RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2019 Year 6

Chapter 13C : Matrix Spaces and Linear Transformation

SYLLABUS INCLUDES

- Basis and dimension of column space, row space, range space and null space.
- Rank of a square matrix and relation between rank, dimension of null space and order of a matrix
- Linear transformations and matrices from $\mathbb{R}^n \to \mathbb{R}^m$

PRE-REQUISITES

- Vectors, Matrices and Functions

CONTENT

1 Null Space of A Matrix

1.1 Null Space

not empty

Let **A** be an $m \times n$ matrix. Then the set of all solutions of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ is called the **null space** of **A**, i.e.

$$\text{Null space of an } m \times n \text{ matrix } \mathbf{A} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{0}\}.$$

For example,

if $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, then the null space of A

$$= \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{Ax} = \mathbf{0}\}$$

$$= \left\{\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} u+2v \\ 2u+4v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 : u+2v=0\right\}$$

$$= \left\{\begin{pmatrix} -2v \\ v \end{pmatrix} : v \in \mathbb{R}\right\} = \left\{v\begin{pmatrix} -2 \\ 1 \end{pmatrix} : v \in \mathbb{R}\right\}$$

Theorem 1.1

The null space of an $m \times n$ matrix **A** forms a subspace of $\mathbb{R}^n$.

Proof :

Let U denote the null space of **A**. Then U is a subset of $\mathbb{R}^n$ since **A** is an $m \times n$ matrix.

Since $\mathbf{A0} = \mathbf{0}$ where $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, so $\mathbf{0} \in U$. Hence U is non-empty.

Let $\mathbf{x} \in U$ and $\mathbf{y} \in U$.

Then $A\underset{\sim}{x} = 0$, $A\underset{\sim}{y} = 0$

$A(\underset{\sim}{x}+\underset{\sim}{y}) = A\underset{\sim}{x} + A\underset{\sim}{y} = \underset{\sim}{0}+\underset{\sim}{0} = \underset{\sim}{0}$

Thus $\underset{\sim}{x}+\underset{\sim}{y} \in U$

Let $\underset{\sim}{x} \in U$. Then $A\underset{\sim}{x} = 0$

$A(\alpha \underset{\sim}{x}) = \alpha(A\underset{\sim}{x}) = \alpha 0 = 0$ where $\alpha \in \mathbb{R}$, thus $\alpha\underset{\sim}{x} \in U$

Hence U forms a subspace of $\mathbb{R}^n$.

1.2 Basis of Null Space of a Matrix

To find the basis of the null space a matrix **A**, we first row-reduce the augmented matrix **[A|0]** (or just **A**) to **reduced row-echelon form.**

Then, solve for the unknown x_i's where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \cdot \\ x_n \end{pmatrix} \in \mathbb{R}^n$.

For example, if the reduced row-echelon form of **A** is $\begin{pmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then the solution to $\mathbf{Ax} = \mathbf{0}$ is

[Note that **[A|0]** in reduced row echelon form is $\left(\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.]

So the null space of **A**, U, consists of the set of vectors

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_3 - x_4 \\ 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \text{ where } \alpha = x_3,\ \beta = x_4 \in \mathbb{R}.$$

Hence $\left\{\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}\right\}$ spans U.

$\left\{\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}\right\}$ is linearly independent as $\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$.

Thus $\left\{\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}\right\}$ is a basis for the null space of matrix **A**.

Alternatively, since the null space is the set of solutions of $\mathbf{Ax} = \mathbf{0}$, we can use Plysmlt2 GC app to help us to find the answer.

Example 1 (9225/June1983/01/Q14, part of question)

Given matrix **A** where $\mathbf{A} = \begin{pmatrix} 3 & 0 & -2 & 3 \\ -4 & 2 & 0 & -8 \\ 1 & -2 & 2 & 5 \\ 3 & -3 & 2 & 9 \end{pmatrix}$, find a basis for the subspace V defined by $V = \{\mathbf{x} \in \mathbb{R}^4 : \mathbf{A}\mathbf{x} = \mathbf{0}\}$.

we want, $A\underset{\sim}{x} = A\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

Find the general solution of the equation $\mathbf{A}\mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}$ and state, justifying your answer, whether the set of all the solutions forms a subspace of $\mathbb{R}^4$.

$A\underset{\sim}{x} - A\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \underset{\sim}{0}$

Solution :

$A\left(\underset{\sim}{x} - \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right) = \underset{\sim}{0}$

Method 1 (Using RREF)

$A\underset{\sim}{x} = \underset{\sim}{0}$

$\left(\begin{array}{cccc|c} \cdot & \cdot & \cdot & \cdot & 3 \\ & & & & -4 \\ & & & & 1 \\ & & & & 3 \end{array}\right)$

Using GC, the reduced row-echelon form of **A** is $\begin{pmatrix} 1 & 0 & -\frac{2}{3} & 1 \\ 0 & 1 & -\frac{4}{3} & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in V$. Then $\begin{pmatrix} 1 & 0 & -\frac{2}{3} & 1 \\ 0 & 1 & -\frac{4}{3} & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \left[\underset{\sim}{x} - \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right] \in K$

$\underset{\sim}{x} \neq \begin{pmatrix}1\\0\\0\\0\end{pmatrix} =$

$x_1 - \frac{2}{3}x_3 + x_4 = 0 \Rightarrow x_1 = \frac{2}{3}x_3 - x_4$

$x_2 - \frac{4}{3}x_3 - 2x_4 = 0 \Rightarrow x_2 = \frac{4}{3}x_3 + 2x_4$

Thus $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}x_3 - x_4 \\ \frac{4}{3}x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = \overset{\alpha}{\frac{1}{3}x_3}\begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + \overset{\beta}{x_4}\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, x_3, x_4 \in \mathbb{R}$

So a basis for V is

$x_3\begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$x_1 - \frac{2}{3}x_3 + x_4 = 1$

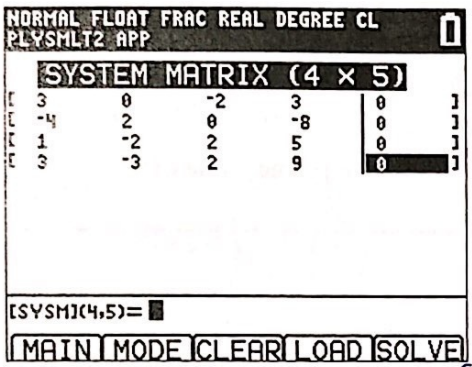
$x_2 - \frac{4}{3}x_3 - 2x_4 = 0$

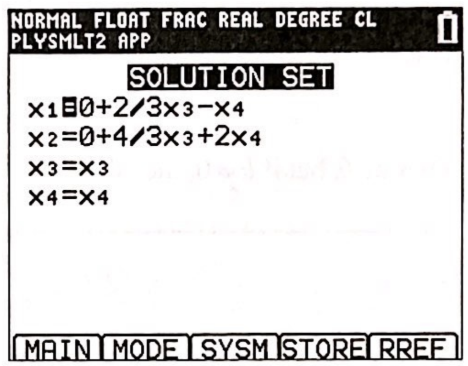
Method 2 (Using Plysmt2 App)

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in V$. Then, $A\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \begin{pmatrix} 3 & 0 & -2 & 3 \\ -4 & 2 & 0 & -8 \\ 1 & -2 & 2 & 5 \\ 3 & -3 & 2 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Using GC, we get $x_1 = \frac{2}{3}x_3 - x_4$, $x_2 = \frac{4}{3}x_3 + 2x_4$

Thus $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}x_3 - x_4 \\ \frac{4}{3}x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = \frac{1}{3}x_3\begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$,





So a ~~basis~~ basis for V is $\left\{ \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a solution of $A\underset{\sim}{x} = \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}$

since

?

Then $A\underset{\sim}{x} = \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow A\underset{\sim}{x} - A\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow A\left(x - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

So, $\underset{\sim}{x} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in V$

Hence $\underset{\sim}{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \alpha, \beta \in \mathbb{R}$

The GS is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$, $\alpha, \beta \in \mathbb{R}$

Let W be the set of all the solns to $A\underset{\sim}{x} = \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}$

Since $A\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in W$

But, $3\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \notin W$ since $A\begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}$

W is not closed under scalar multiplication

W doesnt form a subspace of $\mathbb{R}^4$.

Example 2 (9225/June1989/01/Q9)

The matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} a & 1 & 1 & 2 \\ a & 1 & 2 & 3 \\ 2a & b+2 & 1 & 3 \\ 3a & 2b+3 & 0 & 3 \end{pmatrix}, \ (a, b \in \mathbb{R}, a \neq 0).$$

The null space of $\mathbf{A}$ is denoted by K.

(i) Show that when $b \neq 0$, the dimension of K is 1, and that when $b = 0$, the dimension of K is 2.

(ii) For the case $b \neq 0$, find, in terms of a only, a basis vector $\mathbf{e}_1$ of K.

(iii) For the case $b = 0$, find a vector $\mathbf{e}_2$ such that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for K.

(iv) Hence show that if $b = 0$, then the vector $\begin{pmatrix} a+\theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix}$, where $\theta \in \mathbb{R}$, belongs to K for all values of θ, but if $b \neq 0$, then this vector belongs to K for only one value of θ.

Solution :

$$\begin{pmatrix} a & 1 & 1 & 2 \\ a & 1 & 2 & 3 \\ 2a & b+2 & 1 & 3 \\ 3a & 2b+3 & 0 & 3 \end{pmatrix} \xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 - R_1} \begin{pmatrix} a & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & b & -1 & -1 \\ 0 & 2b & -3 & -3 \end{pmatrix} \xrightarrow[R_4 \to R_4 + 3R_2]{R_3 \to R_3 + R_2} \begin{pmatrix} a & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & b & 0 & 0 \\ 0 & 2b & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{R_4 \to R_4 - 2R_3} \begin{pmatrix} a & 1 & 1 & 2 \\ 0 & b & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{pmatrix} a & 1 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A row-echelon form of $\begin{pmatrix} a & 1 & 1 & 2 \\ a & 1 & 2 & 3 \\ 2a & b+2 & 1 & 3 \\ 3a & 2b+3 & 0 & 3 \end{pmatrix}$ is $\begin{pmatrix} a & 1 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(i) If $b \neq 0$, the reduced row-echelon form of $\mathbf{A}$ can be further reduced to $\begin{pmatrix} 1 & 0 & 0 & \frac{1}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Then, consider $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in K$, so

i.e. $x = -\frac{1}{a}w$, $y = 0$, $z = -w$, $w \in \mathbb{R}$

So, $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{1}{a}w \\ 0 \\ -w \\ w \end{pmatrix} = -w \begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix}$

A basis for K is $\left\{ \begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

So $\dim(K) = 1$

If $b=0$, the rref of A is $\begin{pmatrix} 1 & \frac{1}{a} & 0 & \frac{1}{a} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

~~Then~~

If $b = 0$, then the reduced row-echelon form of **A** is

i.e. $x = -\frac{1}{a}y - \frac{1}{a}w$, $z = -w$, $y, w \in \mathbb{R}$.

So $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{1}{a}y - \frac{1}{a}w \\ y \\ -w \\ w \end{pmatrix} = -y\begin{pmatrix} \frac{1}{a} \\ -1 \\ 0 \\ 0 \end{pmatrix} - w\begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix}$

A basis for K is $\left\{ \begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{a} \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$. So $\dim(K) = 2$.

Note: there's a shorter way to do (i) at the end of notes

(ii) From (i), $e_1 = \begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix}$

(iii) From (i), $e_2 = \begin{pmatrix} \frac{1}{a} \\ -1 \\ 0 \\ 0 \end{pmatrix}$

(iv) If $b=0$, for all values of θ,

$$\begin{pmatrix} a+\theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ a^2 \\ -a^2 \end{pmatrix} + \begin{pmatrix} \theta \\ -\theta a \\ 0 \\ 0 \end{pmatrix} = a^2\begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix} + \theta a\begin{pmatrix} \frac{1}{a} \\ -1 \\ 0 \\ 0 \end{pmatrix} = a^2\underset{\sim}{e_1} + \theta a\underset{\sim}{e_2} \in K$$

If $b \neq 0$, $\begin{pmatrix} a+\theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix} = \alpha\begin{pmatrix} \frac{1}{a} \\ 0 \\ 1 \\ -1 \end{pmatrix}$ for some $\alpha \in \mathbb{R}$

From the 2nd component, $-\theta\alpha = 0 \Rightarrow \theta = 0$

Then, $\alpha = \alpha^2$. So, for $b \neq 0$, $\begin{pmatrix} a+\theta \\ -\theta a \\ a^2 \\ -a^2 \end{pmatrix}$ belongs to K when $\theta = 0$.

2 Row Space and Column Space of a Matrix

2.1 Row Space

Let $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ be an $m\times n$ matrix.

Note that we could consider the rows of the matrix **A** as vectors in $\mathbb{R}^n$.
The **row space** of **A** is the linear **subspace** of $\mathbb{R}^n$ spanned by the rows of the matrix **A**.
The **row rank** of **A** is defined as the **dimension of the row space of A.**

For example,

Row Space of $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} = \left\{\alpha\begin{pmatrix}1\\2\end{pmatrix}+\beta\begin{pmatrix}4\\8\end{pmatrix} : \alpha,\beta\in\mathbb{R}\right\} = \left\{(\alpha+4\beta)\begin{pmatrix}1\\2\end{pmatrix} : \alpha,\beta\in\mathbb{R}\right\}$

$= \left\{\gamma\begin{pmatrix}1\\2\end{pmatrix} : \gamma\in\mathbb{R}\right\}$

which is a straight line passing thru $\begin{pmatrix}0\\0\end{pmatrix}$ and parallel $\begin{pmatrix}1\\2\end{pmatrix}$ in $\mathbb{R}^2$

Row rank of $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ is 1.

Another example,

Row Space of $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \left\{\alpha\begin{pmatrix}1\\2\end{pmatrix}+\beta\begin{pmatrix}1\\0\end{pmatrix} : \alpha,\beta\in\mathbb{R}\right\}$

Row rank of $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ is 2 since $\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\}$ is a linearly independent set

Thus, in fact, row Space of $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \mathbb{R}^2$

Remarks

(a) The **non-zero rows of a matrix in row echelon form are linearly independent.**
(b) The **non-zero rows** of a matrix **A** in **row-echelon form** form a **basis for the row space** of **A**.
(c) A matrix **A** in row-echelon form has row rank equal to the number of non-zero rows.
(d) Row operations do not change the row-space of a matrix. Hence row space of **A** = row space of row-echelon form of **A**.
(e) Any two matrices which are **row equivalent have the same row rank.** (2 matrices are said to be **row equivalent** if one matrix can be obtained from the other matrix by repeatedly applying elementary row operations.)

To find a basis for the row space of a matrix **A**
(a) Reduce matrix **A** to **row-echelon form.**
(b) The **non-zero rows of A in row-echelon form** form a basis for the row space of **A**.

2.2 Column Space

Let $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ be an $m \times n$ matrix.

Note that the columns of the matrix **A** could be regarded as vectors in $\mathbb{R}^m$.
The **column space of A** is the vector **subspace** of $\mathbb{R}^m$ **spanned by the columns of the matrix.**
The **column rank** of **A** is defined as the **dimension of the column space of A.**

For example,

Column Space of $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \end{pmatrix} ; \alpha, \beta \in \mathbb{R} \right\}$

Column rank of $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ is 2 since $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$ is a linearly indepdt set

Thus, in fact, column space of $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \mathbb{R}^2$

Theorem 2.1

For any matrix **A**, the dimension of the row space of **A** is equal to the dimension of the column space of **A**.
Thus, **rank (A) = row rank (A) = column rank (A).**

* A need not be a square matrix

Two Important Facts

(a) Row operations will change the column space of a matrix **A**.
Hence the column space of **A** and the column space of the row-echelon form of **A** are not the same.

(b) Row operations do not change the linear relationship between the columns of **A**. For example, if columns 1, 2 and 4 of a matrix **A** are linearly independent, then columns 1, 2, and 4 of the row-echelon form of **A** are also linearly independent.

<u>Theorem</u>
Let A and B be row equivalent matrices
Then the following statements are true.
(a) A given set of column vectors of A is l.i.
if and only if the set of corresponding column vectors of B is l.i.
(b) A given set of column vectors of A forms a basis for the column space of A.
If and only if the set of corresponding column vectors of B ~~for~~ forms a basis for the column space of B.

Finding a Basis for the Column Space of A

(a) Reduce matrix **A** until it is in row-echelon form.

(b) Let **B** be the row-echelon form of **A**. Look for the columns of **B** which contain a leading "1". Then **the same columns of A form a basis for the column space of A.**
Eg if columns 1 & 4 of **B** contain a leading "1", then columns 1 & 4 of **A** form a basis for the column space of **A**.

Caution : The columns of B containing leading "1" do not form a basis for the column space of A. In general , the column spaces of A & B are different (See Important Facts).

Important Note :
Although the row rank of **A** equals the column rank of **A**, a basis for the row space of **A** is **not** a basis for the column space of **A**.

Example 3

Given matrix **A** where

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & -2 & 3 \\ -4 & 2 & 0 & -8 \\ 1 & -2 & 2 & 5 \\ 3 & -3 & 2 & 9 \end{pmatrix}, \text{ find in any order}$$

(i) a basis for the row space of **A**,
(ii) a basis for the column space of **A**,
(iii) rank **A**.

Solution :

Using GC, the row echelon matrix of **A** is $= \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 2 \\ 0 & 1 & -\frac{4}{3} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(i) A basis for the row space of A is $\left\{ \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\frac{4}{3} \\ 2 \end{pmatrix} \right\}$

Another basis for the row space of A is $\left\{ \begin{pmatrix} 3 \\ 0 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 0 \\ -8 \end{pmatrix} \right\}$, since

rank(A) = 2 and $\left\{ \begin{pmatrix} 3 \\ 0 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 0 \\ -8 \end{pmatrix} \right\}$ is a linearly independent set.

(ii) A basis for the column space of A is $\left\{ \begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -2 \\ -3 \end{pmatrix} \right\}$

Question: Can the following be a basis for the column space of **A**?

(a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ (b) $\left\{ \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\frac{4}{3} \\ 2 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} -2 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -8 \\ 5 \\ 9 \end{pmatrix} \right\}$

Ans: (a) Since $\begin{pmatrix} 3 \\ -4 \\ 1 \\ 3 \end{pmatrix}$ cant be expressed in terms of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, so $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ cannot be a basis for the column space of A.

(b) A basis for the row space of A is not a basis for the column space of A

(c) The choice in this case corresponds to the non leading 1's columns. It's NOT a guarantee that it's a basis for the column space of A. It just happen that the rank is 2 and the these 2 vectors are linearly independent, it wld still be a basis. In this case will need to justify so.

(iii) Rank A $= 2$.

Example 4

elements in the vector space is a linear combination of the elements inside basis.

Find a basis for the vector space spanned by the vectors $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} \right\}$.

Solution :

Form the matrix **A** whose columns are $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}$.

Then

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 & 7 \\ 2 & -1 & -3 & 0 \\ -1 & 2 & 3 & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 2 & 7 \\ 0 & -7 & -7 & -14 \\ 0 & 5 & 5 & 10 \end{pmatrix} \xrightarrow[R_2 \to -\frac{1}{7}R_2]{R_3 \to R_3 + \frac{5}{7}R_2} \begin{pmatrix} 1 & 3 & 2 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence a basis for the given vector space is the same as finding a basis for the column space of A

Hence a basis for the given vector span is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right\}$

3 Linear Transformation

3.1 Representation of a Matrix as a Linear Transformation

Let $\mathrm{T} : V \to W$ be a map from a vector space V to another vector space W.
T is called a **linear map or linear transformation** if the following properties are satisfied:

(i) $\mathrm{T}(\mathbf{v} + \mathbf{u}) = \mathrm{T}(\mathbf{v}) + \mathrm{T}(\mathbf{u})$

(ii) $\mathrm{T}(\alpha\mathbf{v}) = \alpha\mathrm{T}(\mathbf{v})$

for all $\mathbf{u}, \mathbf{v} \in V$ and $\alpha \in \mathbb{R}$.

For example, $\mathrm{T}:\mathbb{R}^2 \to \mathbb{R}$ by $\mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix} = 2x + y$.

Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$.

(i) $\mathrm{T}\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \mathrm{T}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$=$

$=$

$=$

(ii) $\mathrm{T}\left(\alpha\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = \mathrm{T}\begin{pmatrix} \alpha x_1 \\ \alpha y_1 \end{pmatrix} = \quad = \quad =$

Thus, T is a linear transformation

The mapping V defined by $\mathrm{V} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-3 \\ y+2 \end{pmatrix}$ is **not** a linear transformation

since $\mathrm{V}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \mathrm{V}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} =$, but

$$\mathrm{V}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{V}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \neq \mathrm{V}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Theorem 3.1

Let **A** be a fixed $m \times n$ real matrix. Define the map $\mathrm{T}_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$ by

$$\mathrm{T}_{\mathbf{A}}(\mathbf{v}) = \mathbf{Av} \text{ for all } \mathbf{v} \in \mathbb{R}^n.$$

Then $\mathrm{T}_{\mathbf{A}}$ is a linear transformation.

Proof :

From the basic properties of matrices,

(i) $\mathrm{T}_{\mathbf{A}}(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av} = \mathrm{T}_{\mathbf{A}}(\mathbf{u}) + \mathrm{T}_{\mathbf{A}}(\mathbf{v})$

(ii) $\mathrm{T}_{\mathbf{A}}(\alpha\mathbf{v}) = \mathbf{A}(\alpha\mathbf{v}) = \alpha\mathbf{Av} = \alpha\, \mathrm{T}_{\mathbf{A}}(\mathbf{v})$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$.

Consider the linear transformation T such that $T:\mathbb{R}^2 \to \mathbb{R}^2$. Let the standard basis in $\mathbb{R}^2$ be $\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$. Thus, any vector $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$.

Observe the effect of the transformation T on **x**:

$$\mathrm{T}(\mathbf{x}) = \mathrm{T}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = \mathrm{T}(x_1\mathbf{e}_1) + \mathrm{T}(x_2\mathbf{e}_2) = x_1\mathrm{T}(\mathbf{e}_1) + x_2\mathrm{T}(\mathbf{e}_2)$$

Let $\mathrm{T}(\mathbf{e}_1) = \begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}$ and $\mathrm{T}(\mathbf{e}_2) = \begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}$. Then,

$$\mathrm{T}(\mathbf{x}) = x_1\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix} + x_2\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}\mathbf{x}$$

Hence, the matrix representing the above transformation is $\mathbf{A} = \begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}$, which is obtained by observing the effect of T on the elements of the standard basis (i.e. $\mathrm{T}(\mathbf{e}_1)$ and $\mathrm{T}(\mathbf{e}_2)$).

In fact,

if T is a linear transformation such that T: $\mathbb{R}^n \to \mathbb{R}^m$, then T can be represented by an $m\times n$ matrix

$$\mathbf{A} = \begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\a_{21} & a_{22} & \cdots & a_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}, \text{ where}$$

$$\mathrm{T}(\mathbf{e}_1) = \begin{pmatrix}a_{11}\\a_{21}\\ \vdots\\ a_{m1}\end{pmatrix}, \mathrm{T}(\mathbf{e}_2) = \begin{pmatrix}a_{12}\\a_{22}\\ \vdots\\ a_{m2}\end{pmatrix}, \ldots, \mathrm{T}(\mathbf{e}_n) = \begin{pmatrix}a_{1n}\\a_{2n}\\ \vdots\\ a_{mn}\end{pmatrix}, \text{ and } \mathbf{e}_1 = \begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix}0\\1\\ \vdots\\ 0\end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix}0\\0\\ \vdots\\ 1\end{pmatrix}.$$

For example, T: $\mathbb{R}^2 \to \mathbb{R}^3$ by $\mathrm{T}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+2y\\y-3x\\x\end{pmatrix}$.

Then the linear transformation T can be represented by the 3×2 matrix $\begin{pmatrix}1 & 2\\-3 & 1\\1 & 0\end{pmatrix}$ since

$$\mathrm{T}\begin{pmatrix}1\\0\end{pmatrix} = \qquad \text{and } \mathrm{T}\begin{pmatrix}0\\1\end{pmatrix} = \qquad .$$

Alternatively, it can be seen that

$$\mathrm{T}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+2y\\y-3x\\x\end{pmatrix} = \qquad + \qquad = \qquad .$$

3.2 Null Space and Range

Let the linear transformation T : $\mathbb{R}^n \to \mathbb{R}^m$ be represented by the $m\times n$ matrix $\mathbf{A}$,i.e $\mathrm{T}(\mathbf{v}) = \mathbf{Av}$ where $\mathbf{v} \in \mathbb{R}^n$. Then,

(a) the **null space or kernel** of the linear transformation T is defined as the set of all vectors in $\mathbb{R}^n$ which are mapped to the zero vector of $\mathbb{R}^m$. Symbolically,
$\ker \mathrm{T} = \{\mathbf{v} \in \mathbb{R}^n \mid \mathrm{T}(\mathbf{v}) = \mathbf{0}\} = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{Av} = \mathbf{0}\}$.
The **nullity** of T is the dimension of the null space of T.

(b) the **range** of T is defined as the set of vectors which are the images of those vectors in $\mathbb{R}^n$. Symbolically,
range $\mathrm{T} = \{\mathrm{T}(\mathbf{v}) \mid \mathbf{v} \in \mathbb{R}^n\} = \{\mathbf{Av} \mid \mathbf{v} \in \mathbb{R}^n\}$.
The **rank** of T is the dimension of the range of T.
Note : Range $\mathrm{T} \subseteq \mathbb{R}^m$.

For example, if the linear transformation $\mathrm{T}:\mathbb{R}^2 \to \mathbb{R}^3$ be represented by the 3×2 matrix $\begin{pmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{pmatrix}$,

then range of T = $\left\{ x\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} : x, y \in \mathbb{R} \right\}$ is a plane in $\mathbb{R}^3$, parallel to vectors $\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

and passing through the origin $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Let $\begin{pmatrix} 2x+y \\ -3x+2y \\ 3y-x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then

$y = -2x$ ...... (1)

$2y = 3x$ .... (2)

$3y = x$ .... (3)

The solution of these 3 equations give $x = 0$ and $y = 0$. So, only $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is mapped to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ under T.

Thus, Ker T = $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

Theorem 3.2

Let $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, $\ker \mathrm{T}$ is a subspace of $\mathbb{R}^n$ and range T is a subspace of $\mathbb{R}^m$.

Proof :

Note

$\ker \mathrm{T}$ is the same as the solution space to the homogeneous linear system $\mathbf{Av} = \mathbf{0}$. Thus, finding a basis for $\ker \mathrm{T}$ is the same as finding a basis for the null space of $\mathbf{A}$ which is the same as Section 1.

Spanning Set for Range of a Linear Transformation

Let the linear transformation $\mathrm{T}: \mathbb{R}^n \to \mathbb{R}^m$ be represented by the $m\times n$ matrix

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \text{ i.e } \mathrm{T}(\mathbf{v}) = \mathbf{Av} \text{ where } \mathbf{v}=\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n.$$

Then $\mathrm{T}(\mathbf{v}) = \mathbf{Av}$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \ldots + a_{mn}v_n \end{pmatrix}$$

$$= v_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + v_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + v_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Let $T\underset{\sim}{x}$ be represented by $\underset{\sim}{A}\underset{\sim}{x}$ where A is $m\times n$ matrix.

$\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{y}$ → range.

↓ basis

As $\mathbf{v}$ runs through all the possible elements of $\mathbb{R}^n$,

$$\text{range of T} = \text{span}\left\{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \begin{pmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}\right\}$$

= basis of column space.

$= \text{span }\{\text{column vectors of } \mathbf{A}\}$
$= \text{column space of } \mathbf{A}$

Thus,

(i)	**dim(range T) = column rank (A) = rank (A).**
(ii)	to find a basis for the range of T, simply find a basis for the column space of **A** as described in Subsection 2.2.

no of column

It can be proven that for any linear transformation $\mathrm{T}: \mathbb{R}^n \to \mathbb{R}^m$,

dim(ker T) + dim(range T) = n,

i.e., **nullity + rank = n** **(Rank-Nullity Theorem)**

no. of row

Similarly, if **A** is an $m\times n$ matrix, then

dim (null space of A) + rank A = number of columns of A.

Example 5 (9225/1989/96/Q9)

The linear transformations $T_1 : \mathbb{R}^4 \to \mathbb{R}^4$ and $T_2 : \mathbb{R}^4 \to \mathbb{R}^4$ are represented by the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively, where $\mathbf{M}_1 = \begin{pmatrix} 1 & -2 & 3 & 5 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{M}_2 = \begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

The range space of T_1 and T_2 are denoted by R_1 and R_2 respectively.

(i) Write down the dimensions of R_1 and R_2.

(ii) Write down a basis for R_1 and a basis for R_2.

(iii) Determine whether $R_1 \cup R_2$ is a vector space, and justify your conclusion.

(iv) The null space of T_1 and T_2 are denoted by K_1 and K_2 respectively. Find a basis for K_1 and a basis for K_2.

(v) It is given that $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Find a vector $\mathbf{x}_1$ and a vector $\mathbf{x}_2$ such that $\mathbf{M}_1\mathbf{x}_1 = \mathbf{M}_1\mathbf{a}$ and $\mathbf{M}_2\mathbf{x}_2 = \mathbf{M}_2\mathbf{a}$, and such that $\mathbf{x}_1 - \mathbf{x}_2$ is of the form $\begin{pmatrix} 0 \\ p \\ q \\ 0 \end{pmatrix}$, where p and q are non-zero integers.

Solution :

(i) $\dim(R_1) = 3$, $\dim(R_2) = 3$

(ii) Basis for R_1 is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix} \right\}$ and basis for R_2 is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(iii) $R_1 \cup R_2$

$\cancel{\alpha\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \mu\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0}$

$\cancel{R_1 \cup R_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$

$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -11\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 4\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix} \in R_1 \subseteq R_1 \cup R_2$

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in R_2 \subseteq R_1 \cup R_2$

Then consider $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \notin R_1$, $\notin R_2$ $\therefore \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \notin R_1 \cup R_2$

Thus, $R_1 \cup R_2$ not closed under vector addition.

$\therefore$ Not a vector space.

(iv)

$$\mathbf{M}_1\mathbf{x}=\begin{pmatrix}1&-2&3&5\\0&1&4&9\\0&0&1&2\\0&0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad\Rightarrow\quad\begin{aligned}x-2y-3y+5w&=0\\y+4z+9w&=0\\z+2w&=0\end{aligned}$$

Using GC, we get $x=-w, y=-w, z=-2w$

So, $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}-w\\-w\\-2w\\w\end{pmatrix}=-w\begin{pmatrix}1\\1\\2\\-1\end{pmatrix}$. A basis for K_1 is $\left\{\begin{pmatrix}1\\1\\2\\-1\end{pmatrix}\right\}$.

$$\mathbf{M}_2\mathbf{x}=\begin{pmatrix}1&-2&0&-3\\0&1&0&2\\0&0&0&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad\Rightarrow\quad\begin{aligned}x-2y-3w&=0\\y+2w&=0\\z+w&=0\end{aligned}$$

Using GC, we get $x=-w, y=-w, z=-2w$

So, $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}-w\\-2w\\-w\\w\end{pmatrix}=-w\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}$. A basis for K_2 is $\left\{\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}\right\}$.

(v) $\underset{\sim}{M}_1\underset{\sim}{x}_1=\underset{\sim}{M}_1\underset{\sim}{a} \Rightarrow \underset{\sim}{M}_1(\underset{\sim}{x}_1-\underset{\sim}{a})=\underset{\sim}{0} \Rightarrow \underset{\sim}{x}_1-\underset{\sim}{a}\in K_1$

$\underset{\sim}{M}_2\underset{\sim}{x}_2=\underset{\sim}{M}_2\underset{\sim}{a} \Rightarrow \underset{\sim}{M}_2(\underset{\sim}{x}_2-\underset{\sim}{a})=\underset{\sim}{0} \Rightarrow \underset{\sim}{x}_2-\underset{\sim}{a}\in K_2$

So $\underset{\sim}{x}_1-\underset{\sim}{a}=\alpha\begin{pmatrix}1\\1\\2\\-1\end{pmatrix}$ and $\underset{\sim}{x}_2-\underset{\sim}{a}=\beta\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}$ for some $\alpha,\beta\in\mathbb{R}$

Furthermore $\underset{\sim}{x}_1-\underset{\sim}{x}_2=\alpha\begin{pmatrix}1\\1\\2\\-1\end{pmatrix}-\beta\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}=\begin{pmatrix}\alpha-\beta\\\alpha-2\beta\\2\alpha-\beta\\-\alpha+\beta\end{pmatrix}=\begin{pmatrix}0\\p\\q\\0\end{pmatrix}$

$\alpha=\beta$

$\alpha-2\beta=-\alpha=p$

$2\alpha-\beta=\alpha=q$

Let $p=-1$, $q=1$

$\underset{\sim}{x}_1=\begin{pmatrix}1\\1\\2\\-1\end{pmatrix}+\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\2\\3\\0\end{pmatrix}$, $\underset{\sim}{x}_2=\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}+\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\3\\2\\0\end{pmatrix}$

Example 6 (9225/1995/01/Q9)

The linear transformations $T: \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix **A**

$$\begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & -4 & -5 \\ 3 & a^2+5 & 2a-7 & 3a-9 \\ 6 & a^2+12 & 2a-14 & 3a-18 \end{pmatrix}$$

Show that, provided $a \neq -1$ and $a \neq 2$, the dimension of the range space of T is 3.

In the case where $a = 2$,

(i) show that the dimension of K, the null space of T, is 2,

(ii) show that there is a basis of K, which is to be found, of the form $\left\{ \begin{pmatrix} p \\ q \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} r \\ s \\ 0 \\ 1 \end{pmatrix} \right\}$, where p, q, r and s are integers.

(iii) find a solution of $\mathbf{Ax} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}$ of the form $\begin{pmatrix} u \\ 0 \\ v \\ w \end{pmatrix}$, where u, v and w are non-zero integers.

Solution :

After doing row operations, $\begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & -4 & -5 \\ 3 & a^2+5 & 2a-7 & 3a-9 \\ 6 & a^2+12 & 2a-14 & 3a-18 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -2a^2+2a+4 & -3a^2+3a+6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Since $a \neq -1$ and $a \neq 2$, $-2a^2+2a+4 \neq 0$. So the ref of the matrix is

$\begin{pmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and it has 3 non zero rows. Thus the dimension of the range space of T is 3.

(i) Since $a = 2$, $\begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & -4 & -5 \\ 3 & a^2+5 & 2a-7 & 3a-9 \\ 6 & a^2+12 & 2a-14 & 3a-18 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The rank of T is 2. So the dim of the null space of T is $4-2=2$

(ii) For the null space of T,

$$\begin{pmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 - 7x_3 - 10x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{matrix}$$

Let $x_3 = 1$ and $x_4 = 0$, $x_1 = 7$ and $x_2 = -2$

Let $x_1 = 0$ and $x_4 = 1$, $x_1 = 10$ and $x_2 = -3$

So basis for K is $\left\{ \begin{pmatrix} 7 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

(iii) Notice that $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}$ is the first column vector of $\underset{\sim}{A}$.

So clearly $A\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}$

The GS of $\underset{\sim}{A}\underset{\sim}{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}$ would be $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} 7 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 10 \\ -3 \\ 0 \\ 1 \end{pmatrix}$, $\alpha, \alpha \in \mathbb{R}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} 7 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 10 \\ -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 7\alpha + 10\beta \\ -2\alpha - 3\beta \\ \alpha \\ \beta \end{pmatrix}$$

Since the second component of the required soln is 0,

$-2\alpha - 3\beta = 0$

Choose $\alpha = 6$, $\beta = -4$, soln would be $\begin{pmatrix} 3 \\ 0 \\ 6 \\ -4 \end{pmatrix}$.

SUMMARY

eg

Find a basis for the row space and for the column space of $A = \begin{pmatrix} 3 & 0 & 3 & 3 & 3 \\ -3 & -1 & -2 & -4 & -1 \\ 5 & 4 & 9 & 1 & 13 \\ 7 & 6 & 13 & 1 & 19 \end{pmatrix}$

soln

The rref of A is $\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$