2024 H3 Math Prelim Suggested Solution

	Solution		
1(a)	Consider the first term of a special sequence, it must an odd integer and divided into the following mutually exclusive cases. **Case 1:** $a_1 = 1$ Hence the sequence is of the form $\{1, a_2, a_3, ..., a_m\}$ where $m \le n$. Consider the function $\{1, a_2, a_3, ..., a_m\} \xrightarrow{f} \{a_2 - 1, a_3 - 1, ..., a_m - 1\}$. f is a bijection and the number of special sequences of the type $\{a_2 - 1, a_3 - 1, ..., a_m - 1\} = A_{n-1}$ Number of special sequences with $a_1 = 1$ is $A_{n-1} + 1$ (Accounting for $\{1\}$) **Case 2:** $a_1 \ge 3$ is of the form $\{a_1, a_2, a_3, ..., a_m\}$ where $m \le n$. Consider the function $\{a_1, a_2, a_3, ..., a_m\} \xrightarrow{g} \{a_1 - 2, a_2 - 2, a_3 - 2, ..., a_m - 2\}$. g is a bijection and the number of special sequences of the type $\{a_1 - 2, a_2 - 2, a_3 - 3, ..., a_m - 2\} = A_{n-2}$ Hence, $A_n = A_{n-1} + A_{n-2} + 1$ $\therefore A_n = A_{n-1} + A_{n-2} + 1$ $A_1 = 1$ (i.e $\{1\}$) $A_2 = 2$ (i.e $\{1\}$, $\{1, 2\}$)		
1(b)	We want A_{15} By G.C, $A_{15} = 1596$		

	Solution		
2(a)	$g'(x) = e^x f'(x) + e^x f(x)$ $= e^x \left(f'(x) + f(x) \right)$ $\le e^x$		
2(b)	Since $g'(x) \le e^x$, $\int_0^1 g'(x)dx \le \int_0^1 e^x dx$ $g(1) - g(0) \le \left[e^x \right]_0^1$ $ef(1) - f(0) \le e - 1$ $ef(1) \le e - 1 + a$ $f(1) \le \frac{e-1+a}{e}$ The largest possible value of f(1) is $\frac{e-1+a}{e}$, which is attainable when $f'(x) + f(x) = 1$. Let $y = f(x)$, $f'(x) + f(x) = 1$ $\frac{dy}{dx} + y = 1$ $\frac{dy}{dx} = 1 - y$ $\int \frac{1}{1-y} dy = x + C$ $-\ln\lvert 1 - y \rvert = x + C$ $1 - y = Ae^{-x}$ Since $x = 0, y = a$, $1 - a = A$ $f(x) = 1 - (1-a)e^{-x}$		

3(a) (i)	$\mathrm{f}(x)=\dfrac{x}{(1-x)^2}=x(1-x)^{-2}$ $=x\left(1+(-2)(-x)+\ldots+\dfrac{(-2)(-3)\ldots(-(n))}{(n-1)!}(-x)^{n-1}+\ldots\right)$ $=0+x+2x^2+\ldots+\dfrac{(-2)(-3)\ldots(-(n))}{(n-1)!}(-1)^{n-1}x^n+\ldots$ Note that $u_0=0$ and $u_1=1$ For $n\ge 2$, coefficient of x^n in series expansion of $\mathrm{f}(x)$ $=\dfrac{(-2)(-3)\ldots(-(n))}{(n-1)!}(-1)^{n-1}$ $=\dfrac{(-1)^{n-1}n!(-1)^{n-1}}{(n-1)!}$ $=n$ Hence $\mathrm{f}(x)=\dfrac{x}{(1-x)^2}$ is the generating function of $u_n=n$. $\mathrm{f}(x)=\dfrac{x}{(1-x)^3}=x(1-x)^{-3}$ $=x\left(1+(-3)(-x)+\ldots+\dfrac{(-3)(-4)\ldots(-(n+1))}{(n-1)!}(-x)^{n-1}+\ldots\right)$ $=0+x+3x^2+\ldots+\dfrac{(-3)(-4)\ldots(-(n+1))}{(n-1)!}(-1)^{n-1}x^n+\ldots$ For $n\ge 2$, coefficient of x^n in series expansion of $\mathrm{f}(x)$		

	$= \frac{(-3)(-4)\ldots(-(n+1))}{(n-1)!}(-1)^{n-1}$ $= \frac{(-1)^{n-1}(n+1)!(-1)^{n-1}}{2(n-1)!}$ $= \frac{n(n+1)}{2}$ Note that $u_0 = 0 = \frac{0(1)}{2}$ and $u_1 = 1 = \frac{1(2)}{2}$ Hence the sequence with generating function $f(x) = \frac{x}{(1-x)^3}$ is $u_n = \frac{n(n+1)}{2}$, $n \geq 0$.		

3(a) (ii)

Method 1

$$n^2 = 2\left(\frac{n(n+1)}{2}\right) - n$$

Hence the generating function for $u_n = n^2$ is

$$\mathrm{f}(x) = 2\left(\frac{x}{(1-x)^3}\right) - \frac{x}{(1-x)^2}$$

$$= \frac{2x - x(1-x)}{(1-x)^3} = \frac{x+x^2}{(1-x)^3}$$

Method 2

Consider

$$\frac{x}{(1-x)^2} = 0 + x + 2x^2 + \ldots + nx^n + \ldots$$

Differentiate w..r.t. x, we have

$$\frac{(1-x)^2 - x2(1-x)(-1)}{(1-x)^4} = 0 + 1 + 4x + \ldots + n^2 x^{n-1} + \ldots$$

$$\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = 0 + 1 + 4x + \ldots + n^2 x^{n-1} + \ldots$$

$$\frac{1 - x + 2x}{(1-x)^3} = 0 + 1 + 4x + \ldots + n^2 x^{n-1} + \ldots$$

$$\frac{1+x}{(1-x)^3} = 0 + 1 + 4x + \ldots + n^2 x^{n-1} + \ldots$$

Multiplying x throughout, we have

$$\frac{x+x^2}{(1-x)^2} = 0 + x + 4x^2 + \ldots + n^2 x^n + \ldots$$

Hence the generating function for $u_n = n^2$ is

$$\mathrm{f}(x) = \frac{x+x^2}{(1-x)^3}$$

3(b)	Let the generating function of the sequence be $\mathrm{f}(x)$. $\mathrm{f}(x) = \sum_{n=0}^{\infty} u_n x^n$ $= 1 + \sum_{n=1}^{\infty} u_n x^n$ $= 1 + \sum_{n=1}^{\infty} \left(2u_{n-1} + 1\right) x^n$ $= 1 + \sum_{n=1}^{\infty} 2u_{n-1} x^n + \sum_{n=1}^{\infty} x^n$ $= 2x \sum_{n=1}^{\infty} u_{n-1} x^{n-1} + \sum_{n=0}^{\infty} x^n$ $= 2x \sum_{n=0}^{\infty} 2u_n x^n + \frac{1}{(1-x)}$ $= 2x\mathrm{f}(x) + \frac{1}{(1-x)}$ Therefore $(1-2x)\mathrm{f}(x) = \frac{1}{(1-x)}$ $\Rightarrow \mathrm{f}(x) = \frac{1}{(1-2x)(1-x)}$.		

<table>
<tr><td></td><td>Solution</td><td></td><td></td></tr>
<tr><td>4(a)</td><td>$\tau(1)=1$, $\tau(3)=2$, $\tau(12)=6$.</td><td></td><td></td></tr>
<tr><td>4(b)</td><td>We note that each p_i has α_1+1 choices to be included in a factor, since the choices are $p_i^0, p_i^1, p_i^2, \dots, p_i^{\alpha_i}$. Hence considering all the prime factors, we have
$\tau(n)=(\alpha_1+1)(\alpha_2+1)\dots(\alpha_k+1)$.</td><td></td><td></td></tr>
<tr><td>4(c)</td><td>By part (b), we have $\tau(n)=(\alpha_1+1)(\alpha_2+1)\dots(\alpha_k+1)$. Hence
$\tau(n)$ is odd
$\Leftrightarrow (\alpha_1+1)(\alpha_2+1)\dots(\alpha_k+1)$ is odd
$\Leftrightarrow \forall 1\le i\le k.\ (\alpha_i+1)$ is odd
$\Leftrightarrow \forall 1\le i\le k.\ \alpha_i$ is even
Now,
$\tau(n)$ is odd
$\Leftrightarrow n = p_1^{\alpha_1} p_2^{\alpha_2}\dots p_k^{\alpha_k} = p_1^{2\beta_1} p_2^{2\beta_2}\dots p_k^{2\beta_k}$
$\Leftrightarrow n = \left(p_1^{\beta_1} p_2^{\beta_2}\dots p_k^{\beta_k}\right)^2$
$\Leftrightarrow n$ is a perfect square</td><td></td><td></td></tr>
<tr><td>4(d)</td><td>
<table>
<tr><td>n</td><td colspan="8">Divisors of k</td></tr>
<tr><td>…</td><td colspan="8">…</td></tr>
<tr><td>$k=8$</td><td>1</td><td>2</td><td></td><td>4</td><td></td><td></td><td></td><td>8</td></tr>
<tr><td>$k=7$</td><td>1</td><td></td><td></td><td></td><td></td><td></td><td>7</td><td></td></tr>
<tr><td>$k=6$</td><td>1</td><td>2</td><td>3</td><td></td><td></td><td>6</td><td></td><td></td></tr>
<tr><td>$k=5$</td><td>1</td><td></td><td></td><td></td><td>5</td><td></td><td></td><td></td></tr>
<tr><td>$k=4$</td><td>1</td><td>2</td><td></td><td>4</td><td></td><td></td><td></td><td></td></tr>
<tr><td>$k=3$</td><td>1</td><td></td><td>3</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>$k=2$</td><td>1</td><td>2</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>$k=1$</td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
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4(e)	Notice that n gives the number of rows of the table in part **(c)**, and that each $k \leq n$ occurs every kth row, for a total of $\left\lfloor \frac{n}{k} \right\rfloor$ rows. Hence $\sum_{k=1}^{n} \tau(k) =$ sum of number of members of each row $=$ sum of number of members of each column $= \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$		

	Solution		
5(a)	$\int_0^\infty e^{-2x}\sin x dx$ $=\left[-\frac{1}{2}e^{-2x}\sin x\right]_0^\infty-\int_0^\infty -\frac{1}{2}e^{-2x}\cos x dx$ $=\left[-\frac{1}{2}e^{-2x}\sin x\right]_0^\infty+\frac{1}{2}\left(\left[-\frac{1}{2}e^{-2x}\cos x\right]_0^\infty-\int_0^\infty -\frac{1}{2}e^{-2x}(-\sin x)\ dx\right)$ $=\left[-\frac{1}{2}e^{-2x}\sin x-\frac{1}{4}e^{-2x}\cos x\right]_0^\infty-\frac{1}{4}\int_0^\infty e^{-2x}\sin x dx$ Since $\sin x$ and $\cos x$ are bounded, $\lim_{x\to\infty} e^{-2x}\sin x=0$, $\lim_{x\to\infty} e^{-2x}\cos x=0$, $\frac{5}{4}\int_0^\infty e^{-2x}\sin x dx=\left[-\frac{1}{2}e^{-2x}\sin x-\frac{1}{4}e^{-2x}\cos x\right]_0^\infty=0-(0-\frac{1}{4})=\frac{1}{4}$ $\int_0^\infty e^{-2x}\sin x dx=\frac{1}{5}$		
5(b)	$\int_0^\infty e^{-2x}\sin x\ dx<\int_0^\infty e^{-2x}\lvert\sin x\rvert\ dx<\int_0^\infty e^{-2x}\ dx$ $\int_0^\infty e^{-2x}\ dx=\left[-\frac{1}{2}e^{-2x}\right]_0^\infty=0-(-\frac{1}{2})=\frac{1}{2}$ Hence, $\frac{1}{5}<\int_0^\infty e^{-2x}\lvert\sin x\rvert\ dx<\frac{1}{2}$ and $\int_0^\infty e^{-2x}\lvert\sin x\rvert\ dx$ exists.		

5(c)

$$\int_0^{n\pi} \mathrm{e}^{-2x} \left|\sin x\right| \, \mathrm{d}x$$

$$= \int_0^{\pi} \mathrm{e}^{-2x} \sin x \, \mathrm{d}x - \int_{\pi}^{2\pi} \mathrm{e}^{-2x} \sin x \, \mathrm{d}x + \int_{2\pi}^{3\pi} \mathrm{e}^{-2x} \sin x \, \mathrm{d}x +$$

$$\ldots + \left(-1\right)^{n-1} \int_{(n-1)\pi}^{n\pi} \mathrm{e}^{-2x} \sin x \, \mathrm{d}x$$

$$= \sum_{k=1}^{n} \left(-1\right)^{k-1} \int_{(k-1)\pi}^{k\pi} \mathrm{e}^{-2x} \sin x \, \mathrm{d}x$$

$$\frac{5}{4}\int \mathrm{e}^{-2x} \sin x \mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-2x} \sin x - \frac{1}{4}\mathrm{e}^{-2x} \cos x$$

$$\int \mathrm{e}^{-2x} \sin x \mathrm{d}x = -\frac{2}{5}\mathrm{e}^{-2x} \sin x - \frac{1}{5}\mathrm{e}^{-2x} \cos x + C$$

$$\int_{(k-1)\pi}^{k\pi} \mathrm{e}^{-2x} \sin x \, \mathrm{d}x = \left[-\frac{2}{5}\mathrm{e}^{-2x} \sin x - \frac{1}{5}\mathrm{e}^{-2x} \cos x \right]_{(k-1)\pi}^{k\pi}$$

$$= \left[0 - \frac{1}{5}\mathrm{e}^{-2k\pi} \cos k\pi \right] - \left[0 - \frac{1}{5}\mathrm{e}^{-2(k-1)\pi} \cos(k-1)\pi \right]$$

$$= \begin{cases} \frac{1}{5}\mathrm{e}^{-2k\pi} + \frac{1}{5}\mathrm{e}^{(-2k+2)\pi} & \text{if } k \text{ is odd} \\ -\frac{1}{5}\mathrm{e}^{(-2k+2)\pi} - \frac{1}{5}\mathrm{e}^{-2k\pi} & \text{if } k \text{ is even} \end{cases}$$

$$= \begin{cases} \frac{1}{5}\mathrm{e}^{-2k\pi}(1+\mathrm{e}^{2\pi}) & \text{if } k \text{ is odd} \\ -\frac{1}{5}\mathrm{e}^{-2k\pi}(1+\mathrm{e}^{2\pi}) & \text{if } k \text{ is even} \end{cases}$$

$$= \sum_{k=1}^{n} \frac{1}{5}\mathrm{e}^{-2k\pi}(1+\mathrm{e}^{2\pi})$$

$$= \frac{1}{5}(1+\mathrm{e}^{2\pi})\left(\mathrm{e}^{-2\pi} + \mathrm{e}^{-4\pi} + \ldots + \mathrm{e}^{-2n\pi}\right)$$

$$= \frac{1}{5}(1+\mathrm{e}^{2\pi})\frac{\mathrm{e}^{-2\pi}\left(1-\mathrm{e}^{-2n\pi}\right)}{1-\mathrm{e}^{-2\pi}}$$

$$= \frac{1}{5}\frac{\left(1+\mathrm{e}^{-2\pi}\right)\left(1-\mathrm{e}^{-2n\pi}\right)}{1-\mathrm{e}^{-2\pi}}$$

	As $n \to \infty, \mathrm{e}^{-2n\pi} \to 0$ $\int_0^{\infty} \mathrm{e}^{-2x} \lvert \sin x \rvert \, \mathrm{d}x$ $= \lim_{n\to\infty} \int_0^{n\pi} \mathrm{e}^{-2x} \lvert \sin x \rvert \, \mathrm{d}x$ $= \lim_{n\to\infty} \frac{1}{5} \frac{\left(1+\mathrm{e}^{-2\pi}\right)\left(1-\mathrm{e}^{-2n\pi}\right)}{1-\mathrm{e}^{-2\pi}}$ $= \frac{1+\mathrm{e}^{-2\pi}}{5\left(1-\mathrm{e}^{-2\pi}\right)}$		

	Solution		
6(a)	By the Division Algorithm, we have $n=5k,5k+1,5k+2,5k+3,5k+4$. Hence $n^2=5(5k^2),5(5k^2+2k)+1,5(5k^2+4k)+4,$ $5(5k^2+6k+1)+4,5(5k^2+8k+3)+1,$ which in turn implies $n^2\equiv 0,1,4 \pmod 5$. Hence $r=0$ or $r=1$ or $r=4$, where $n^2\equiv r \pmod 5$. Now, $n^2\equiv 0,1,4 \pmod 5$ $\Rightarrow 3n^2\equiv 0,3,12 \pmod 5$ $\Rightarrow 3n^2\equiv 0,3,2 \pmod 5$		
6(b)	Let $(a,b,c)\in\mathbb{Z}^3$. We note that $3a^2+b^2=5c^2\Rightarrow 3a^2+b^2\equiv 0 \pmod 5$. From part **(a)**, we note that $a\not\equiv 0 \pmod 5$ $\Rightarrow a^2\equiv 1,4 \pmod 5$ $\Rightarrow 3a^2\equiv 3,2 \pmod 5$ $\Rightarrow 3a^2+b^2\equiv 3+0,3+1,3+4,2+0,2+1,2+4 \pmod 5$ $\Rightarrow 3a^2+b^2\equiv 3,4,2,2,3,1 \pmod 5$ $\Rightarrow 3a^2+b^2\not\equiv 0 \pmod 5$ From part **(a)**, we note that $b\not\equiv 0 \pmod 5$ $\Rightarrow b^2\equiv 1,4 \pmod 5$ $\Rightarrow 3a^2+b^2\equiv 0+1,3+1,2+1,0+4,3+4,2+4 \pmod 5$ $\Rightarrow 3a^2+b^2\equiv 1,4,3,4,2,1 \pmod 5$ $\Rightarrow 3a^2+b^2\not\equiv 0 \pmod 5$ Hence $a\not\equiv 0 \pmod 5$ or $b\not\equiv 0 \pmod 5 \Rightarrow 3a^2+b^2\not\equiv 0 \pmod 5$, which in turn gives us $3a^2+b^2=5c^2\Rightarrow a\equiv 0 \pmod 5$ and $b\equiv 0 \pmod 5$.		

6(c)	Let $X=\{(x,y,z)\in\mathbb{Z}^3 : x\neq 0 \text{ or } y\neq 0 \text{ or } z\neq 0\}$. Suppose there exists $(a,b,c)\in X$ such that $3a^2+b^2=5c^2$. Without loss of generality, suppose $a\neq 0$. By part **(b)**, $3a^2+b^2=5c^2$ becomes $3(5x)^2+(5y)^2=5c^2$ for some $x,y\in\mathbb{Z}$. Dividing throughout by 5, we have $3\cdot 5(x)^2+5(y)^2=c^2$, which in turn implies $c^2\equiv 0 \pmod 5$, which in turn implies $c\equiv 0 \pmod 5$ by part **(a)**, i.e., $c^2\equiv 0 \pmod 5 \Leftrightarrow c\equiv 0 \pmod 5$. Thus, we have $3\cdot 5(x)^2+5(y)^2=25z^2$ for some $z\in\mathbb{Z}$. Dividing throughout by 5 again, we have $3x^2+y^2=5z^2$, which is similarly in structure to $3a^2+b^2=5c^2$. By part **(b)** again, we have $x\equiv 0 \pmod 5$. We note that this procedure produces the following: Step 1: $a=5x$ where $x\in\mathbb{Z}$ Step 2: $a=5\cdot 5x_1$ where $x_1\in\mathbb{Z}$ Step 3: $a=5\cdot 5\cdot 5x_2$ where $x_2\in\mathbb{Z}$ ... Hence we deduce that a is a nonzero integer that is infinitely divisible by 5, which is a contradiction. Thus there does not exist $(a,b,c)\in X$ such that $3a^2+b^2=5c^2$. Hence $(0,0,0)$ must be the only solution for $3a^2+b^2=5c^2$.		
6(d)	No. We have $4(1^2)+4(2^2)=5(2^2)$.		

	Solution		
7(a)	$F_{n+2}F_{n+5}-F_{n+3}F_{n+4}$ $=\left(F_n+F_{n+1}\right)\left(F_{n+3}+F_{n+4}\right)-\left(F_{n+1}+F_{n+2}\right)\left(F_{n+2}+F_{n+3}\right)$ $=F_nF_{n+3}+F_nF_{n+4}+F_{n+1}F_{n+3}+F_{n+1}F_{n+4}$ $\quad -\left(F_{n+1}F_{n+2}+F_{n+1}F_{n+3}+F_{n+2}F_{n+2}+F_{n+2}F_{n+3}\right)$ $=F_nF_{n+3}-F_{n+1}F_{n+2}+\left(F_n+F_{n+1}\right)F_{n+4}-F_{n+2}\left(F_{n+2}+F_{n+3}\right)$ $=F_nF_{n+3}-F_{n+1}F_{n+2}+F_{n+2}F_{n+4}-F_{n+2}F_{n+4}$ $=F_nF_{n+3}-F_{n+1}F_{n+2}$ (shown)		
7(b)	When n is odd, i.e. $n=2k+1$, $k\geq 0$, $F_nF_{n+3}-F_{n+1}F_{n+2}$ $=F_{2k+1}F_{2k+4}-F_{2k+2}F_{2k+3}$ $=F_{2k-1}F_{2k+2}-F_{2k}F_{2k+1}$ (from (a)) $\vdots$ $=F_1F_4-F_2F_3$ $=1(3)-1(2)$ $=1$ When n is even, i.e. $n=2k$, $k\geq 0$, $F_nF_{n+3}-F_{n+1}F_{n+2}$ $=F_{2k}F_{2k+3}-F_{2k+1}F_{2k+2}$ $=F_{2k-2}F_{2k+1}-F_{2k-1}F_{2k}$ (from (a)) $\vdots$ $=F_0F_3-F_1F_2$ $=0(2)-1(1)$ $=-1$		

7(c)	Consider $\tan\left(\tan^{-1}\left(\frac{1}{F_{2r+1}}\right)+\tan^{-1}\left(\frac{1}{F_{2r+2}}\right)\right)$ $=\frac{\tan\left(\tan^{-1}\left(\frac{1}{F_{2r+1}}\right)\right)+\tan\left(\tan^{-1}\left(\frac{1}{F_{2r+2}}\right)\right)}{1-\tan\left(\tan^{-1}\left(\frac{1}{F_{2r+1}}\right)\right)\tan\left(\tan^{-1}\left(\frac{1}{F_{2r+2}}\right)\right)}$ $=\frac{\frac{1}{F_{2r+1}}+\frac{1}{F_{2r+2}}}{1-\frac{1}{F_{2r+1}F_{2r+2}}}$ $=\frac{F_{2r+2}+F_{2r+1}}{F_{2r+1}F_{2r+2}-1}$ $=\frac{F_{2r+2}+F_{2r+1}}{F_{2r+1}F_{2r+2}+\left(F_{2r}F_{2r+3}-F_{2r+1}F_{2r+2}\right)}$ $=\frac{F_{2r+3}}{F_{2r}F_{2r+3}}$ $=\frac{1}{F_{2r}}$ Hence we have $\tan^{-1}\left(\frac{1}{F_{2r}}\right)=\tan^{-1}\left(\frac{1}{F_{2r+1}}\right)+\tan^{-1}\left(\frac{1}{F_{2r+2}}\right)$ (shown)		

7(d)	Consider $\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{F_{2r+1}}\right)$ $= \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{F_{2r}}\right) - \tan^{-1}\left(\frac{1}{F_{2r+2}}\right)$ $= \tan^{-1}\left(\frac{1}{F_2}\right) - \tan^{-1}\left(\frac{1}{F_4}\right)$ $- \tan^{-1}\left(\frac{1}{F_4}\right) - \tan^{-1}\left(\frac{1}{F_6}\right)$ $\vdots$ $- \tan^{-1}\left(\frac{1}{F_{2n-2}}\right) - \tan^{-1}\left(\frac{1}{F_{2n}}\right)$ $- \tan^{-1}\left(\frac{1}{F_{2n}}\right) - \tan^{-1}\left(\frac{1}{F_{2n+2}}\right)$ $= \tan^{-1}\left(\frac{1}{F_2}\right) - \tan^{-1}\left(\frac{1}{F_{2n+2}}\right)$ Therefore $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2r+1}}\right)$ $= \lim_{n\to\infty}\left(\tan^{-1}\left(\frac{1}{F_2}\right) - \tan^{-1}\left(\frac{1}{F_{2n+2}}\right)\right)$ $= \tan^{-1}(1) - \tan^{-1}(0)$ $= \frac{\pi}{4}$		

	Solution		
8(a)	Let A be the set of numbers s.t. $w=2$ Let B be the set of numbers s.t. $x=4$ Let C be the set of numbers s.t. $y=6$ Let D be the set of numbers s.t. $z=8$ $\lvert A\rvert=\lvert B\rvert=\lvert C\rvert=\lvert D\rvert=(1)(8)(7)(6)=336$ $\lvert A\cap B\rvert=\lvert A\cap C\rvert=\lvert A\cap D\rvert=\lvert B\cap C\rvert=\lvert B\cap D\rvert=\lvert C\cap D\rvert$ $=(1)(1)(7)(6)=42$ $\lvert A\cap B\cap C\rvert=\lvert A\cap B\cap D\rvert=\lvert A\cap C\cap D\rvert=\lvert B\cap C\cap D\rvert$ $=(1)(1)(1)(6)=6$ $\lvert A\cap B\cap C\cap D\rvert=1$ No. of ways $=\lvert P\rvert-(\lvert A\rvert+\lvert B\rvert+\lvert C\rvert+\lvert D\rvert)+(\lvert A\cap B\rvert+\lvert A\cap C\rvert+\lvert A\cap D\rvert+\lvert B\cap C\rvert+\lvert B\cap D\rvert+\lvert C\cap D\rvert)$ $-(\lvert A\cap B\cap C\rvert+\lvert A\cap B\cap D\rvert+\lvert A\cap C\cap D\rvert+\lvert B\cap C\cap D\rvert)+\lvert A\cap B\cap C\cap D\rvert$ $=(9)(8)(7)(6)-(4)(336)+(6)(42)-(4)(6)+1$ $=1909$		
8(b) (i)	Consider $x_1\geq 1$, (Class A has at least 1 notepad) $(x_1-1)+x_2+x_3+x_4=98$ Let $y_1=x_1-1$ $y_1+x_2+x_3+x_4=98$ Number of ways where y_1,x_2,x_3,x_4 are non-empty $={}^{98+4-1}C_3=166650$ Consider $x_1\geq 1, x_2\geq 9$, (Class A has at least 1 and Class B has 9 or more) $(x_1-1)+(x_2-9)+x_3+x_4=89$ $y_1+y_2+x_3+x_4=89$ Number of ways $={}^{89+4-1}C_3=125580$ Required answer $=166650-125580=41070$		

8(b) (ii)	x_3 is odd and x_4 is even **or** x_4 is odd and x_3 is even $x_1+x_2+x_3+x_4=99$ $(2y_1+1)+(2y_2+1)+(2y_3+1)+2y_4=99$ $y_1+y_2+y_3+y_4=48$ Required answer $= {}^{48+4-1}C_3\times 2=20825\times 2=41650$		
8(b) (iii)	No. of ways $=S(4,1)+S(4,2)+S(4,3)+S(4,4)$ $=1+7+6+1$ $=15$		

8(c)

For a student, let p and q be the number of students in his class before and after orientation respectively. For every student i, we tag him with $\frac{1}{p_i}$ before orientation and $\frac{1}{q_i}$ after orientation. We also define $k_i = \frac{1}{q_i} - \frac{1}{p_i}$.

1. $k_i = \frac{1}{q_i} - \frac{1}{p_i} \leq 1 - \frac{1}{p_i} < 1$
2. Note that if $q < p \Rightarrow \frac{1}{q} > \frac{1}{p} \Rightarrow \frac{1}{q} - \frac{1}{p} > 0$, our goal is to show there are at least $b+1$ students with positive k_i.

Consider the sum of tags for all the students in a class of size n is $n\left(\frac{1}{n}\right) = 1$.

Hence the total sum of tags for all the students **<u>before</u>** orientation is the total number of classes $= a$

Similarly the total sum of tags for all the students **<u>after</u>** orientation $= a + b$

The difference in the total sums of tags before and after orientation $= a + b - a = b$

Now we consider total difference in tags for every student

$$= \sum\left(\frac{1}{q} - \frac{1}{p}\right) = \sum k_i = b$$

Suppose there are at most b students with positive k_i ,

Since $k_i < 1$, $\sum k_i < b(1) = b$. (Contradiction).

Hence, there are at least $b+1$ students with positive k_i i.e, who have fewer classmates after orientation than before orientation.

	Solution		
9(a)	Let P(n) be the statement $\frac{1\times3\times5\times...(2n-1)}{2\times4\times6\times...(2n)} < \frac{1}{\sqrt{2n+1}}$ for $n\in\mathbb{Z}^+$. Consider n = 1, $\frac{1}{2} < \frac{1}{\sqrt{3}}$ Therefore P(1) is true. Assume P(k) is true for some $k\in\mathbb{Z}^+$ i.e. $\frac{1\times3\times5\times...(2k-1)}{2\times4\times6\times...(2k)} < \frac{1}{\sqrt{2k+1}}$ To prove P(k+1) is true i.e. $\frac{1\times3\times5\times...(2k-1)(2k+1)}{2\times4\times6\times...(2k)(2k+2)} < \frac{1}{\sqrt{2k+3}}$ LHS : $\frac{1\times3\times5\times...(2k-1)(2k+1)}{2\times4\times6\times...(2k)(2k+2)} < \frac{1}{\sqrt{2k+1}}\times\frac{2k+1}{2k+2}$ $\frac{1}{\sqrt{2k+1}}\times\frac{2k+1}{2k+2} = \sqrt{\frac{2k+1}{(2k+2)^2}} = \sqrt{\frac{2k+1}{4k^2+8k+4}}$ -------(1) $\sqrt{\frac{2k+1}{4k^2+8k+4}} < \sqrt{\frac{2k+1}{4k^2+8k+3}}$ ------------------(2) $\sqrt{\frac{2k+1}{4k^2+8k+3}} = \sqrt{\frac{2k+1}{(2k+1)(2k+3)}} = \frac{1}{\sqrt{2k+3}}$ Therefore $\frac{1\times3\times5\times...(2k-1)(2k+1)}{2\times4\times6\times...(2k)(2k+2)} < \frac{1}{\sqrt{2k+3}}$ P(k+1) is true Since P(1) is true, P(k) is true implies P(k+1) is true, by PMI, P(n) is true for all $n\in\mathbb{Z}^+$.		

9(b) (i)	Based on Cauchy-Schwarz inequality $(1x_1+1x_2+1x_3+...+1x_{10})^2 \le (1^2+1^2+...+1^2)(x_1{}^2+x_2{}^2+x_3{}^2+...+x_{10}{}^2)$ $(x_1+x_2+x_3+...+x_{10})^2 \le 10(x_1{}^2+x_2{}^2+...+x_{10}{}^2)$ $10^2 \le 10\times 10$ Here the equal sign is valid for the inequality, Therefore $x_1=x_2=...=x_{10}=1$ is the only solution.		
9(b) (ii)	Based on Cauchy-Schwarz inequality $(1x_2+1x_3+...+1x_{10})^2 \le (1^2+1^2+...+1^2)(x_2{}^2+x_3{}^2+...+x_{10}{}^2)$ $(x_2+x_3+...+x_{10})^2 \le 9(x_2{}^2+x_3{}^2+...+x_{10}{}^2)$ $(10-x_1)^2 \le 9(11-x_1{}^2)$ ------------------(1) Simplify to $10x_1{}^2-20x_1+1\le 0$ $\frac{20-\sqrt{360}}{20} \le x_1 \le \frac{20+\sqrt{360}}{20}$ $\frac{10-3\sqrt{10}}{10} \le x_1 \le \frac{10+3\sqrt{10}}{10}$ If $x_1=\frac{10-3\sqrt{10}}{10}$, For the Cauchy-Schwarz inequality to take the equal sign, $x_2=x_3=...x_{10}=\frac{10-\frac{10-3\sqrt{10}}{10}}{9}=\frac{90+3\sqrt{10}}{90}=\frac{30+\sqrt{10}}{30}$		

9(c)	To prove $ab+bc+ac \le \frac{3}{2}$ We can change the above inequality to $ab+bc+ac = \frac{(a+b+c)^2-(a^2+b^2+c^2)}{2} \le \frac{3}{2}$ $(a+b+c)^2 \le (a^2+1+b^2+1+c^2+1)$ -----------(1) If let $x=a^2+1, \quad y=b^2+1, \quad z=c^2+1$ $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2 \quad \Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$ Inequality (1) will become: $\left(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\right)^2 \le (x+y+z)$ $\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1} \le \sqrt{(x+y+z)}$ ----------------(2) Now we try to prove (2) Since $\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=3-\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=3-2=1$ $\left(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\right)^2=\left(\sqrt{x}\sqrt{\frac{x-1}{x}}+\sqrt{y}\sqrt{\frac{y-1}{y}}+\sqrt{z}\sqrt{\frac{z-1}{z}}\right)^2$ By Cauchy-Schwarz inequality $\left(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\right)^2 \le (x+y+z)\left(\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}\right)$ $\left(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\right)^2 \le (x+y+z)$ $\left(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\right) \le \sqrt{(x+y+z)}$ Therefore inequality (2) is true, inequality (1) is equivalently true as well. Therefore $ab+bc+ac \le \frac{3}{2}$		