AHMAD IBRAHIM SECONDARY SCHOOL
GCE O-LEVEL PRELIMINARY EXAMINATION 2023

SECONDARY 4 EXPRESS

Name: SOLUTION	Class:	Register No.:

ADDITIONAL MATHEMATICS **4049/01**
Paper 1 **7 August 2023**

Candidates answer on the Question Paper**.** **2 hours 15 minutes**

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 90.

For Examiner's Use
/90

This document consists of **19** printed pages.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

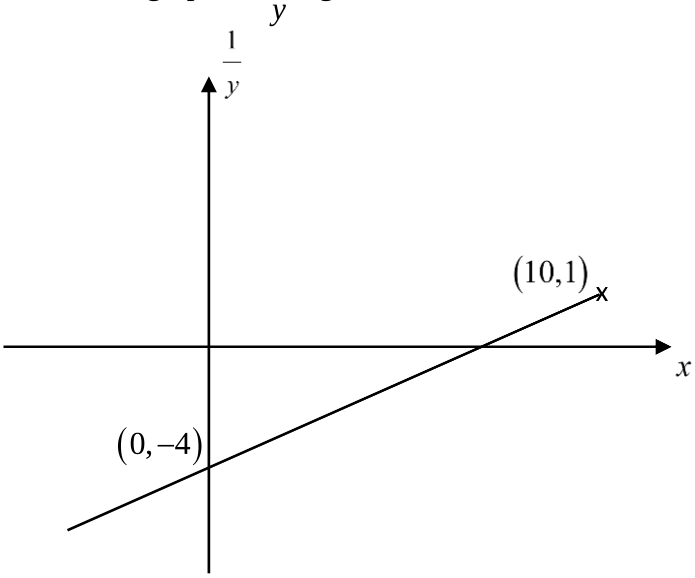
$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\Delta = \frac{1}{2} bc \sin A$$

1 The variables x and y are related by the equation $y = \dfrac{h}{2x-k}$. The diagram below shows the graph of $\dfrac{1}{y}$ against x.



Calculate the value of h and of k. [4]

$$y = \frac{h}{2x-k}$$

$$\frac{1}{y} = \frac{2x-k}{h} = \frac{2}{h}x - \frac{k}{h}$$

B1: setting up linear form

Gradient of line = $\dfrac{1-(-4)}{10-0} = \dfrac{1}{2} = \dfrac{2}{h}$

M1: finding gradient

$h = 4$

A1

y-intercept at $-4 = -\dfrac{k}{h}$

$k = 16$

A1

2 The Richter scale measures the intensity of an earthquake using the formula $M = \lg\left(\frac{I}{I_0}\right)$, where M is the magnitude of the earthquake, I is the intensity of the earthquake, and I_0 is the intensity of the smallest earthquake that can be measured.

(a) Calculate the magnitude of an earthquake if its intensity is 1000 times the intensity of the smallest earthquake that can be measured. [1]

(b) In February 2011, an earthquake with magnitude 6.2 was recorded in Christchurch, New Zealand. Few weeks later, an earthquake with magnitude 9.0 was detected in Fukushima, Japan. How many times stronger in intensity was the Japan's earthquake as compared to the New Zealand's earthquake? Give your answer to 2 decimal places. [3]

$$M = \lg\left(\frac{I}{I_0}\right)$$

$$M = \lg\left(\frac{1000I_0}{I_0}\right) = 3 \qquad \text{B1}$$

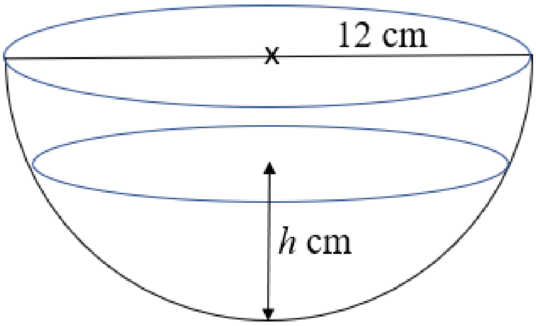
$$6.2 = \lg\left(\frac{I_{NZ}}{I_0}\right) \Rightarrow I_{NZ} = 10^{6.2} I_0 \qquad \text{M1: substitution and making I the subject}$$

$$9.0 = \lg\left(\frac{I_J}{I_0}\right) \Rightarrow I_J = 10^9 I_0$$

$$\frac{I_J}{I_{NZ}} = 10^{9-6.2} = 630.96 \text{ (2 d.p.)} \qquad \text{M1, A1}$$

The Japan's earthquake is 630.96 times stronger than the New Zealand's earthquake.

3 The diagram shows a hemispherical bowl of radius 12 cm. Water is poured into the bowl and at any time t seconds, the height of the water level from the lowest point of the hemisphere is h cm. The rate of change of the height of the water level is 0.4 cm/s.



(a) Show that the area of the water surface, A, is given by $A = \pi h(24 - h)$. [2]

Let radius of water surface be r cm

$r^2 + (12-h)^2 = 12^2$

M1: use of Pythagoras thm

$r^2 = 12^2 - (12-h)^2$

$= (12-(12-h))(12+(12-h))$

$= h(24-h)$

M1: simplification of r^2 and getting the result

Area, $A = \pi r^2 = \pi h(24-h)$

(b) Find the rate of change of A when $h = 5$ cm.
Leave your answer in terms of π. [3]

$\frac{dA}{dh} = \pi(24-2h)$

M1: correct differentiation

$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$

$= \pi(24-2h) \times 0.4$

M1: chain rule

$= \pi(24-10) \times 0.4$ when $h = 5$ cm

$= 5.6\pi$

A1: simplification of r^2 and getting the result

Rate of change of surface area = 5.6π cm^2/s

4 **(a)** Explain why there is only one solution to the equation $\log_5(13-4x)=\log_{\sqrt{5}}(2-x)$. [5]

$$\log_5(13-4x)=\log_{\sqrt{5}}(2-x)$$

$$=\frac{\log_5(2-x)}{\log_5\sqrt{5}}$$

$$=\frac{\log_5(2-x)}{\frac{1}{2}}$$

M1: correct change of base

$$=2\log_5(2-x)$$

$$\log_5(13-4x)=\log_5(2-x)^2$$

$$(13-4x)=(2-x)^2$$

M1: obtaining quadratic equation

$$(13-4x)=x^2-4x+4$$

$$x^2=9$$

$$x=3 \quad or \quad -3$$

M1: correctly solving quadratic eqn

When $x=3$, $\log_{\sqrt{5}}(2-3)=\log_{\sqrt{5}}(-1)$ is undefined, so reject $x=3$.

The only solution is $x=-3$ (ans)

A1, A1: explanation of undefined function and concluding only 1 final answer

(b) Solve the simultaneous equations

$$4^{x+3}=32\left(2^{x+y}\right),$$

$$9^{x}+3^{y}=10\ .$$ [7]

$4^{x+3}=32\left(2^{x+y}\right)$

$2^{2x+6}=2^{5}\left(2^{x+y}\right)=2^{5+x+y}$

$2x+6=5+x+y$ — M1: linear equation relating x and y

$y=x+1$ ---------- (1)

$9^{x}+3^{y}=10$

$3^{2x}+3^{y}=10$

$3^{2x}+3^{x+1}=10$ — M1: equation with 1 variable

$\left(3^{x}\right)^{2}+3\left(3^{x}\right)-10=0$

Let $u=3^{x}$:

$u^{2}+3u-10=0$ — M1: use of substitution

$(u+5)(u-2)=0$

$u=2$ or $u=-5$ — M1: solving quadratic equation

$3^{x}=2$ or $3^{x}=-5$ (reject since $3^{x}>0$)

$x\lg 3=\lg 2$ — M1: Taking lg on both sides to find x

$x=\dfrac{\lg 2}{\lg 3}=0.631$ (3 sf) — A1: correct x

$y=\dfrac{\lg 2}{\lg 3}+1=1.63$ (3 s.f) — A1: correct y

5 **(a)** Prove the identity $\cot 2x = \dfrac{1}{2\tan x} - \dfrac{1}{2}\tan x$. [2]

$$\cot 2x = \frac{1}{2\tan x} - \frac{1}{2}\tan x$$

$$\begin{aligned} LHS &= \cot 2x \\ &= \frac{1}{\tan 2x} \\ &= \frac{1}{\frac{2\tan x}{1-\tan^2 x}} \\ &= \frac{1-\tan^2 x}{2\tan x} \\ &= \frac{1}{2\tan x} - \frac{\tan^2 x}{2\tan x} \\ &= \frac{1}{2\tan x} - \frac{1}{2}\tan x \end{aligned}$$

M1: double angle formula

M1: splitting terms and getting result

(b) Hence solve the equation $\tan x(3 - 4\cot 2x) = 3$ for $0° \leq x \leq 360°$. [5]

$$\tan x(3 - 4\cot 2x) = 3$$

$$\tan x\left(3 - 4\left(\frac{1}{2\tan x} - \frac{1}{2}\tan x\right)\right) = 3$$

$$\tan x\left(3 - \frac{2}{\tan x} + 2\tan x\right) = 3$$

$$3\tan x - 2 + 2\tan^2 x = 3$$

$$2\tan^2 x + 3\tan x - 5 = 0$$

$$(\tan x - 1)(2\tan x + 5) = 0$$

$\tan x = 1$ or $\tan x = -2.5$

basic angle = 45° or 68.199°

$x = 45°,\ 225°,\ 111.8°,\ 291.8°$

M1: simplification to trigo quadratic

M1: 2 answers

M1: correct basic angles

A2: 2 pairs correct answers
A1: any 1 pair correct

(c) Without further solving, explain why there are 6 roots to the equation $\tan\frac{x}{2}\left(3-4\cot x\right)=3$ for $-360° \le x \le 720°$. [2]

There are 4 roots to the equation $\tan x\left(3-4\cot 2x\right)=3$ for $0° \le x \le 360°$ from (b).

Since the **period** of $\tan\frac{x}{2}\left(3-4\cot x\right)$ is **doubled** of $\tan x\left(3-4\cot 2x\right)$, there will be 4/2 = 2 roots to the equation for $0° \le x \le 360°$. [B1]

For $-360° \le x \le 720°$, the graph of $\tan\frac{x}{2}\left(3-4\cot x\right)$ would have **repeated 3 cycles**, thereby giving 3 x 2 = 6 roots to the equation. [B1]

OR

$\tan\frac{x}{2}\left(3-4\cot x\right)=3$, $-360° \le x \le 720°$

Let $y=\frac{x}{2}$, then $\tan y\left(3-4\cot 2y\right)=3$, $-180° \le y \le 360°$

y has 4 solutions in the domain $0° \le y \le 360°$, **1 from each quadrant**, from (b). [B1]

Therefore, for the domain $-180° \le y \le 360°$, the graph would have entered **another half a cycle**, giving rise to 2 additional roots. [B1]

Therefore, there will be 6 solutions for y in the given domain, and thus, 6 roots to the equation.

6 The curve $y = e^{2x}\sqrt{1-3x}$ intersects the y-axis at the point P. The tangent and the normal to the curve at P meet the x-axis at A and B respectively. Find the exact area of triangle PAB. [7]

$y = e^{2x}\sqrt{1-3x}$

At P, $x = 0$: $y = 1$.

M1: coor of P

$$\frac{dy}{dx} = 2e^{2x}\sqrt{1-3x} + e^{2x}\left(\frac{1}{2}\right)(1-3x)^{-1/2}(-3)$$

M1: correct differentiation with product rule

$$= 2e^{2x}\sqrt{1-3x} - \left(\frac{3}{2\sqrt{1-3x}}\right)e^{2x}$$

$$= \frac{e^{2x}}{\sqrt{1-3x}}\left(2(1-3x) - \frac{3}{2}\right)$$

Note: students may not simplify and just subst x values to find gradient

$$= \frac{e^{2x}}{\sqrt{1-3x}}\left(\frac{1}{2} - 6x\right)$$

At P, $\frac{dy}{dx} = \frac{1}{2}$.

M1: either gradient of tangent or normal

Equation of tangent: $y - 1 = \frac{1}{2}(x - 0)$

M1: equation of tangent

$$y = \frac{1}{2}x + 1$$

At A, $y = 0$: $x = -2$

Equation of normal: $y - 1 = -2(x - 0)$

M1: equation of normal

$$y = -2x + 1$$

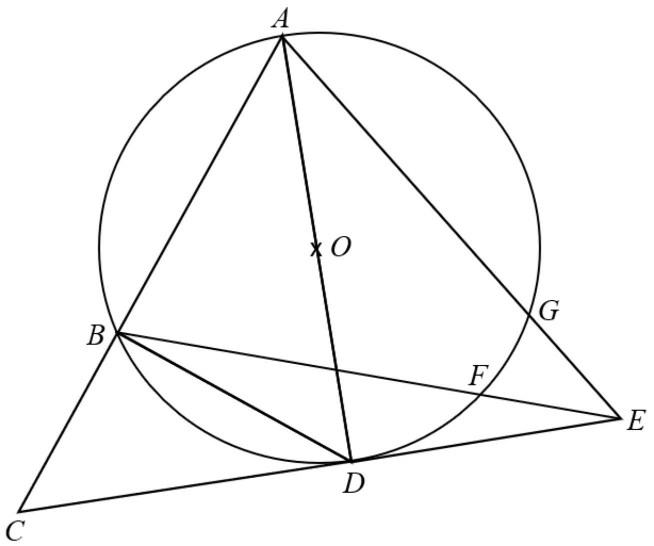
At B, $y = 0$: $x = \frac{1}{2}$

M1: finding coordinates of A and B

Area of triangle PAB = $\frac{1}{2}(1)\left(2 + \frac{1}{2}\right) = \frac{5}{4}$ units^2

A1

7 In the diagram, CE is a tangent that touches the circle of centre O at D.
AD is the diameter of the circle, EA cuts the circle at points G and A, and EB cuts the circle at points F and B.



(a) Given that ABC is a straight line, show that triangle ABD and triangle DBC are similar. [3]

$\measuredangle ABD = 90°$ (right angle in semicircle)
$\measuredangle DBC = 90°$ (adjacent angles on a straight line)
Therefore $\measuredangle ABD = \measuredangle DBC$.

$\measuredangle CDB = \measuredangle DAB$ (angles in alt. segment)

B3: all 3 reasoning used on correct angles + concluding with correct test B2: any 2 correct reasoning and angles B1: any 1 correct reasoning and angle

Since there are 2 pairs of corresponding angles that are equal, triangle ABD and triangle DBC are similar.

(b) If $BE = AE$, show that $EF = EG$. [4]

$\measuredangle EFG = 180° - \measuredangle BFG$ (adj angles on a str. line)
$= 180° - (180° - \measuredangle BAG)$ (angles in opp. segment)
$= \measuredangle BAG$
$= \measuredangle EAB$

B1: establishing angle EFG = angle EAB

$\measuredangle EGF = 180° - \measuredangle FGA$ (adj angles on a str. line)
$= 180° - (180° - \measuredangle FBA)$ (angles in opp. segment)
$= \measuredangle FBA$
$= \measuredangle EBA$

B1: establishing angle EGF = angle EBA

Given $BE = AE$, triangle EAB is isosceles.
$\measuredangle EBA = \measuredangle EAB$ (base angle of isos. triangle)

B1: establishing angle EBA = angle EAB

Therefore $\measuredangle EGF = \measuredangle EFG$
Hence triangle EGF is isosceles, and $EF = EG$

B1: using base angles of isosceles triangle argument

8 **(a)** Write down the first three terms in the expansion, in ascending powers of x, of $\left(2-\frac{x}{4}\right)^n$, where n is a positive integer greater than 2. [3]

$$\left(2-\frac{x}{4}\right)^n = 2^n + \binom{n}{1}(2)^{n-1}\left(-\frac{x}{4}\right) + \binom{n}{2}(2)^{n-2}\left(-\frac{x}{4}\right)^2 + \ldots$$

B1

$$= 2^n - \frac{1}{8}nx\left(2^n\right) + \frac{n(n-1)}{32}\left(2^n\right)x^2 + \ldots$$

B1: simplifying binomial coefficient

$$= 2^n - nx\left(2^{n-3}\right) + n(n-1)\left(2^{n-5}\right)x^2 + \ldots$$

B1

(b) The first two terms in the expansion, in ascending powers of x, of $(1+x)^2\left(2-\frac{x}{4}\right)^n$ are $a+bx^2$, where a and b are constants.

Find the value of n. [3]

$$(1+x)^2\left(2-\frac{x}{4}\right)^n = \left(1+2x+x^2\right)\left(2^n - nx\left(2^{n-3}\right) + n(n-1)\left(2^{n-5}\right)x^2 + \ldots\right)$$

$$= 2^n - nx\left(2^{n-3}\right) + n(n-1)\left(2^{n-5}\right)x^2 + 2^{n+1}x - nx^2\left(2^{n-2}\right) + \left(2^n\right)x^2$$

$$= 2^n + x\left(2^{n+1} - n\left(2^{n-3}\right)\right) + x^2\left[n(n-1)\left(2^{n-5}\right) - n\left(2^{n-2}\right) + \left(2^n\right)\right]$$

$$= a + bx^2$$

M1: coeff of x

Comparing coefficient of x: $2^{n+1} - n\left(2^{n-3}\right) = 0$

M1: equating to 0

$$2^{n-3}\left(2^4 - n\right) = 0$$

Since $2^{n-3} > 0$ for all real values of n, $n = 2^4 = 16$

A1

(c) Hence find the value of a and of b. [3]

Comparing constant: $a = 2^{16} = 65536$

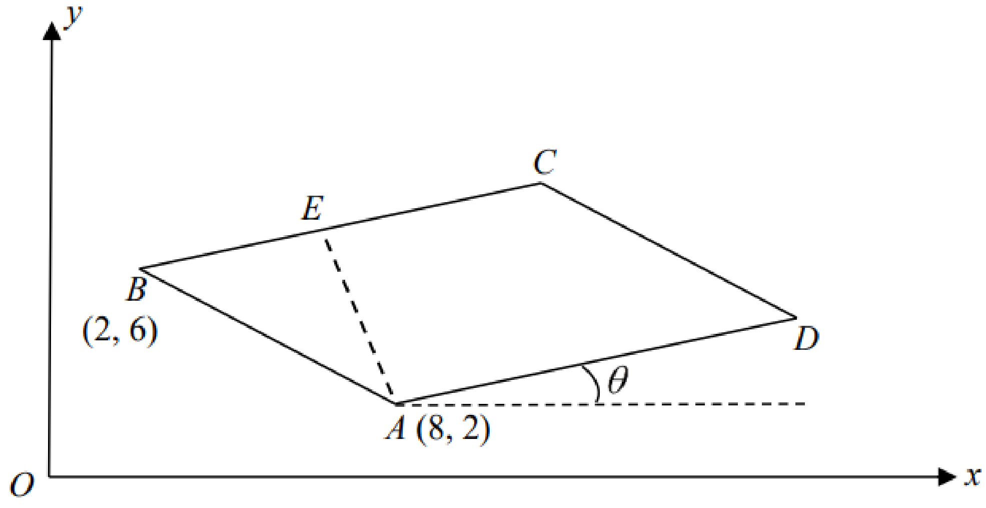
B1

Comparing coefficient of x^2 :

$b = 16(15)\left(2^{16-5}\right) - 16\left(2^{16-2}\right) + \left(2^{16}\right) = 294912$

M1: coeff of x^2
A1

9 The diagram shows a parallelogram $ABCD$ in which the coordinates of the points A and B are (8, 2) and (2, 6) respectively. The line AD makes an angle θ with the horizontal and $\tan\theta = 0.5$. The point E lies on BC such that AE is the shortest distance from A to BC.



(a) Show that the equation of line BC is $2y = x + 10$. [2]

Gradient of BC = Gradient of AD ($BC//AD$)

$$= \tan\theta = \frac{1}{2}$$

M1: identify gradient

Equation of BC: $y - 6 = \frac{1}{2}(x - 2)$

$$y = \frac{1}{2}x - 1 + 6$$

$$y = \frac{1}{2}x + 5$$

$$2y = x + 10$$

M1: equation formed and simplification to answer given

(b) Find the equation of line AE and the coordinates of E. [3]

Gradient of AE = -2

Equation of AE: $y - 2 = -2(x - 8)$

$$y = -2x + 18$$

M1: equation formed with correct gradient

At intersection: $\frac{1}{2}x + 5 = -2x + 18$

M1: equating and solving x or y

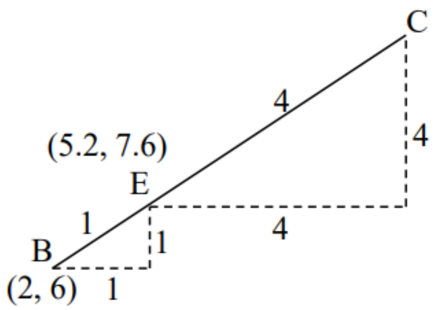
$x = 5.2$, $y = 7.6$

Coordinates of E: $(5.2, 7.6)$

A1

(c) Given that $\frac{BE}{BC} = \frac{1}{5}$, find the coordinates of *C* and *D*. [4]

Let coordinates of *C* be (x, y).
Using similar triangles,



$$\frac{1}{4} = \frac{5.2-2}{x-5.2} = \frac{7.6-6}{y-7.6}$$ M1

$$x = 4(3.2) + 5.2 = 18$$
$$y = 4(1.6) + 7.6 = 14$$

Coor of C: (18, 14) A1

Midpoint of *AC*: (13, 8) = Midpoint of *BD* (p, q)

$$\left(\frac{2+p}{2}, \frac{6+q}{2}\right) = (13,8)$$ M1

$$p = 24,\ q = 10$$

Coordinates of D (24, 10) A1

(d) Find the area of the figure *OBEA*, where *O* is the origin. [2]

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 0 & 8 & 5.2 & 2 & 0 \\ 0 & 2 & 7.6 & 6 & 0 \end{vmatrix}$$ M1

$$= \frac{1}{2}(60.8 + 31.2 - 10.4 - 15.2) = 33.2 \text{ units}^2$$ A1

10 **(a)** Solve the equation $2\cos 3x+1=0$ for $0 \le x \le \pi$. [3]

$2\cos 3x+1=0$

$\cos 3x=-\dfrac{1}{2}$

Basic angle = $\cos^{-1}\left(\dfrac{1}{2}\right)=\dfrac{\pi}{3}$

M1: finding correct basic angle

Since $0<x<\pi$, $0<3x<3\pi$

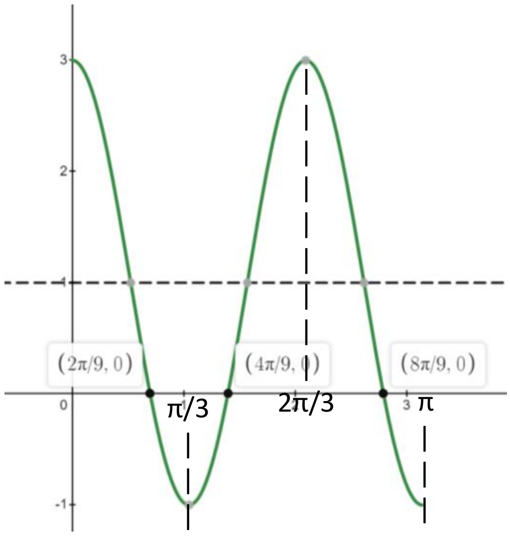
$3x=\dfrac{2\pi}{3},\dfrac{4\pi}{3},\dfrac{8\pi}{3}$

M1: correct 3x values

$x=\dfrac{2\pi}{9},\dfrac{4\pi}{9},\dfrac{8\pi}{9}$

A1: correct

(b) Sketch the graph of $y=2\cos 3x+1$ for $0 \le x \le \pi$. [3]



B1: correct shape
B1: correct period
B1: correct amplitude

(c) The equation of a curve is $y = \dfrac{\sin 3x}{2 + \cos 3x}$, where $0 \le x \le \pi$.

Using **(a)** and **(b)**, find the range of values of x for which y is a decreasing function. [5]

$$y = \frac{\sin 3x}{2 + \cos 3x}$$

$$\frac{dy}{dx} = \frac{(2+\cos 3x)3\cos 3x - \sin 3x(-3\sin 3x)}{(2+\cos 3x)^2}$$

M1: correct differentiation

$$= \frac{6\cos 3x + 3\cos^2 3x + 3\sin^2 3x}{(2+\cos 3x)^2}$$

$$= \frac{6\cos 3x + 3}{(2+\cos 3x)^2}$$

For decreasing function, $\dfrac{dy}{dx} < 0$

$$\frac{6\cos 3x + 3}{(2+\cos 3x)^2} < 0$$

M1: setting to <0

Since $(2+\cos x)^2 > 0$ for all values of x, $6\cos 3x + 3 < 0$

$$2\cos 3x + 1 < 0$$

M1: simplification to (i) expression

$\dfrac{2\pi}{9} < x < \dfrac{4\pi}{9}$ or $\dfrac{8\pi}{9} < x \le \pi$

A2

11 **(a)** Express $\dfrac{3x^2+4x-20}{(2x+1)\left(x^2+4\right)}$ in partial fractions. [5]

$$\frac{3x^2+4x-20}{(2x+1)\left(x^2+4\right)}=\frac{A}{2x+1}+\frac{Bx+C}{x^2+4}$$

M1: identify correct form

$$3x^2+4x-20=A(x^2+4)+(Bx+C)(2x+1)$$

Let $x=-\dfrac{1}{2}$ $\quad 3\left(-\dfrac{1}{2}\right)^2+4\left(-\dfrac{1}{2}\right)-20=A(\left(-\dfrac{1}{2}\right)^2+4)+0$

M1: substitution or expand and compare coeff

$$A=-5$$

A2: any correct 2 constants

A1: any correct 1 constant

Let $x=0$ $\quad -20=A(4)+C(1)$

$$C=0$$

Let $x=1$ $\quad -13=5A+3(B+C)$

$$B=4$$

$$\frac{3x^2+4x-20}{(2x+1)\left(x^2+4\right)}=\frac{4x}{x^2+4}-\frac{5}{2x+1}$$

A1

(b) Differentiate $\ln\left(x^2+4\right)$ with respect to x. [2]

$$\frac{d}{dx}\ln\left(x^2+4\right)=\frac{2x}{x^2+4}$$

B1: chain rule to get denominator,
B1: numerator

(c) The gradient function of a curve is $\dfrac{3x^2+4x-20}{(2x+1)\left(x^2+4\right)}$.

Given that the y-intercept of the curve is $\left(0,\ln 4\right)$, using part **(a)** and **(b)**, find the equation of the curve. [4]

$$\frac{dy}{dx}=\frac{3x^2+4x-20}{(2x+1)\left(x^2+4\right)}=\frac{4x}{x^2+4}-\frac{5}{2x+1}$$

$$y=\int\frac{4x}{x^2+4}-\frac{5}{2x+1}\,dx$$

$$=\int\frac{4x}{x^2+4}\,dx-\int\frac{5}{2x+1}\,dx$$

$$=2\int\frac{2x}{x^2+4}\,dx-5\int\frac{1}{2x+1}\,dx$$

$$=2\ln\left(x^2+4\right)-\frac{5}{2}\ln\left(2x+1\right)+C$$

M1: factorising with intent to use (b)

M1: correctly integrating $\dfrac{1}{2x+1}$

Since $\left(0,\ln 4\right)$ is on the curve: $\ln 4=2\ln\left(4\right)-\frac{5}{2}\ln\left(1\right)+C$ M1

$$C=-\ln 4$$

$$y=2\ln\left(x^2+4\right)-\frac{5}{2}\ln\left(2x+1\right)-\ln 4$$ A1

END OF PAPER