1

(a) Length of semi-major axis, $a = 13$. Length of semi-minor axis, $b = 5$.

Distance from centre to focus, . $c = \sqrt{a^2 - b^2} = \sqrt{13^2 - 5^2} = 12$.

Centre of ellipse: (12, 0)

Since the principal axis is along the *x*-axis, one of the foci has to be (0, 0), the origin.

(b) $e = \frac{c}{a} = \frac{12}{13}$. Then the polar equation of the ellipse is $r = \dfrac{k}{1 - \frac{12}{13}\cos\theta}$ for some *k*.

When $\theta = 0$, $r = \frac{13k}{13 - 12(1)} = 13k$. When $\theta = \pi$, $r = \frac{13k}{13 - 12(-1)} = \frac{13}{25}k$.

So $13k + \frac{13}{25}k = 2a = 26 \quad \Rightarrow \quad k = \frac{25}{13}$.

Hence, the polar equation of the ellipse is $r = \frac{25}{13 - 12\cos\theta}$.

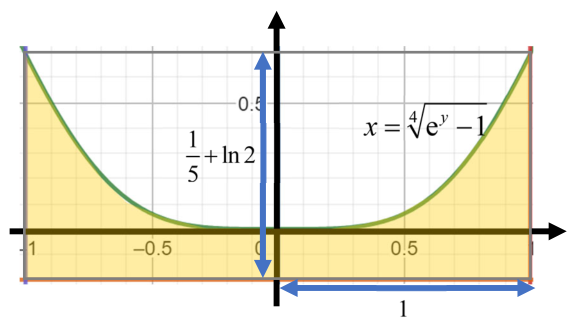
2

$y = \ln(x^4 + 1) \Rightarrow x^2 = \sqrt{e^y - 1}$

$$\text{Volume of bowl} = \pi(1^2)\left(\frac{1}{5} + \ln 2\right) - \pi\int_0^{\ln 2} x^2 \, dy$$

$$= \pi\left(\frac{1}{5} + \ln 2\right) - \pi\int_0^{\ln 2} \sqrt{e^y - 1} \, dy$$

Let $u^2 = e^y - 1 \Rightarrow 2u\frac{du}{dy} = e^y$.



When $y = 0,\ u = 0.$ When $y = \ln 2,\ u = 1.$

Then $\int_0^{\ln 2} \sqrt{e^y - 1} \, dy = \int_0^1 u \cdot \frac{2u}{e^y} \, du$

$$= \int_0^1 \frac{2u^2}{u^2 + 1} \, du$$

$$= 2\int_0^1 \left(1 - \frac{1}{u^2 + 1}\right) du$$

$$= 2\left[u - \tan^{-1} u\right]_0^1$$

$$= 2\left(1 - \frac{\pi}{4} - 0\right) = 2 - \frac{\pi}{2}$$

$$\text{Volume of bowl} = \pi\left(\frac{1}{5} + \ln 2\right) - \pi\left(2 - \frac{\pi}{2}\right)$$

$$= \pi\left(\frac{\pi}{2} + \ln 2 - \frac{9}{5}\right) \text{ units}^3$$

3

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(1+\cos\theta)\cos\theta-\sin\theta\sin\theta}{-(1+\cos\theta)\sin\theta-\sin\theta\cos\theta}=\dfrac{\cos\theta+\cos^2\theta-\sin^2\theta}{-\sin\theta-2\sin\theta\cos\theta}$

When $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-1$, $\cos\theta+\cos 2\theta=\sin\theta+\sin 2\theta$ (double-angle formulas).

(b) Using factor formula,

$$2\cos\frac{3\theta}{2}\cos\frac{\theta}{2}=2\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$$

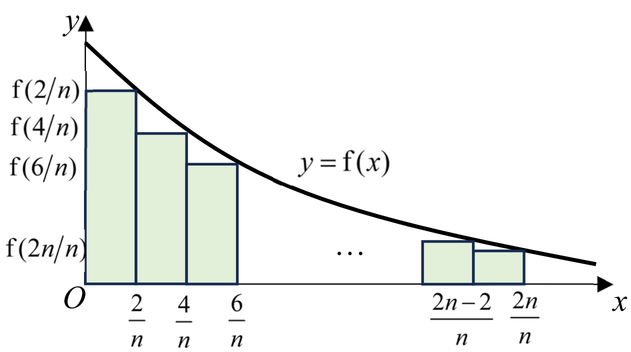
$$\left(\cos\frac{3\theta}{2}-\sin\frac{3\theta}{2}\right)\cos\frac{\theta}{2}=0$$

$\Rightarrow \cos\dfrac{3\theta}{2}=\sin\dfrac{3\theta}{2}$ or $\cos\dfrac{\theta}{2}=0$ (no solutions for $0<\dfrac{\theta}{2}<\dfrac{\pi}{2}$)

$$\tan\frac{3\theta}{2}=1\Rightarrow\frac{3\theta}{2}=\frac{\pi}{4}\text{ or }\frac{5\pi}{4}$$

$$\theta=\frac{\pi}{6}\text{ or }\frac{5\pi}{6}$$

4



$$U_n=\sum_{r=1}^{n}\underbrace{\frac{2}{n}}_{\text{width of rectangle}}\cdot\underbrace{\mathrm{f}\left(\frac{2r}{n}\right)}_{\text{height of }r^{\text{th}}\text{ rectangle}}$$

(a)

= Total area of the n rectangles

Since f is a strictly decreasing function, the total area will be an underestimate of the actual area under the curve from $x=0$ to 2. So $U_n<I$.

Also, the total area of the n rectangles approaches the actual area, which is denoted by the definite integral $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$. Therefore, for the limiting case, $U_\infty=I$.

(b) V_n is an over-estimate of I. Therefore, $V_n>I$.

(c) Let $\mathrm{f}(x)=\dfrac{1}{2(x+1)}$.

$$U_n = \frac{2}{n}\left[\frac{1}{2\left(\frac{2}{n}+1\right)}+\frac{1}{2\left(\frac{4}{n}+1\right)}+\frac{1}{2\left(\frac{6}{n}+1\right)}+\ldots+\frac{1}{2\left(\frac{2n}{n}+1\right)}\right]$$

$$=\frac{1}{n}\left[\frac{1}{1+\frac{2}{n}}+\frac{1}{1+\frac{4}{n}}+\frac{1}{1+\frac{6}{n}}+\ldots+\frac{1}{1+\frac{2n}{n}}\right]$$

$$=\frac{1}{n+2}+\frac{1}{n+4}+\frac{1}{n+6}+\ldots+\frac{1}{n+2n}$$

Therefore,

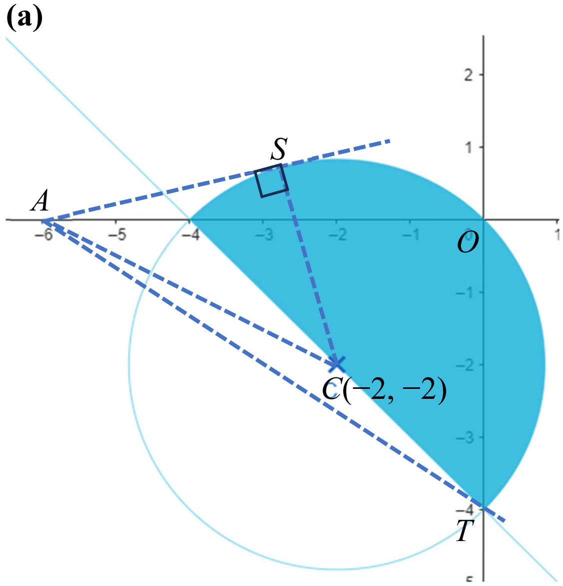
$$\lim_{n\to\infty}\left(\frac{1}{n+2}+\frac{1}{n+4}+\frac{1}{n+6}+\ldots+\frac{1}{3n}\right)=U_\infty$$

$$=I$$

$$=\int_0^2 \frac{1}{2(x+1)}\,\mathrm{d}x$$

$$=\frac{1}{2}\ln 3$$

5 **(a)**



(b) $|\mathrm{i}z+k|=|\mathrm{i}||z-k\mathrm{i}|=|z-k\mathrm{i}|$

$$\min|\mathrm{i}z+k|=\sqrt{(2+k)^2+2^2}-2\sqrt{2}$$

$$=\sqrt{4+4k+k^2+4}-2\sqrt{2}$$

$$=\sqrt{k^2+4k+8}-2\sqrt{2}$$

(c) $\angle OAT = \tan^{-1}\frac{4}{6} = \tan^{-1}\frac{2}{3} \approx 0.588$ rad

$\angle OAC = \tan^{-1}\frac{2}{4} = \tan^{-1}\frac{1}{2}$

$\angle CAS = \sin^{-1}\frac{CS}{AC} = \sin^{-1}\frac{2\sqrt{2}}{\sqrt{4^2+2^2}} = \sin^{-1}\frac{2}{\sqrt{10}}$

Therefore, $\angle OAS = \angle CAS - \angle OAC = \sin^{-1}\frac{2}{\sqrt{10}} - \tan^{-1}\frac{1}{2} \approx 0.221$

So $-0.588 \le \arg(z+6) \le 0.221$ (3 s.f.).

(d) Multiplying w with $e^{i\theta}$ rotates the point W representing w about the origin anticlockwise by an angle of θ. If the point upon rotation does not lie on the locus of z, we require $|w|$ to be greater than $\max(|z-0|) = 4$. Hence $|w| > 4$.

6

(a)(i) ${}^{17}P_{11} = 4.940103168 \times 10^{11}$

(ii) ${}^{12}P_6 \times 5! = 79833600$

(the last row is filled by the 5 particular people in $5!$ ways; the 12 seats in the front 3 rows can be filled by the remaining 6 people in ${}^{12}P_6$ ways)

(iii) ${}^{3}C_2 \times {}^{7}C_1 = 21$

(the 2 married couples can be chosen from the 3 in ${}^{3}C_2$ ways; the 1 other person can be chosen from the remaining 7 people in ${}^{7}C_1$ ways)

(b)(i) Treat the married couple and family of 4 as one unit each; together with the 2 women, there are 4 units to arrange in a circle. For each circular arrangement, the married couple and family of 4 can each be further arranged in $2!$ and $4!$ ways respectively.

$(4-1)! \times 2! \times 4! = 288$

(ii) First, find the number of circular arrangements with the married couple seated together. Next, find the number of circular arrangements with the married couple seated together and the 2 women seated directly opposite each other. Subtract the second from the first.

$(7-1)! \times 2! - 1 \times 4 \times 2! \times 4! = 1440 - 192 = 1248$

7

(a) $\ln(y+1) = 1 - \sin x$

$\frac{1}{y+1}\frac{dy}{dx} = -\cos x \Rightarrow \frac{dy}{dx} = -(y+1)\cos x$

$$\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)\cos x - (y+1)(-\sin x) = -\cos x\frac{dy}{dx} + (y+1)\sin x$$

When $x = 0$, $y+1 = e^{1-\sin 0} = e \Rightarrow y = e - 1$

$$\frac{dy}{dx} = -e$$

$$\frac{d^2y}{dx^2} = -(-e) - e.0 = e$$

Maclaurin's series for y is $e - 1 + (-e)x + \frac{1}{2}ex^2 = e - 1 - ex + \frac{1}{2}ex^2$

(b) Given x is small such that x^3 and higher powers of x can be neglected,

$$\ln(y+1) = 1 - \sin x \approx 1 - x$$

$$y + 1 \approx e^{1-x} = e.e^{-x} = e\left(1 + (-x) + \frac{(-x)^2}{2} + \ldots\right)$$

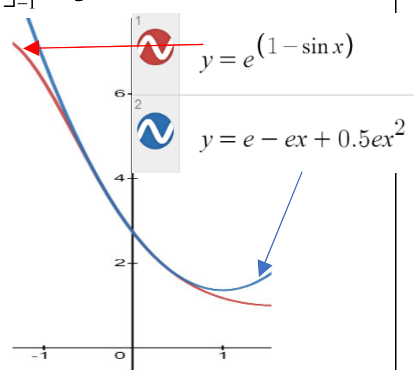
$$y = e - 1 - ex + \frac{1}{2}ex^2 \text{ (verified)}$$

(c) $\int_{-1}^{1} e^{1-\sin x} dx \approx \int_{-1}^{1}\left(e - ex + \frac{1}{2}ex^2\right)dx = \left[ex - \frac{1}{2}ex^2 + \frac{1}{6}ex^3\right]_{-1}^{1} = \frac{7}{3}e$

(d) From the sketch, it is clear that

$$\int_{-1}^{1} e^{1-\sin x} dx < \int_{-1}^{1}\left(e - ex + \frac{1}{2}ex^2\right)dx$$

therefore, the answer in (iii) is an over-estimation.



8

(a) Possible sketch: A graph with a turning (or sharp) point as x-intercept

(b) $f(x_3) = -0.28383 < 0$ and $f(3) = 8.88751 > 0$. The root is in $(x_3, 3)$.

$$x_4 = \frac{x_3 f(3) - 3f(x_3)}{f(3) - f(x_3)} \approx 1.47254 = 1.47 \text{ (2 d.p.)}$$

$f(x_4) = -0.16086 < 0$. Root is in $(x_4, 3)$.

$$x_5 = \frac{x_4 f(3) - 3f(x_4)}{f(3) - f(x_4)} \approx 1.499696 = 1.50 \text{ (2 d.p.)}$$

$f(1.495) = -0.1012 < 0$, $f(1.505) = -0.07407 < 0$. There is no change of sign.

Therefore, the required accuracy correct to 2 decimal places have not been achieved.

(c) The initial value needs to be larger than the x-coordinate of the turning point of f.

$f'(x) = x(2\ln x + 1) = 0 \Rightarrow x = 0$ or $x = e^{-\frac{1}{2}}$. Turning point occurs at $x = \frac{1}{\sqrt{e}}$.

	Set of values of u_1 is $\left\{u_1 : u_1 > \frac{1}{\sqrt{e}},\, u_1 \in \mathbb{R}\right\}$. (d) $u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)}$ $= u_n - \frac{{u_n}^2 \ln(u_n) - 1}{u_n(2\ln(u_n)+1)}$ (e) Using the recurrence relation, $u_2 = 2,\ u_3 = 1.62859 \approx 1.63,\ u_4 = 1.53734 \approx 1.54,\ u_5 = 1.53161 \approx 1.53,\ u_6 = 1.53158 \approx 1.53$ The required root is 1.53 (2 d.p.).
9	(a) Asymptote: $y = \frac{b}{a}x$, Perpendicular line passing through *F*: $y = -\frac{a}{b}(x-c)$. $\frac{b}{a}x = -\frac{a}{b}(x-c)$ $\left(\frac{b}{a}+\frac{a}{b}\right)x = \frac{ac}{b}$ $x = \frac{ac}{b}\left(\frac{ab}{a^2+b^2}\right)$ $= \frac{a^2c}{a^2+b^2} = \frac{a^2}{\sqrt{a^2+b^2}}$ (since $c^2 = a^2+b^2$) (b) $\frac{x^2}{a^2} - \frac{(mx+k)^2}{b^2} = 1$ $b^2x^2 - a^2(m^2x^2 + 2mkx + k^2) = a^2b^2$ $(a^2m^2 - b^2)x^2 + 2a^2mkx + a^2(b^2+k^2) = 0$ If the line is a tangent to *H*, discriminant = 0. $(2a^2mk)^2 - 4(a^2m^2-b^2)a^2(b^2+k^2) = 0$ $a^2m^2k^2 - (a^2m^2-b^2)(b^2+k^2) = 0$ $a^2m^2k^2 - (a^2m^2b^2 - b^4 + a^2m^2k^2 - b^2k^2) = 0$ $-a^2m^2b^2 + b^4 + b^2k^2 = 0$ $k^2 = a^2m^2 - b^2$ (shown) (b) A line passing through $(c, 0)$ and perpendicular to tangent is $y = -\frac{1}{m}(x-c)$. The foot of perpendicular is the intersection between this line and the tangent.

	$y=-\frac{1}{m}(x-c)\Rightarrow my=-x+c\Rightarrow (my+x)^2=c^2$ --- (1) $y=mx+k\Rightarrow (y-mx)^2=k^2=a^2m^2-b^2$ --- (2) Adding (1) and (2), $m^2y^2+2mxy+x^2+y^2-2mxy+m^2x^2=a^2m^2+c^2-b^2$ $(m^2+1)(x^2+y^2)=a^2m^2+a^2$ [since $c^2=a^2+b^2$] $=a^2(m^2+1)$ $x^2+y^2=a^2$ The tangents cannot become the asymptotes, so we need to exclude this limiting case. The two points to be excluded both have *x*-coordinates $\frac{a^2}{\sqrt{a^2+b^2}}$.
10	(a)

+	1	2	3
1	2	3	4
2	3	4	5
3	4	5	6

Let T be Tim's score.

$\mathrm{P}(T\geq 3)=1-\mathrm{P}(T\leq 2)=1-\frac{1}{9}=\frac{8}{9}\left(=0.889\,(3\text{sf})\right)$

(b) (i) $X=\{1,2,3\}$

$\mathrm{P}(X=1)=\mathrm{P}(1111)=\left(\frac{1}{3}\right)^4=\frac{1}{81}$

$\mathrm{P}(X=2)=\mathrm{P}(1112)\times {}^4C_1+\mathrm{P}(1122)\times {}^4C_2+\mathrm{P}(1222)\times {}^4C_3+\mathrm{P}(2222)$

$=\left(\frac{1}{3}\right)^4\times 4+\left(\frac{1}{3}\right)^4\times 6+\left(\frac{1}{3}\right)^4\times 4+\left(\frac{1}{3}\right)^4$

$=\frac{4}{81}+\frac{6}{81}+\frac{4}{81}+\frac{1}{81}$

$=\frac{15}{81}$

$\mathrm{P}(X=3)=1-\mathrm{P}(X\leq 2)=\frac{65}{81}$

x	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{81}$	$\frac{15}{81}$	$\frac{65}{81}$

(ii) $\mathrm{E}(X)=\sum_{x=1}^{3}x\mathrm{P}(X=x)=\frac{1}{81}+\frac{30}{81}+\frac{195}{81}=\frac{226}{81}$ (shown)

	$\mathrm{E}\left(X^2\right)=\sum_{x=1}^{3}x^2\mathrm{P}\left(X=x\right)$ $=\frac{1}{81}+\frac{60}{81}+\frac{585}{81}=\frac{646}{81}\left(=7.98\left(3\text{sf}\right)\right)$ $\mathrm{Var}\left(X\right)=\mathrm{E}\left(X^2\right)-\left(\mathrm{E}\left(X\right)\right)^2=0.191\left(3\text{sf}\right)$ (c) $\mathrm{P}\left(\text{total is }8\cap\left(X=3\right)\right)=\mathrm{P}\left(3311\right)\times{}^4C_2+\mathrm{P}\left(3221\right)\times\frac{4!}{2!}$ $=\left(\frac{1}{3}\right)^4\times6+\left(\frac{1}{3}\right)^4\times12=\frac{18}{81}$ $\mathrm{P}\left(\text{total is }8\middle|\left(X=3\right)\right)=\frac{\mathrm{P}\left(\text{total is }8\cap\left(X=3\right)\right)}{\mathrm{P}\left(X=3\right)}=\frac{\frac{18}{81}}{\frac{65}{81}}=\frac{18}{65}\left(=0.277\left(3\text{sf}\right)\right)$
11	(a) $L_n=L_{n-1}+\frac{\lambda}{100}L_{n-1}-R=\left(1+\frac{\lambda}{100}\right)L_{n-1}-R$ so $a=1+\frac{\lambda}{100}$ and $b=-R$ (b) Try $L_n=ka^n+p$ where k, p are constants $\therefore\ L_{n-1}=ka^{n-1}+p$ so $ka^n+p=a\left(ka^{n-1}+p\right)+b=ka^n+ap+b$ $\Rightarrow\ p-ap=b$ $\Rightarrow p=\frac{b}{1-a}$ Given $L_0=C$, $C=ka^0+p=k+\frac{b}{1-a}\Rightarrow k=C-\frac{b}{1-a}$ Therefore, $L_n=\left(C-\frac{b}{1-a}\right)a^n+\frac{b}{1-a}$. (c) Repayment scheme terminates if $C-\frac{b}{1-a}<0$ $\Rightarrow C-\frac{-R}{-\lambda/100}<0$ $\Rightarrow\frac{R}{\lambda/100}>C$ $\Rightarrow R>\frac{\lambda C}{100}$

	(d) (i) $L_n = \left(50000 - \frac{5000}{0.08}\right)(1.08)^n + \frac{5000}{0.08}$ $= 62500 - 12500(1.08)^n$ $L_n < 0 \Rightarrow n > \frac{\ln 5}{\ln 1.08} = 20.9$ Repayment takes 21 years. (ii) $L_{21} = -422.92$ $21 \times 5000 - 422.92 = 104577.08$ Total repayment is \$104 577.