $x=\frac{\mu}{2},\ y=-\frac{8}{\mu^2}$ is a stationary point 

$b(x-\frac{\mu}{2})^2+a^2y^2=a^2b$

$\frac{(x-\frac{\mu}{2})^2}{a^2}+\frac{y^2}{b}=1$. Since $a=\frac{\mu}{2}$,

Ellipse centre: $(\frac{\mu}{2},0)$ x radius a, and y radius $\sqrt{b}$.



The two graphs will intersect twice if $\sqrt{b}>\frac{8}{\mu^2}\Rightarrow b>\frac{64}{\mu^4}$.

16

(i) $y=\frac{x^2+kx+1}{x-2}$

$$\frac{dy}{dx}=\frac{(2x+k)(x-2)-(x^2+kx+1)}{(x-2)^2}=\frac{x^2-4x-(2k+1)}{(x-2)^2}$$

$$\frac{dy}{dx}=0 \quad\Rightarrow\quad x^2-4x-(2k+1)=0$$

2 stationary points $\Rightarrow$ Discriminant > 0

$\Rightarrow 16+4(2k+1)>0$

$\Rightarrow k>-\frac{5}{2}$

<u>Alternatively</u>:

$$y=\frac{x^2+kx+1}{x-2}=x+(k+2)+\frac{2k+5}{x-2}$$

$$\frac{dy}{dx}=1-\frac{(2k+5)}{(x-2)^2}$$

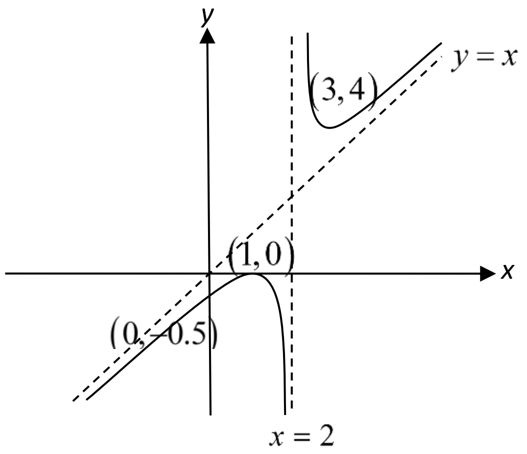
$$\frac{dy}{dx}=0 \quad\Rightarrow\quad (x-2)^2=2k+5$$

2 stationary points: $2k+5>0 \Rightarrow \quad k>-\frac{5}{2}$

(ii) $y=x+(k+2)+\frac{2k+5}{x-2}$

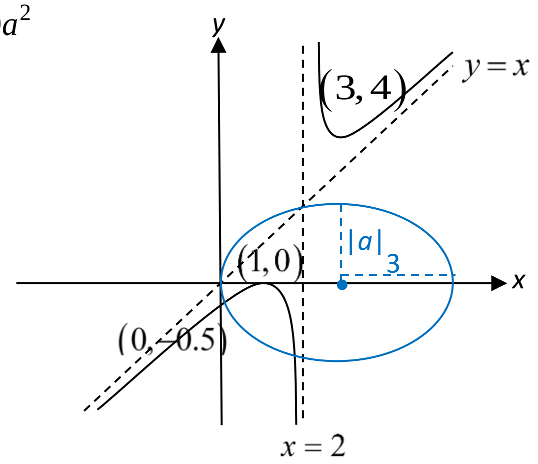
$k+2=0 \quad\Rightarrow\quad k=-2$

(iii) When $k=-2$, $y=\dfrac{(x-1)^2}{x-2}$



(iv) $a^2(x-3)^2+9y^2=9a^2$

$$\frac{(x-3)^2}{3^2}+\frac{y^2}{a^2}=1$$



(v) Substitute $y=\dfrac{(x-1)^2}{x-2}$ in $9y^2=9a^2-a^2(x-3)^2$:

$$9\frac{(x-1)^4}{(x-2)^2}=9a^2-a^2(x-3)^2$$

$$9(x-1)^4=a^2(x-2)^2\left(9-(x-3)^2\right)$$

From graph, the ellipse cuts the graph of C at exactly 3 points when $|a|=4$.

So $a=4$ or $a=-4$.