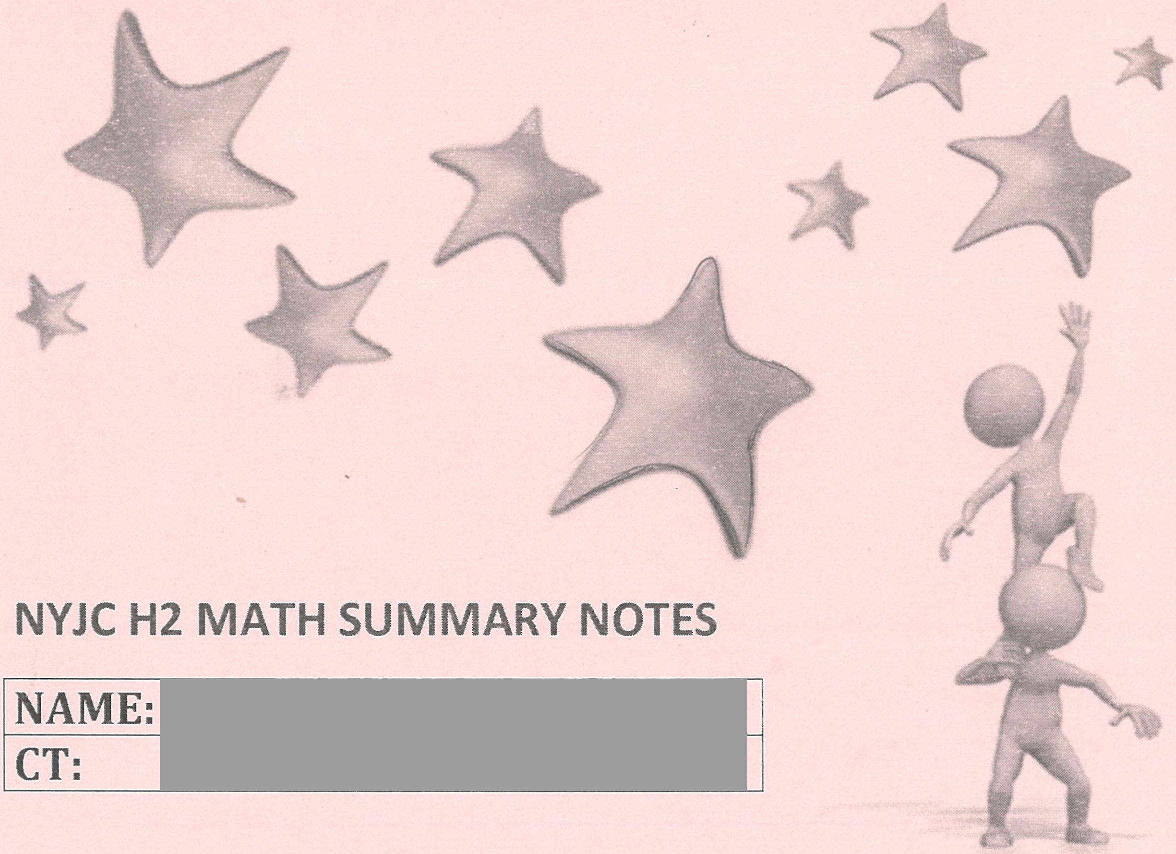

NYJC H2 MATH SUMMARY NOTES

NAME:	
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Contents

Section A: Pure Mathematics

Chapter 1: Vectors

1. Vectors (I) – Collinearity, Scalar and Vector Products, Area of Triangle and the Ratio Theorem

(i) Collinearity:

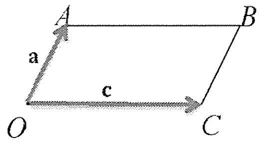
Three distinct points A, B and C are collinear if and only if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some real scalar k.
Do note that the equation must have a common point, in this case, the point B.

(ii) Scalar and Vector Products

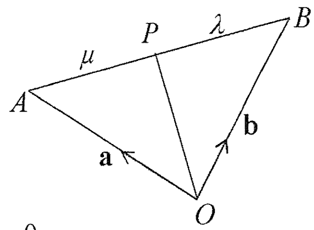
Scalar Product	Vector Product
$\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$, θ is the angle between the vectors **a** and **b** (scalar product to find the angle between two vectors)	$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$, θ is the angle between the vectors **a** and **b**
$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
If **a** and **b** are non-zero vectors, then $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$	If **a** and **b** are non-zero vectors, then $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{a} = \lambda\mathbf{b}, \quad \lambda \in \mathbb{R}$
$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$	$\mathbf{a} \times \mathbf{a} = \mathbf{0}$

(iii) Area of parallelogram $OABC$ is $|\mathbf{a}\times\mathbf{c}|$

(iv) Area of triangle OAC is $\frac{1}{2}|\mathbf{a}\times\mathbf{c}|$



(v) Ratio theorem: $\overrightarrow{OP}=\dfrac{\lambda\mathbf{a}+\mu\mathbf{b}}{\lambda+\mu}$



(vi) If **a** and **b** are non-zero and non-parallel vectors, then $\lambda\mathbf{a}=\mu\mathbf{b}\Rightarrow\lambda=\mu=0$

(vii) If **a** and **b** are non-zero and non-parallel vectors then $\lambda_1\mathbf{a}+\mu_1\mathbf{b}=\lambda_2\mathbf{a}+\mu_2\mathbf{b}\Rightarrow\lambda_1=\lambda_2$ and $\mu_1=\mu_2$

2. Vectors (II) – Lines

(i) Equation of a line: Equation of a line in

- Vector form: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{m},\ \lambda\in\mathbb{R}$ where $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix},\ \mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\ \mathbf{m}=\begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix}$.

  Note: a and **r** are position vectors of points on the line while **m** is a direction vector of the line

- Parametric form: $x=a_1+\lambda m_1,\ \ y=a_2+\lambda m_2,\ \ z=a_3+\lambda m_3,\ \lambda\in\mathbb{R}$
- Cartesian form: $\dfrac{x-a_1}{m_1}=\dfrac{y-a_2}{m_2}=\dfrac{z-a_3}{m_3}\ \ (=\lambda)$

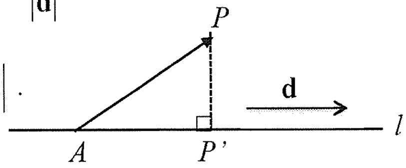
(ii) Equation of a line passing through two points A and B:

l: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}, \lambda\in\mathbb{R}$ where $\mathbf{d}=\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$

Special case: Line passing through O has the form $\mathbf{r}=\lambda\mathbf{d}, \lambda\in\mathbb{R}$ where **d** is the direction vector of the line.

(iii) Length of projection of $\overrightarrow{AP}$ onto l $= AP' = \left|\overrightarrow{AP}\bullet\hat{\mathbf{d}}\right|$ where $\hat{\mathbf{d}}=\dfrac{\mathbf{d}}{|\mathbf{d}|}$ is a unit direction vector of l

The perpendicular distance from P to the line $l = PP' = \left|\overrightarrow{AP}\times\hat{\mathbf{d}}\right|$.



(iv) One line and one point

(a) Foot of the Perpendicular from a point P to a Line l: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}, \lambda\in\mathbb{R}$

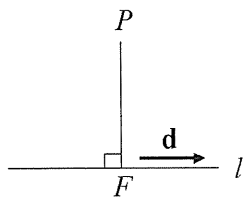
Method:

Step 1: F lies on the line $\Leftrightarrow$ $\overrightarrow{OF}=\mathbf{a}+\lambda\mathbf{d}$ for some $\lambda\in\mathbb{R}$.

Step 2: $\overrightarrow{PF}\perp\mathbf{d}$ $\Rightarrow$ $\overrightarrow{PF}\bullet\mathbf{d}=0$.

Step 3: Solve the above equation for λ.

Step 4: Substitute λ into $\overrightarrow{OF}=\mathbf{a}+\lambda\mathbf{d}$ to obtain $\overrightarrow{OF}$.

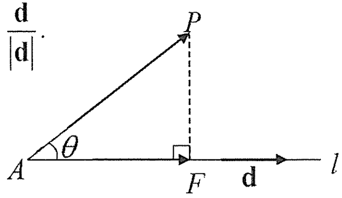


(b) Perpendicular Distance from a Point P to a Line l: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \lambda \in \mathbb{R}$

Method 1:

In triangle PAF, PF is the opposite side to angle θ.

$\therefore$ the perpendicular distance PF is $AP\sin\theta = \left|\overrightarrow{AP} \times \hat{\mathbf{d}}\right|$ where $\hat{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|}$.



Method 2:

Find $AF = |\overrightarrow{AP} \cdot \hat{\mathbf{d}}|$ and apply Pythagoras Theorem

Method 3:

Step 1: Find F the foot of the perpendicular from point P to the line as in **(a)** above.

Step 2: Find $\left|\overrightarrow{PF}\right|$.

(c) Point of reflection of a Point P in a Line l: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \lambda \in \mathbb{R}$

Method:

Step 1: Find F the foot of the perpendicular from point P to the line as in **(a)**.

Step 2: Apply Ratio Theorem to find the position vector of the image P' using

$\overrightarrow{OF} = \frac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$ where F is the mid-point of the line segment joining P and P'.

(v) Two lines:

(a) Parallel, intersecting and skew lines

For two distinct lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$, $\mathbf{b} \neq \mathbf{d}$

<u>Case 1: Parallel and do not intersect</u>

If $\mathbf{b} = k\mathbf{d}$ for some $k \in \mathbb{R}$, then the lines are parallel and do not intersect.

Distance between parallel lines: Take any point P on a line and find its perpendicular distance to the other line as in **(b)** above.

<u>Case 2: Intersecting lines</u>

The lines are not parallel and intersect at one point. In this case, set $\mathbf{a} + \lambda\mathbf{b} = \mathbf{c} + \mu\mathbf{d}$.

This gives rise to a system of simultaneous equations in λ and μ. Use GC to solve for λ and μ, and then substitute into the equation of line to get the point of intersection.

<u>Case 3: Skew lines</u>

Non-parallel lines that do not intersect are called skew lines, i.e. the system of simultaneous equations are not consistent (equations have no solutions).

Note: For both skew and intersecting lines (i.e cases 2 and 3), we assume they intersect first. Equate the two position vectors $\mathbf{r}$ and solve for λ and μ. If solution exists, then they are intersecting, otherwise they are skew.

(b) Acute angle between 2 lines

The *acute* angle between 2 lines is $\theta = \cos^{-1}\left(\frac{|\mathbf{b} \bullet \mathbf{d}|}{|\mathbf{b}||\mathbf{d}|}\right)$.

Caution: The modulus applied to $\mathbf{b} \bullet \mathbf{d}$ forces the expression $\frac{|\mathbf{b} \bullet \mathbf{d}|}{|\mathbf{b}||\mathbf{d}|}$ to be non-negative so that θ is guaranteed acute.
However, in finding angle AOB (say) we *do not* apply the modulus to $\overrightarrow{OA} \bullet \overrightarrow{OB}$ since the angle can be obtuse.

3. Vectors (III) – Planes

(i) One plane:

- The equation of a plane which contains the point A with position vector $\mathbf{a}$ and having a normal vector $\mathbf{n}$ is given by $\mathbf{r} \bullet \mathbf{n} = D \ (= \mathbf{a} \bullet \mathbf{n})$

- Equation of a plane in
  - Scalar product form: $\mathbf{r} \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} = D$, where $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
  - Cartesian form: $ax + by + cz = D$
    If $D = 0$, the origin O lies on the plane.

- Parametric form: $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}_1 + \mu \mathbf{d}_2$ where **a** is the position vector of a particular point on the plane, $\mathbf{d}_1$ and $\mathbf{d}_2$ are two vectors parallel to the plane (sometimes called *direction vectors* parallel to the plane).
  Special case: Plane passing through origin O has equation $\mathbf{r} = \lambda \mathbf{d}_1 + \mu \mathbf{d}_2$

- **Using GC to check vector product of two non-parallel vectors a and b**

Using GC to check: **APPS** > **4:PlySmlt2** > **ENTER** > **2:SIMULT EQN SOLVER** > in the SIMULT EQN SOLVER MODE select **EQUATIONS 2** and **UNKNOWNS 3** > **NEXT** > key in the values of the two vectors and key in 0 in the last column > press **SOLVE**

Note: The vector product of two non-parallel direction vectors of a plane gives a normal vector to the plane. That is, $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$.

(ii) One plane and one point:

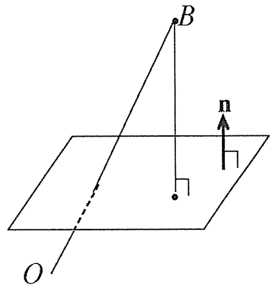
If a point B lies on the plane $\mathbf{r} \bullet \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = D$, then $\overrightarrow{OB} \bullet \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = D$.

(a) Perpendicular distance from the origin O to a plane

Given a plane with equation $\mathbf{r} \bullet \mathbf{n} = D$ (= $\mathbf{a} \bullet \mathbf{n}$), the perpendicular distance from O to the plane is $\dfrac{|D|}{|\mathbf{n}|}$.

(b) Perpendicular distance from a point B to a plane $\mathbf{r} \bullet \mathbf{n} = D$ is given by

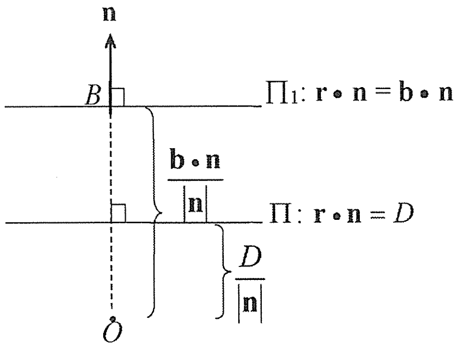
$$\frac{|\mathbf{b} \bullet \mathbf{n} - D|}{|\mathbf{n}|} \text{ or } \frac{|D - \mathbf{b} \bullet \mathbf{n}|}{|\mathbf{n}|}$$



OR Construct a plane Π_1 containing the point B and parallel to Π: $\mathbf{r} \bullet \mathbf{n} = D$

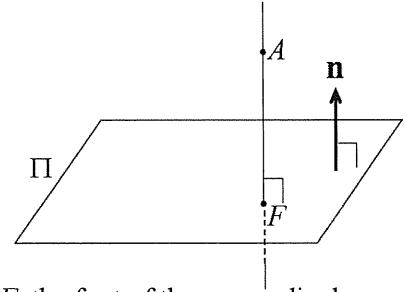
Then Π_1: $\mathbf{r} \bullet \mathbf{n} = \mathbf{b} \bullet \mathbf{n}$. (Compare with perpendicular distance between two parallel planes)

Perpendicular distance from a point B to a plane $\mathbf{r} \bullet \mathbf{n} = D$ is $\left| \frac{\mathbf{b} \bullet \mathbf{n}}{|\mathbf{n}|} - \frac{D}{|\mathbf{n}|} \right| = \frac{|\mathbf{b} \bullet \mathbf{n} - D|}{|\mathbf{n}|}$ as before.



(c) Foot of perpendicular of a point to a plane and the point of reflection

Method:
Let A be a point with position vector $\mathbf{a}$ and Π a plane with equation $\mathbf{r} \bullet \mathbf{n} = D$.
Let F be the foot of the perpendicular from A to Π.



To find the position vector of F, the foot of the perpendicular:
Note that F is the point of intersection of line AF and the plane Π.

Step 1: Since F lies on line AF, $\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{n}$ for some $\lambda \in \mathbb{R}$.

Step 2: F lies on the plane Π $\Rightarrow$ $\overrightarrow{OF} \bullet \mathbf{n} = D$ $\Rightarrow$ $(\mathbf{a} + \lambda\mathbf{n}) \bullet \mathbf{n} = D$ for some λ.
Solve the equation for λ.

Step 3: On knowing the value of λ, we can substitute λ into $\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{n}$ to find $\overrightarrow{OF}$.

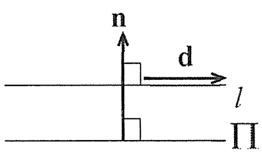
Step 4: After obtaining $\overrightarrow{OF}$, we can apply ratio theorem to find the reflection A' of A in Π using $\overrightarrow{OF} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$.

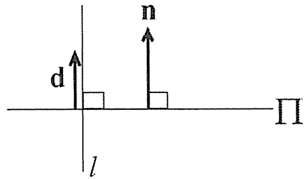
(iii) One plane and one line:

(a) Line is parallel/perpendicular to plane

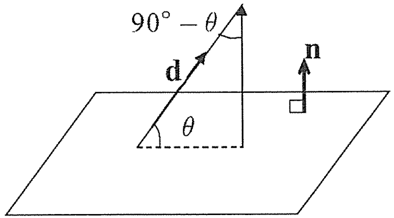
Given a line l with direction vector $\mathbf{d}$ and a plane Π with normal vector $\mathbf{n}$,

- the line and the plane are *parallel* if $\mathbf{n} \perp \mathbf{d}$, i.e. $\mathbf{n} \bullet \mathbf{d} = 0$;
- the line and the plane are *perpendicular* if $\mathbf{n} \parallel \mathbf{d}$, i.e. $\mathbf{n} = \lambda \mathbf{d}$ for some $\lambda \in \mathbb{R}$.





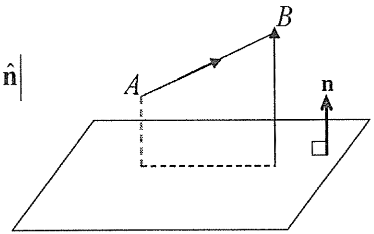
(b) Angle between a line and a plane



The acute angle θ between a line with direction vector $\mathbf{d}$ and a plane with normal vector $\mathbf{n}$ is given by:

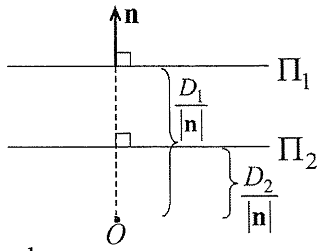
$$\sin\theta = \frac{|\mathbf{d} \bullet \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}\left(= \cos(90^\circ - \theta)\right).$$

(c) The length of projection of a vector $\overrightarrow{AB}$ onto a plane $\Pi = \left|\overrightarrow{AB} \times \hat{\mathbf{n}}\right|$



(iv) Two planes:

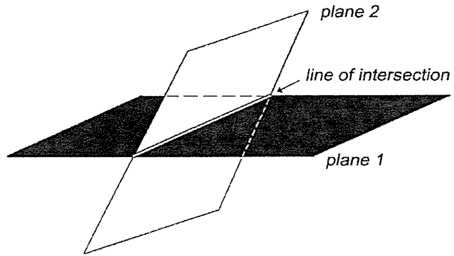
(a) Parallel planes and perpendicular distance between them



Given planes $\Pi_1: \ \mathbf{r} \bullet \mathbf{n} = D_1$ and $\Pi_2: \ \mathbf{r} \bullet \mathbf{n} = D_2$,
the distance between them is given by $\frac{|D_1 - D_2|}{|\mathbf{n}|}$.

Note: Both equations (in scalar product form) must have the same normal $\mathbf{n}$.

(b) Two non-parallel planes intersect at a line



Note: Two non-parallel planes must intersect along a line
To find the line of intersection of two non-parallel planes:

Method 1 (By GC):
If there are no unknowns in the equation of the planes, obtain the Cartesian equations of the planes and use GC: **APPS > 4:PlySmlt2 > ENTER > 2:SIMULT EQN SOLVER** to solve for the vector equation of the line.

Method 2 (By Calculation):

Step 1: Direction vector of the line is $\mathbf{n}_1 \times \mathbf{n}_2$ since it is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$.

Step 2: Find the position vector of a point, say $\mathbf{a}$ on the line, by letting $x = 0$ (or $y = 0$ or $z = 0$) in the Cartesian equations of the planes and solving simultaneously.

The vector equation of the line of intersection is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{n}_1 \times \mathbf{n}_2)$.

(c) Acute angle θ between two non-parallel planes are given by $\cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$

Note: The modulus applied to $\mathbf{n}_1 \bullet \mathbf{n}_2$ forces the expression $\dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$ to be non-negative so that θ is guaranteed acute.

Chapter 2: Curve Sketching

1. Graphing of rational function $y=\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$ where f(x) and g(x) are polynomials

A sketch of the curve $y=\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$ should convey the general shape of the curve, showing the following information:

- Asymptotes
- Intercepts on the axes
- Stationary points and their nature
- Symmetry (if any)

(i) How to find the equations of vertical asymptotes

Given a rational function $y=\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$, then $y=\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$ is undefined when $\mathrm{g}(x)=0$. The equations of vertical asymptotes are found by finding the roots of the equation $\mathrm{g}(x)=0$.

i.e. If $\alpha, \beta, \ldots$ are the roots of $\mathrm{g}(x)=0$, then $x=\alpha, x=\beta, \ldots$ are the equations of vertical asymptotes of the graph.

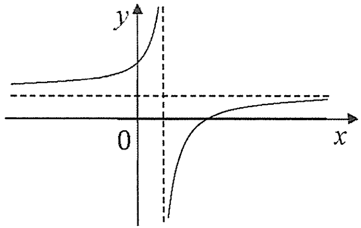
NOTE: $y=\ln x$ has $x=0$ as the vertical asymptote.

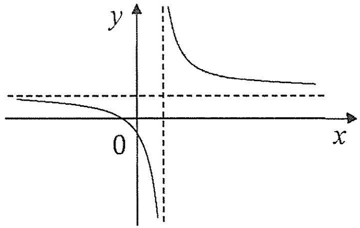
(ii) How to find the equation of oblique or horizontal asymptote

- If $y=\frac{\mathrm{f}(x)}{\mathrm{g}(x)}$ is proper [i.e. degree of f(x) < degree of g(x)] , then $y = 0$ is the horizontal asymptote.
- If $y=\frac{\mathrm{f}(x)}{\mathrm{g}(x)}$ is improper [i.e. degree of f(x) ≥ degree of g(x)] , first perform long division to change it to $y=\mathrm{p}(x)+\frac{\mathrm{q}(x)}{\mathrm{g}(x)}$, where $\frac{\mathrm{q}(x)}{\mathrm{g}(x)}$ is a proper fraction.
- If $\mathrm{p}(x)=k$, where k is a real constant, then $y=k$ is the horizontal asymptote.
- If $\mathrm{p}(x)=ax+b$, where a and b are real constants, then $y=ax+b$ is the oblique asymptote.

NOTE: $y=\mathrm{e}^x$ has $y=0$ as the horizontal asymptote.

Graph of $y=\frac{ax+b}{cx+d}$, $a\neq 0$, $c\neq 0$, $x\neq -\frac{d}{c}$, has two asymptotes **(1 vertical and 1 horizontal)**.
Some possible shapes of the graph are illustrated as follow:





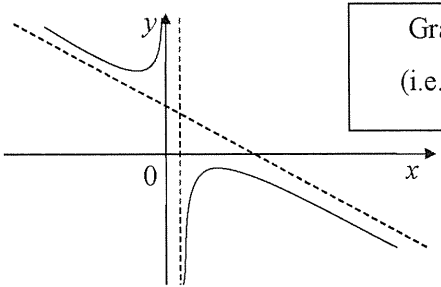
If $a=0$ but $b\neq 0$, then the horizontal asymptote is $y = 0$ (i.e. x-axis).

The vertical asymptote is $x=-\frac{d}{c}$. If $d=0$, then the vertical asymptote is $x=0$. (i.e. y-axis)

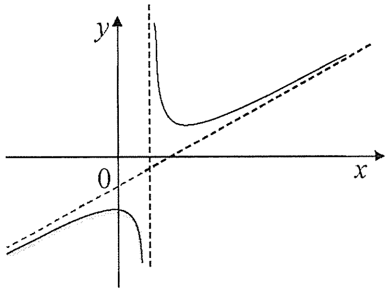
Graph of $y=\frac{ax^2+bx+c}{dx+e}$, $a\neq 0$, $d\neq 0$, $x\neq -\frac{e}{d}$, has two asymptotes **(1 vertical and 1 oblique)**.

Some possible shapes of the graph are illustrated as follow:

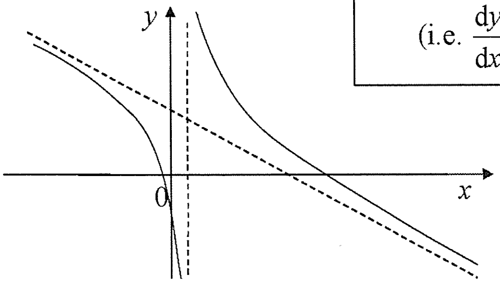
Case 1:



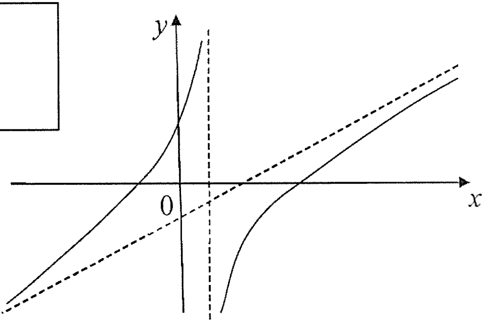
Graph has two turning points
(i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ has two real roots)



Case 2:



Graph has no turning point
(i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ has no real root)



(iii) How to find the range of values of y for Case 1

Rewrite the equation $y=\dfrac{ax^2+bx+c}{dx+e}$ into a quadratic equation of x. Solve the inequality $D \geq 0$ in terms of y.

2. Graph of $y = |f(x)|$ and $f(|x|)$

(i) Graph of $y = |f(x)|$

$y = |f(x)|$ is obtained from the graph of f(x) as follows:

Step 1: Retain the portion of the graph of f(x) on and above the x-axis (i.e. for $y \geq 0$).

Step 2: Reflect the portion of the graph of f(x) below the x-axis in the x-axis.

(ii) Graph of $y = f(|x|)$

$y = f(|x|)$ is from the graph of f(x) as follows:

Step 1: Remove the portion of the graph of f(x) on the left side of the y-axis (i.e. for $x < 0$).

Step 2: Retain the portion of the graph of f(x) on the right of the y-axis (i.e. for $x \geq 0$) and reflect it in the y-axis to get a resulting graph that is symmetrical about the y-axis.

(Tip: A way to remember the transformation is **"Remove left, Reflect right".**
Right side of the graph will still remain.)

3. Graphing of $y=\frac{1}{f(x)}$ from graph of $y=f(x)$

Step	$y=f(x)$	$y=\frac{1}{f(x)}$	Remarks
1	x-intercept at $(a,0)$	Vertical asymptote at $x=a$	
	Vertical asymptote at $x=a$	x-intercept at $(a,0)$	
2	Point (a, b)	Point $\left(a,\frac{1}{b}\right)$	Important: $b \neq 0$ Take reciprocal of all the values of y of the graph. Note that if $b > 0$, then $\frac{1}{b} > 0$ & if $b < 0$, then $\frac{1}{b} < 0$.
	Maximum point (a, b)	Minimum point $\left(a,\frac{1}{b}\right)$	
	Minimum point (a, b)	Maximum point $\left(a,\frac{1}{b}\right)$	
	Horizontal asymptote: $y=b$	Horizontal asymptote: $y=\frac{1}{b}$	
	Oblique asymptote: $y=ax+b$	Horizontal asymptote $y=0$	

3	$f(x) \to 0^+$ $f(x) \to 0^-$	$\frac{1}{f(x)} \to +\infty$ $\frac{1}{f(x)} \to -\infty$	Use the fact: when $f(x) \to 0$, then $\frac{1}{f(x)} \to \infty$; when $f(x) \to \infty$, then $\frac{1}{f(x)} \to 0$.
4	$f(x) > 0$ $f(x) < 0$	$\frac{1}{f(x)} > 0$ $\frac{1}{f(x)} < 0$	If the graph of $y = f(x)$ lies above (or below) the x-axis for $c < x < d$, then the graph of $y = \frac{1}{f(x)}$ will lie above (or below) the x-axis for $c < x < d$.
5	$f(x)$ is increasing $f(x)$ is decreasing	$\frac{1}{f(x)}$ is decreasing $\frac{1}{f(x)}$ is increasing	

Note: Steps 4 and 5 are more for checking that graph is correct.

4. Linear Transformation

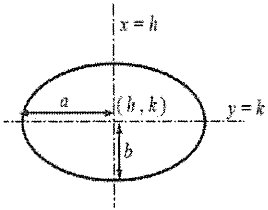
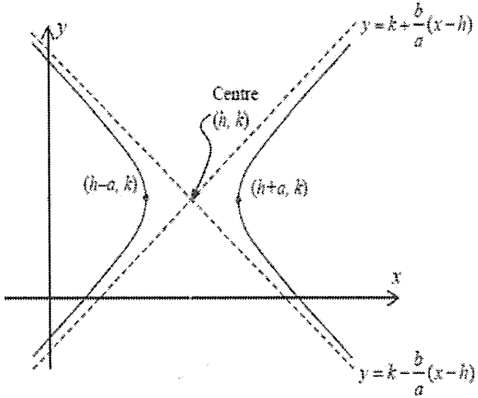
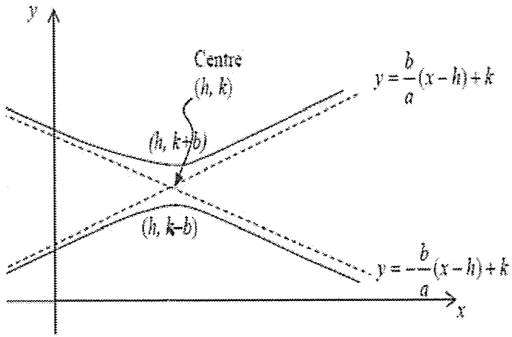
Given $y = \mathrm{f}(x)$ and $a > 0$.

Type	Equation	Description of transformation
T_x	$y = \mathrm{f}(x-a)$	**Translation** of $y = \mathrm{f}(x)$ by ***a* units** in the direction of the ***x*-axis**
	$y = \mathrm{f}(x+a)$	**Translation** of $y = \mathrm{f}(x)$ by $-$***a* units** in the direction of the ***x*-axis**
T_y	$y = \mathrm{f}(x)+a$	**Translation** of $y = \mathrm{f}(x)$ by ***a* units** in the direction of the ***y*-axis**
	$y = \mathrm{f}(x)-a$	**Translation** of $y = \mathrm{f}(x)$ by $-$***a* units** in the direction of the ***y*-axis**
S_x	$y = \mathrm{f}(ax)$	**Scaling** of $y = \mathrm{f}(x)$ by **factor** $\frac{1}{a}$ parallel to the ***x*-axis**
	$y = \mathrm{f}(-x)$ (i.e. when $a = -1$)	**Reflection** of $y = \mathrm{f}(x)$ in the ***y* -axis**
S_y	$y = a\mathrm{f}(x)$	**Scaling** of $y = \mathrm{f}(x)$ by **factor *a*** parallel to the ***y*-axis**
	$y = -\mathrm{f}(x)$ (i.e. when $a = -1$)	**Reflection** of $y = \mathrm{f}(x)$ in the ***x* -axis**

Recommended order of Transformation:

$$\mathrm{T}_x \mathrm{S}_x \mathrm{S}_y \mathrm{T}_y$$

Conics

Ellipse	Hyperbola	
$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$	$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$	$\dfrac{(y-k)^2}{b^2}-\dfrac{(x-h)^2}{a^2}=1$
		
When $a=b$, it will be a **circle**.	To find the equation of the asymptotes for the hyperbolas, let $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=0$ and make y the subject.	

Note: <u>For ellipse</u>, label the coordinates of the centre and the length of the semi-major and semi-minor axis.
<u>For hyperbola</u>, label the coordinates of the centre, the vertices, and equations of the asymptotes.

Chapter 3: Functions

1. Definition

A function f is a rule that maps elements of X to elements of Y such that every element of X is mapped to a unique element of Y. We denote this by f: $X \mapsto Y$.

A function is well-defined if and only if it has *both* a **rule** and a **domain.**

The set X is called the domain of f, denoted by D_f and the set Y (graphically, it is the y-coordinates of the graph of the function $y = f(x)$) is called the range of f, denoted by R_f.

Note:
1) D_f and R_f are both sets so must be presented in **set notation**.
2) Use a graph to find the range of a function

When sketching graphs for functions, 1) only sketch for the required domain
2) take note of the endpoints (use ● if point is included,
and ○ if point is excluded).

2. Inverse Function

(i) Condition

A function f has an inverse, denoted by f^{-1}, if and only if f is **1-1.**

(ii) To test for 1-1: Horizontal line test

To show that a function is **not** one-one, state and provide one horizontal line $y = k$, where $k \in \mathbb{R}$, that intersects the graph more than once. Use a **specific** value for k.

Eg: Since the line $y =$_____intersects the graph of $y = f(x)$ **more than once**, f is not one-one.

(iii) To find f^{-1}, we set $y = f(x)$ and write x in terms of y. Then we use $f^{-1}(y) = x$.

(iv) When f^{-1} exists,

- $D_{f^{-1}} = R_f$, $R_{f^{-1}} = D_f$,
- The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.
- $f^{-1}f(x) = x$, $x \in D_f$,

  $ff^{-1}(x) = x$, $x \in D_{f^{-1}}$,

  $(f^{-1})^{-1}(x) = f(x)$
- To solve $f(x) = f^{-1}(x)$, often suffice to solve $f(x) = x$

3. Composite Function

(i) Condition

The composite function fg exists if and only if $R_g \subseteq D_f$.

(ii) If fg is a composite function, then $D_{fg} = D_g$ and $R_{fg} \subseteq R_f$

(iii) To find R_{fg},

1. Sketch the graph of $y = g(x)$ to determine R_g based on the given domain.
2. Get a sketch of $y = f(x)$
3. Apply the range of g (R_g) onto the graph of $y = f(x)$ as the 'domain'. The corresponding set of y values is R_{fg}.

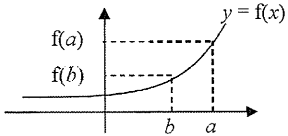
Chapter 4: Inequalities and Equations

1. Rules for Manipulating Inequalities

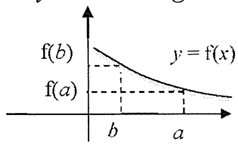
- Multiplying/dividing both sides of an inequality by a positive real number *preserves* the inequality sign
- Do not "cross-multiply"/ "cancel" a term on both sides unless you are sure that the term is positive.
- Given $a > b$,

  $f(a) > f(b)$, inequality sign *unchanged* if f is a *strictly increasing* function,

  $f(a) < f(b)$, inequality sign *changed* if f is a *strictly decreasing* function.



$a > b$ and $f(a) > f(b)$



$a > b$ and $f(a) < f(b)$

Example: Let $f(x) = e^x$

- If $a > b$,

  then $e^a > e^b$,

  and $\ln a > \ln b$

Example: Let $f(x) = \frac{1}{x}$

- If $a > b$, $a, b \neq 0$,

  then $\frac{1}{a} < \frac{1}{b}$.

2. General steps in solving polynomial inequalities or inequalities involving rational functions

(i) One side of the inequality must be 0: $f(x) > 0$ or $\frac{g(x)}{h(x)} > 0$.

(ii) Factorize $f(x)$, $g(x)$ and $h(x)$ into linear factors where possible.

- If a quadratic expression cannot be factorized, complete the square.
- After completing the square, if the quadratic expression is always positive, e.g. $(x+1)^2 + 3$, divide both sides of the inequality by the quadratic expression to get rid of it.
- If not always positive, factorise using the "difference of square" formula i.e. $a^2 - b^2 = (a+b)(a-b)$.
  E.g. $(x+1)^2 - 3 = \left(x+1+\sqrt{3}\right)\left(x+1-\sqrt{3}\right)$.

(iii) Once all factors are linear, apply test point method.

(iv) For fractions, remember to exclude from the final answer those values for which the denominator is zero.

3. Properties of the Modulus Function

For $x, y \in \mathbb{R}$,	For $x, y \in \mathbb{R}$, and $a > 0$
(i) $\lvert x^2 \rvert = \lvert x \rvert^2 = x^2$	(i) $\lvert x \rvert < a \Leftrightarrow -a < x < a$
(ii) $\lvert xy \rvert = \lvert x \rvert \lvert y \rvert$	(ii) $\lvert x \rvert > a \Leftrightarrow x < -a$ or $x > a$
(iii) $\left\lvert \frac{x}{y} \right\rvert = \frac{\lvert x \rvert}{\lvert y \rvert}$, provided $y \neq 0$	(iii) $\lvert x \rvert < \lvert y \rvert \Leftrightarrow x^2 < y^2$
(iv) $\sqrt{x^2} = \lvert x \rvert$	(iv) $\lvert x - y \rvert = \lvert y - x \rvert$

4. Inequalities involving the Modulus Function

(i) $|f(x)| >$ constant or $|f(x)| <$ constant

Example:
- $|1+x| < 7 \Rightarrow -7 < 1+x < 7$
- $|1+x| > 7 \Rightarrow 1+x < -7$ <u>or</u> $1+x > 7$
- $|1+x| < -7 \Rightarrow$ no solution
- $|1+x| > -7 \Rightarrow x \in \mathbb{R}$
- $-3 < |1+x| < 4 \Rightarrow |1+x| < 4 \Rightarrow -4 < 1+x < 4$

(ii) $|f(x)| > g(x)$

Solve using graphical method.

(iii) $|f(x)| > |g(x)|$

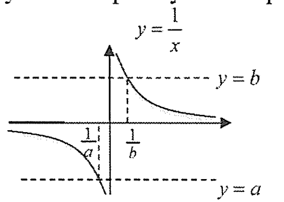
Either 1) Solve using graphical method
Or 2) Square both sides of the inequality

(iv) $f(|x|) > 0$

Solve $f(x) > 0$ and then substitute x with $|x|$ to solve for final answer.

Note: Unless specifically prohibited, GC can be used to solve the inequality using a graphical approach.

5. Some useful results

If $a>0$, then • $a^x>0 \Rightarrow x\in \mathbb{R}$ • $a^x\le 0 \Rightarrow$ no solution	If $a>1$, then • $\log_a x\ge 0 \Rightarrow x\ge 1$ • $\log_a x<0 \Rightarrow 0<x<1$	• $\frac{1}{x}>0 \Rightarrow x>0$ • $\frac{1}{x}<0 \Rightarrow x<0$ • $\frac{1}{x}<a$, $\frac{1}{x}>b$, or $a<\frac{1}{x}<b$ (a and b can be positive or negative). Use graph to determine the interval(s) of x that satisfy the inequality. Example: 

Remember to take into account the domain for which the function is valid.
Example: $\sqrt{x} \Rightarrow x\ge 0$, $\ln x \Rightarrow x>0$.

6. Solving Practical Problems Using Systems of Linear Equations

(i) *Define the variables* based on the context of the question.

(ii) *Formulate* the systems of linear equations.

(iii) Key the equations into the GC (under APPS, Plysmlt2, Simultaneous Eqn Solver).

(iv) Give answers *in context.*

Chapter 5: Sequences and Series

1. Key ideas, glossary and formulae:

(i) Sequence – a list (usually of numbers), in *a definite order.*

1,	3,	7,	15,	31, …			
↕	↕	↕	↕	↕			
u_1,	u_2,	u_3,	u_4,	u_5, …	u_k,	u_{k+1}, …	u_n, …

u_1, u_2, u_3, ... is used to denote a sequence; u_k (or u_n or u_r) is used to denote the general k^{th} term (or n^{th} or r^{th} term respectively) of the sequence.

(ii) Series – a *sum of terms of a sequence*

$1+3+7+15+31$ (finite series)

$1+0.1+0.01+0.001+\ldots.$ (infinite series)

$\sum_{r=1}^{n} u_r$ or S_n is used to denote the series $u_1+u_2+u_3+\cdots+u_n$,

i.e. sum of the *first n terms* of a sequence.

$\sum_{r=1}^{\infty} u_r$ or S_∞ is used to denote the series $u_1+u_2+u_3+\cdots$,

i.e. sum to infinity of a sequence.

(iii) For all sequences: if S_n is given, then u_n can be found using $\boldsymbol{u_n = S_n - S_{n-1}}$.

(iv) $S_\infty = \lim_{n\to\infty} S_n$, if S_∞ exists. In general, to show a series (not restricted to GP) converges, we show that S_∞ has a finite value.

2. Key results for AP and GP

AP	GP
Each term is obtained from the earlier one by **adding a common difference (d),** the first term is usually denoted by a. $a,\ a+d,\ a+2d,\ \ldots,\ a+(n-1)d,\ \ldots$	Each term is obtained from the earlier one by **multiplying by a common ratio (r),** the first term is usually denoted by a. $a,\ ar,\ ar^2,\ \ldots,\ ar^{n-1},\ \ldots$
n^{th} term, $u_n = a+(n-1)d$	n^{th} term, $u_n = ar^{n-1}$
Sum of first n terms, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ or $S_n = \frac{n}{2}(a + l)$ (use the second result when first term a and last term l are known)	Sum of first n terms, $S_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{a\left(r^n-1\right)}{r-1},\ r \neq 1$ If $r = 1$, S_n = first term × no. of terms
Result applies for $S_n = \sum_{r=1}^{n} r = \frac{n}{2}(1+n)$, where common difference = 1	Result applies for $S_n = \sum_{r=1}^{n} a^r = \frac{a(a^n-1)}{a-1}$, where common ratio = $a \neq 1$.
For an AP, $\sum_{r=n}^{m} U_r = \frac{1}{2}(m-n+1)(U_n+U_m)$ where $m-n+1$ is the number of terms in the sum	For a GP, $\sum_{r=n}^{m} U_r = \frac{U_n(1-r^{m-n+1})}{1-r}$ where $m-n+1$ is the number of terms in the sum

Sum to infinity, S_∞, does not exist for AP.	$S_\infty = \begin{cases} \dfrac{a}{1-r}, & \text{if } \lvert r\rvert < 1 \text{ or } -1<r<1 \\ \text{diverges}, & \text{if } \lvert r\rvert > 1 \text{ and } a \neq 0 \end{cases}$ An infinite geometric series is said to be convergent if $\lvert r\rvert < 1$; we also say the sum to infinity S_∞ exists when $\lvert r\rvert < 1$.
To show sequence is an AP: Show that for all $n \in \mathbb{Z}^+$ $u_{n+1} - u_n$ is a constant ($=d$)	To show sequence is a GP: Show that for all $n \in \mathbb{Z}^+$ $\dfrac{u_{n+1}}{u_n}$ is a constant ($= r$)

3. Properties and Standard Results of Summation

(i) Properties of Summation

For any two integers m, n such that $m \le n$, and for any constant a,

1. $\sum_{r=m}^{n}[\mathrm{f}(r) \pm \mathrm{g}(r)] = \sum_{r=m}^{n}\mathrm{f}(r) \pm \sum_{r=m}^{n}\mathrm{g}(r)$
2. $\sum_{r=m}^{n}[a\mathrm{f}(r)] = a\left[\sum_{r=m}^{n}\mathrm{f}(r)\right]$
3. $\sum_{r=m}^{n}\mathrm{f}(r) = \sum_{r=1}^{n}\mathrm{f}(r) - \sum_{r=1}^{m-1}\mathrm{f}(r)$ (for $m > 1$) where $\mathrm{f}(r)$ and $\mathrm{g}(r)$ are functions of r.

(ii) Standard Results of Summation

- $\sum_{r=m}^{n} a = (n-m+1)a$ where a is a constant, $n \geq 1$ and $m \leq n$.

- $\sum_{r=1}^{n} r = 1+2+3+\ldots+n = \frac{n}{2}(1+n)$ **(AP)**

  note that lower limit must be 0 or 1 to apply this formula

  $\sum_{r=m}^{n} r = m+(m+1)+(m+2)+\ldots+n = \frac{(n-m+1)}{2}(m+n)$ **(AP)**

- $\sum_{r=1}^{n} a^r = a+a^2+a^3+\ldots+a^n = \frac{a(1-a^n)}{1-a}$ **(GP)**

  note that lower limit must be 1 to apply this formula

  $\sum_{r=m}^{n} a^r = a^m+a^{m+1}+a^{m+2}+\ldots+a^n = \frac{a^m(1-a^{n-m+1})}{1-a}$ **(GP)**

The following results will be provided in the question if you are required to use them.

- $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$

  note that lower limit must be 0 or 1 to apply these two formulae.

- $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{r=1}^{n} r\right)^2$

4. Method of Difference

$$\sum_{k=1}^{n}(u_k - u_{k+1}) = \begin{array}{rccc} & u_1 & - & u_2 \\ + & u_2 & - & u_3 \\ & & \vdots & \\ + & u_n & - & u_{n+1} \end{array} = u_1 - u_{n+1}$$

Note: Cancellation might not happen with successive rows.

(i) Common forms:

- Partial Fractions
- Trigonometric Functions
- Factorial

(ii) Extension of Questions:

- Use of translation to find another sum. For example, if we replace r by $k+1$,

  $$\sum_{r=2}^{r=n}\frac{r-1}{(2r-1)(2r+1)} = \sum_{k+1=2}^{k+1=n}\frac{(k+1)-1}{(2(k+1)-1)(2(k+1)+1)} = \sum_{k=1}^{n-1}\frac{k}{(2k+1)(2k+3)}.$$

  Remember to change corresponding lower and upper limit.
- Find S_∞ by letting $n \to \infty$. Justify the terms involving n will converge to a finite value.
- Establish inequality for 2 sums. Key idea: establish inequality for the general terms of the sums involved.

Chapter 6: Differentiation and Applications

	Basic	General
Powers of x	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right)=nx^{n-1}$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]^n=n[\mathrm{f}(x)]^{n-1}\cdot \mathrm{f}'(x)$
Trigonometric Functions (angle x is in radians)	$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x$ $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)=-\sin x$ $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x)=\sec^2 x$ $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x)=\sec x\tan x$ **(MF26)** $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$ **(MF26)** $\frac{\mathrm{d}}{\mathrm{d}x}(\cot x)=-\operatorname{cosec}^2 x$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin \mathrm{f}(x)\right]=\cos \mathrm{f}(x)\cdot \mathrm{f}'(x)$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos \mathrm{f}(x)\right]=-\sin \mathrm{f}(x)\cdot \mathrm{f}'(x)$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan \mathrm{f}(x)\right]=\sec^2 \mathrm{f}(x)\cdot \mathrm{f}'(x)$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sec \mathrm{f}(x)\right]=\sec \mathrm{f}(x)\tan \mathrm{f}(x)\cdot \mathrm{f}'(x)$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{cosec} \mathrm{f}(x)\right]=-\operatorname{cosec} \mathrm{f}(x)\cot \mathrm{f}(x)\cdot \mathrm{f}'(x)$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cot \mathrm{f}(x)\right]=-\operatorname{cosec}^2 \mathrm{f}(x)\cdot \mathrm{f}'(x)$
Inverse Trigonometric Functions (MF26)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right)=\frac{1}{\sqrt{1-x^2}},\ -\frac{\pi}{2}\le \sin^{-1} x\le \frac{\pi}{2},\ -1\le x\le 1$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1} x\right)=-\frac{1}{\sqrt{1-x^2}},\ 0\le \cos^{-1} x\le \pi,\ -1\le x\le 1$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right)=\frac{1}{1+x^2},\ -\frac{\pi}{2}< \tan^{-1} x< \frac{\pi}{2},\ x\in \mathbb{R}$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^{-1} \mathrm{f}(x)\right]=\frac{1}{\sqrt{1-\mathrm{f}(x)^2}}\cdot \mathrm{f}'(x)$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos^{-1} \mathrm{f}(x)\right]=-\frac{1}{\sqrt{1-\mathrm{f}(x)^2}}\cdot \mathrm{f}'(x)$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1} \mathrm{f}(x)\right]=\frac{1}{1+\mathrm{f}(x)^2}\cdot \mathrm{f}'(x)$

Logarithmic Functions	$\frac{\mathrm{d}}{\mathrm{d}x}\ln x=\frac{1}{x}$ $\frac{\mathrm{d}}{\mathrm{d}x}\log_a x=\frac{1}{x\ln a}=\frac{1}{x}\log_a \mathrm{e}$	$\frac{\mathrm{d}}{\mathrm{d}x}\ln \mathrm{f}(x)=\frac{1}{\mathrm{f}(x)}\cdot \mathrm{f}'(x)$ $\frac{\mathrm{d}}{\mathrm{d}x}\log_a \mathrm{f}(x)=\frac{1}{\mathrm{f}(x)\ln a}\cdot \mathrm{f}'(x)$
Exponential Functions	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x=\mathrm{e}^x$ $\frac{\mathrm{d}}{\mathrm{d}x}a^x=a^x\ln a$	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{f}(x)}=\mathrm{e}^{\mathrm{f}(x)}\cdot \mathrm{f}'(x)$ $\frac{\mathrm{d}}{\mathrm{d}x}a^{\mathrm{f}(x)}=a^{\mathrm{f}(x)}\ln a\cdot \mathrm{f}'(x)$

1. Rules of Differentiation

(1) $\frac{\mathrm{d}}{\mathrm{d}x}\left(k\mathrm{f}(x)\right)=k\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)]$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{f}(x)\pm \mathrm{g}(x)\right)=\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)]\pm\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{g}(x)]$

(3) **Product Rule**: $\frac{\mathrm{d}}{\mathrm{d}x}(uv)=u\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}x}$

(4) **Quotient Rule**: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)=\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

(5) **Chain Rule**: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$

(6) **Inverse/Reciprocal**: $\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{1}{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}$

2. Implicit Differentiation

$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(y)=\mathrm{f}'(y)\frac{\mathrm{d}y}{\mathrm{d}x}$

3. Logarithmic Differentiation

This method is used for differentiating

- expression of the form u^v, where u, v are functions of x, e.g. $x^{\sin x}$
- complicated products and quotients

Steps involved: **simplify logarithms and then differentiate w.r.t. x**.

4. Parametric Differentiation

The **derivative of a curve** defined parametrically by $x=\mathrm{f}(t)$ and $y=\mathrm{g}(t)$ is given by $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\Big/\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{g}'(t)}{\mathrm{f}'(t)}$

5. Sign of First Derivative

$\frac{\mathrm{d}y}{\mathrm{d}x}>\mathbf{0}$ (+ve gradient) on an interval $\Rightarrow y$ is **increasing**

$\frac{\mathrm{d}y}{\mathrm{d}x}<\mathbf{0}$ (–ve gradient) on an interval $\Rightarrow y$ is **decreasing**

$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathbf{0}$ (zero gradient) at $x=a \Rightarrow y$ is **stationary at that point**

APPLICATIONS OF DIFFERENTIATION

1. Concavity

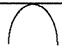
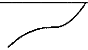
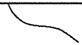
$\frac{d^2 y}{dx^2} > 0$ on an interval $\Rightarrow y$ is **concaving upwards**

$\frac{d^2 y}{dx^2} < 0$ on an interval $\Rightarrow y$ is **concaving downwards**

2. Stationary Points

To Check Nature of Stationary Points

First Derivative Test (provide the values of a^-, a, a^+)

x	a^-	a	a^+	Nature of Stationary Point
$\frac{dy}{dx}$	>0	0	<0	 Maximum Point
$\frac{dy}{dx}$	<0	0	>0	 Minimum Point
$\frac{dy}{dx}$	>0	0	>0	 Stationary Point of Inflexion
$\frac{dy}{dx}$	<0	0	<0	 Stationary Point of Inflexion

Second Derivative Test

$\frac{d^2 y}{dx^2}$	Nature of Stationary Point
>0	Minimum
<0	Maximum
$=0$	No conclusion. Use first derivative test to check.

3. Maximum and Minimum

1. Identify variables (draw diagram if possible)
2. Form the equations involving these variables
3. Express dependent variable in terms of **one variable** only
4. Differentiate to find the stationary values
5. Determine if these values are maximum or minimum.

4. Tangents and Normals

Two lines with gradients m_1 and m_2 are parallel (perpendicular) if $m_1 = m_2$ ($m_1 m_2 = -1$).

Equation of tangent to the curve at the point (x_o, y_o),

$y - y_o = f'(x_o)(x - x_o)$, where $f'(x_o) = \left.\frac{dy}{dx}\right|_{(x_o, y_o)}$

Equation of normal to the curve at the point (x_o, y_o),

$y - y_o = -\frac{1}{f'(x_o)}(x - x_o)$.

For parametric equations, $x = f(t)$, $y = g(t)$,

Gradient of curve at $t = a$ is $\left.\frac{dy}{dx}\right|_{t=a} = \frac{g'(a)}{f'(a)} = m_1$

Equation of tangent: $y - g(a) = m_1(x - f(a))$

Equation of normal: $y - g(a) = -\frac{1}{m_1}(x - f(a))$

5. Rate of Change

Variables related by $y = f(x)$ vary with time t.
By **Chain Rule**,

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Steps:

1. Identify the rate required
2. Obtain an equation involving that variable
3. Differentiate equation w.r.t. time

Chapter 7: Maclaurin's Series and Binomial Series

1. Maclaurin's Series (Polynomial estimation of a function, in MF26)

$$\mathrm{f}(x)=\mathrm{f}(0)+x\mathrm{f}'(0)+\frac{x^2}{2!}\mathrm{f}''(0)+\cdots+\frac{x^n}{n!}\mathrm{f}^{(n)}(0)+\cdots$$

2. Standard Series (in MF26)

(i) $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{r!}x^r+\cdots,\ |x|<1$

(ii) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^r}{r!}+\cdots.$ (all x)

(iii) $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+\frac{(-1)^{r+1}x^r}{r}+\cdots.$ $(-1<x\le 1)$

(iv) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{(-1)^r x^{2r+1}}{(2r+1)!}+\cdots.$ (all x)

(v) $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+\frac{(-1)^r x^{2r}}{(2r)!}+\cdots.$ (all x)

* angle x is in radians for (iv) and (v).

We can use standard series to obtain series expansion of product or quotient of functions.

Application:

Suppose the Maclaurin's series of $f(x)$ is obtained.

- Use a suitable x value to estimate $f(x)$ using the series expansion. Values of x near 0 is preferred.
- Using differentiation or integration to get series expansion of $f'(x)$ and $\int f(x)\,dx$.
- Terms up to power 1 of the Maclaurin' series is the equation of tangent to $y=f(x)$ at $x=0$.

3. Binomial Series

(i) For ascending order, rewrite expression to the form $a(1+bx)^n$ and apply standard series (i).

Expansion is valid if $|bx|<1 \Rightarrow |x|<\frac{1}{|b|} \Rightarrow -\frac{1}{|b|}<x<\frac{1}{|b|}$

(ii) For descending order, rewrite expression to the form $a\left(1+\frac{b}{x}\right)^n$ and apply standard series (i).

Expansion is valid if $\left|\frac{b}{x}\right|<1 \Rightarrow |x|>|b| \Rightarrow x>|b|$ or $x<-|b|$

4. Small Angles

If x^3 and higher powers can be ignored

(i) $\sin x \approx x$

(ii) $\cos x \approx 1-\frac{x^2}{2!}$

(iii) $\tan x \approx x$

Important: angle x must be in **RADIANS**

Questions will often be related to circles and triangles. Thus knowledge of geometrical results related to circles and triangles (e.g. arc length, area of sector, sine rule, cosine rule etc) is required.

Chapter 8: Integration and Applications

1. Integration by Standard form

Step 1: Identify $\mathrm{f}(x)$.
Step 2: Verify you have $\mathrm{f}'(x)$ (usually multiply by a constant).
Step 3: Integrate.

(a) $\int \mathrm{f}'(x)[\mathrm{f}(x)]^n \, dx = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + C, \quad n \neq -1$	**(c)** $\int \mathrm{f}'(x) \, \mathrm{e}^{\mathrm{f}(x)} dx = \mathrm{e}^{\mathrm{f}(x)} + C$
(b) $\int \mathrm{f}'(x)[\mathrm{f}(x)]^{-1} \, dx = \ln\lvert\mathrm{f}(x)\rvert + C$	**(d)** $\int a^x \, dx = \frac{1}{\ln a} a^x + C$, where C is a constant

Note that for (e) - (h) **the coefficient of x must be 1 or –1** to apply the formula from MF26.

(e) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \ (\lvert x\rvert < a)$ **(MF26)**	**(i)** $\int \frac{\mathrm{f}'(x)}{\sqrt{a^2 - [\mathrm{f}(x)]^2}} dx = \sin^{-1}\left(\frac{\mathrm{f}(x)}{a}\right) + C$, $(\lvert\mathrm{f}(x)\rvert < a)$
(f) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ **(MF26)**	**(j)** $\int \frac{\mathrm{f}'(x)}{a^2 + [\mathrm{f}(x)]^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{\mathrm{f}(x)}{a}\right) + C$
(g) $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + C$ **(MF26)**	**(k)** $\int \frac{\mathrm{f}'(x)}{a^2 - [\mathrm{f}(x)]^2} dx = \frac{1}{2a}\ln\left\lvert\frac{a+\mathrm{f}(x)}{a-\mathrm{f}(x)}\right\rvert + C$
(h) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$ **(MF26)**	**(l)** $\int \frac{\mathrm{f}'(x)}{[\mathrm{f}(x)]^2 - a^2} dx = \frac{1}{2a}\ln\left\lvert\frac{\mathrm{f}(x)-a}{\mathrm{f}(x)+a}\right\rvert + C$

2. Integration of Trigonometric Function

(a) $\int \sin x \, dx = -\cos x + C$	(b) $\int \cos x \, dx = \sin x + C$
(c) $\int \sec^2 x \, dx = \tan x + C$	(d) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$
(e) $\int \sec x \tan x \, dx = \sec x + C$	(f) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$
(g) $\int \tan x \, dx = \ln\lvert\sec x\rvert + C$ **(MF26)**	(h) $\int \cot x \, dx = \ln\lvert\sin x\rvert + C$ **(MF26)**
(i) $\int \sec x \, dx = \ln\lvert\sec x + \tan x\rvert + C$ **(MF26)**	(j) $\int \operatorname{cosec} x \, dx = -\ln\lvert\operatorname{cosec} x + \cot x\rvert + C$ **(MF26)**

(i) Important formulae to know

Identities

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \operatorname{cosec}^2 x$

Double Angle (MF26)

- $\sin 2x = 2\sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x$

$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx$

$\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} dx$

Factor Formulae

- $\sin P + \sin Q = 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q) \Rightarrow \sin(A+B) + \sin(A-B) = 2\sin A\cos B$
- $\sin P - \sin Q = 2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q) \Rightarrow \sin(A+B) - \sin(A-B) = 2\cos A\sin B$
- $\cos P + \cos Q = 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q) \Rightarrow \cos(A+B) + \cos(A-B) = 2\cos A\cos B$
- $\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q) \Rightarrow \cos(A+B) - \cos(A-B) = -2\sin A\sin B$

****Note that only the 'LHS' formulae for <u>Factor Formulae</u> are given in MF26**

(ii) Rules for Trigonometric Integral

- Check that the trigonometric functions are of the forms found in the table in Section 2. Otherwise, use the MF26 or the basic trigonometric identities and change the integrand so that it matches one of the expressions found in the table.
- Check for $f'(x)$ by differentiating the expression inside the trigonometric function. For example $\int (4x)\cos(2x^2-3)\,dx$ has $f(x)=2x^2-3$ and $f'(x)=4x$.
- 'Drop' $f'(x)$ and apply the integral above in Section 2.
- Therefore, $\int (4x)\cos(2x^2-3)\,dx = \sin(2x^2-3)+C$

3. Integration by Rational Functions $\int \frac{f(x)}{g(x)}dx$ and $\int \frac{f(x)}{\sqrt{g(x)}}dx$

Step 1:	Proper fractions? If no, do a long division.
Step 2:	Check for $g'(x)$. If yes, integrate using standard form.
Step 3:	Numerator is a constant, denominator is quadratic expression or $\sqrt{\text{quadratic}}$? If yes, complete the square and use formulae in MF26.
Step 4:	Numerator is a linear expression, denominator is a quadratic expression? If yes, split numerator into the form $a\,g'(x)+b$ and integrate via standard form.
Step 5:	If denominator is cubic, most of the time we will try partial fractions.

4. Integration by substitution

Step 1: Differentiate the substitution.
Step 2: Change limits using the substitution if limits are given.
Step 3: Total replacement (including limits and dx).
Step 4: Integrate, and change back to the original variables only if no limits given.

5. Integration by parts

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

Note : In general, you can use **LIATE** to determine u and $\mathrm{d}v$ but it is **not always true!** Identify which term is a derivative of some function and let it be $\mathrm{d}v/\mathrm{d}x$.

Applications of Integration

1A. Area of regions bounded between the curve $y = \mathrm{f}(x)$ and the x-axis or y-axis

Steps	Using x-axis	Using y-axis
1.	Draw a rough sketch of the graph and identify the region needed carefully using the descriptions given in the question.	
2.	The equations of curves must be in terms of x. i.e. $y=\mathrm{f}(x)$.	The equations of the curves must be in terms of y. i.e. $x=\mathrm{g}(y)$.
3.	Area of the region $R=\int_a^b y\,\mathrm{d}x=\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ (above x-axis) Area of the region $R=-\int_a^b y\,\mathrm{d}x=-\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ (below x-axis)	Area of the region $R=\int_c^d x\,\mathrm{d}y=\int_c^d \mathrm{g}(y)\,\mathrm{d}y$ (RHS of y-axis) Area of the region $R=-\int_c^d x\,\mathrm{d}y=-\int_c^d \mathrm{g}(y)\,\mathrm{d}y$ (LHS of y-axis)

Definite integrals involving absolute functions (in this case $\int_{-1}^{1} x|2x-1|\,\mathrm{d}x$)

Step 1 : Draw the function **inside** the modulus, i.e. $y=\mathrm{f}(x)=2x-1$.

Step 2 : Identify the two portions: above the x-axis ($y=\mathrm{f}(x)$ when $x>1/2$) and below the x-axis ($y=-\mathrm{f}(x)$ when $x<1/2$)

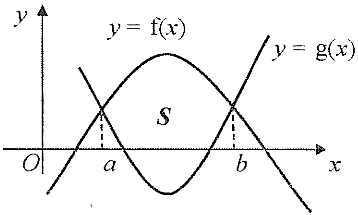
Step 3 : Split the interval of integration into two parts corresponding to $y=\mathrm{f}(x)$ and $y=-\mathrm{f}(x)$

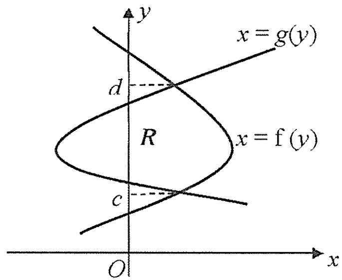
Step 4 : Evaluate the two integrals separately

1B. Area of regions bounded between two curves $y = f(x)$ and $y = g(x)$

<table>
<tr><th>Step</th><th>Using x-axis</th><th>Using y-axis</th></tr>
<tr><td>1.</td><td colspan="2">Do a rough sketch of the 2 graphs and identify the region needed carefully using the descriptions given in the question.</td></tr>
<tr><td>2.</td><td>The equations of curves must be in terms of x. i.e. $y = f(x)$ and $y = g(x)$.</td><td>The equations of the curves must be in terms of y. i.e. $x = f(y)$ and $x = g(y)$.</td></tr>
<tr><td>3.</td><td>Find the x-coordinate of the points of intersection between the two curves (let's say the points are $x = a$ and $x = b$).</td><td>Find the y-coordinate of the points of intersection between the two curves (let's say the points are $y = c$ and $y = d$).</td></tr>
<tr><td>4.</td><td>Establish the "upper" and "lower" curves for the region between $x = a$ and $x = b$ using the sketch done in Step 1.</td><td>Establish the "right-side" and "left-side" curves for the region between $y = c$ and $y = d$ using the sketch done in Step 1.</td></tr>
<tr><td>5.</td><td>Use the "upper" curve minus the "lower" curve.

Area of the region $S = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
$= \int_a^b [f(x) - g(x)]\,dx$
where $f(x) - g(x) \geq 0$ for $a \leq x \leq b$.</td><td>Use the "right-side" curve minus the "left-side" curve.

Area of the region $R = \int_c^d [f(y) - g(y)]\,dy$</td></tr>
<tr><td></td><td>

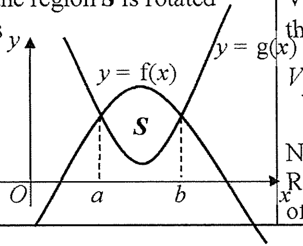
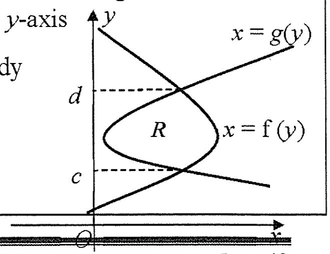
Note: Part of region S can be below the x-axis.</td><td>

Note: Part of region R can be on the left-hand side of the y-axis.</td></tr>
</table>

2A. Volume of solid of revolution for regions bounded by a curve and the x- or y- axis

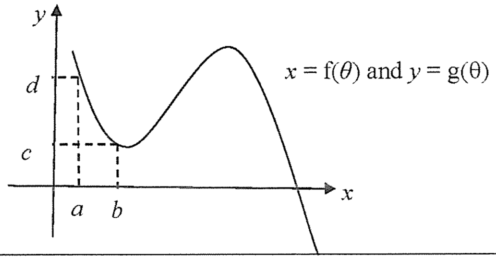
Steps	Rotation about x-axis	Rotation about y-axis
1.	Draw a rough sketch of the graph(s) concerned and identify the region needed carefully using the descriptions given in the question. Spot keywords on the *axis of rotation* as the solid obtained will be very different.	
2.	The equation of curve must be in terms of x. i.e. $y = \mathrm{f}(x)$	The equation of the curve must be in terms of y. i.e. $x = \mathrm{f}(y)$
3.	Volume of the solid formed when the region S is rotated through 2π radians about the x-axis $V_x = \pi \int_a^b [\mathrm{f}(x)]^2 \, \mathrm{d}x$	Volume of the solid formed when the region R is rotated through 2π radians about the y-axis $V_y = \pi \int_c^d [\mathrm{f}(y)]^2 \mathrm{d}y$

2B. Volume of solid of revolution for regions bounded between two curves

Steps	Rotation about x-axis	Rotation about y-axis
1.	Draw a rough sketch of the graph(s) concerned and identify the region needed carefully using the descriptions given in the question. Spot keywords on the *axis of rotation* as the solid obtained will be very different.	
2.	The equations of curves must be in terms of x. i.e. $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$.	The equations of the curves must be in terms of y. i.e. $x = \mathrm{f}(y)$ and $x = \mathrm{g}(y)$.
3.	Find the x-coordinate of the points of intersection between the two curves (let's say the points are $x = a$ and $x = b$).	Find the y-coordinate of the points of intersection between the two curves (let's say the points are $y = c$ and $y = d$).
4.	Establish the "upper" and "lower" curves for the region between $x = a$ and $x = b$ using the sketch done in Step 1.	Establish the "right-side" and "left-side" curves for the region between $y = c$ and $y = d$ using the sketch done in Step 1.
5.	Volume of the solid formed when the region ***S*** is rotated through 2π radians about the x-axis $V_x = \pi \int_a^b \left\{[\mathrm{f}(x)]^2 - [\mathrm{g}(x)]^2\right\} \mathrm{d}x$  Note: Region S must be above the x-axis.	Volume of the solid formed when the region ***R*** is rotated through 2π radians about the y-axis $V_y = \pi \int_c^d \left\{[\mathrm{f}(y)]^2 - [\mathrm{g}(y)]^2\right\} \mathrm{d}y$  Note: Region R must be on RHS of the y-axis.

3. Area for curves given in the parametric form $x = \mathrm{f}(\theta)$ and $y = \mathrm{g}(\theta)$

The technique or solving strategy to finding area for curves given in parametric form is quite similar to the idea of substitution technique in definite integrals; converting the integral into parametric form.

Steps	Area bounded by x-axis	Area bounded by y-axis
1.	Draw a rough sketch of the graph(s) concerned and identify the region needed carefully using the descriptions given in the question. For volume, you will have to convert the parametric equations into cartesian equation (as guided by the question) and follow the steps in **section 2A**. 	
2.	Write out the general expression needed.	
	Area $= \int_a^b y \, \mathrm{d}x$	Area $= \int_c^d x \, \mathrm{d}y$
3.	Apply substitution to change integral with θ as the variable.	
	Differentiate x w.r.t. θ and replace $\mathrm{d}x$ with $\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)\mathrm{d}\theta$, replace y by $\mathrm{g}(\theta)$ and change limits: $\int_a^b y \, \mathrm{d}x = \int_{\theta_1}^{\theta_2} \mathrm{g}(\theta)\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)\mathrm{d}\theta$	Differentiate y w.r.t. θ and replace $\mathrm{d}y$ with $\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)\mathrm{d}\theta$, replace x by $\mathrm{f}(\theta)$ and change limits: $\int_c^d x \, \mathrm{d}y = \int_{\theta_2}^{\theta_1} \mathrm{f}(\theta)\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)\mathrm{d}\theta$
4.	Integrate and solve!	

Remember to read the question carefully to determine if exact area is required. If not, use GC!

Chapter 9: Differential Equations

Definition

If y is a function of x, then any equation relating x, y and one or more of the derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc. is called a **differential equation**.

The **order of a differential equation** is the **highest derivative** that occurs in it.

Solving First Order Differential Equations

Variables Separable Differential Equations

Make $\frac{dy}{dx}$ the subject.

- The DE of the form $\frac{dy}{dx} = f(x)$ can be solved by $\int dy = \int f(x)\, dx$
- The DE of the form $\frac{dy}{dx} = g(y)$ can be solved by $\int \frac{1}{g(y)}\, dy = \int dx$

Substitution

- Some first order differential equations are not variables separable.
- They can be transformed into variables separable with a **given suitable substitution of variables.**
- If $\frac{dy}{dx}$ is the derivative, then after substitution, all y must "disappear" and x remains.

Solving Second Order Differential Equations

Integrating twice w.r.t. x

- Make $\frac{d^2y}{dx^2}$ the subject.
- By integrating w.r.t. x twice, we will have 2 constants of integration, e.g. $Ax + B$, where A and B are constants.

Solutions of Differential Equations

General Solution

- Express y in terms of x only if stated in the question.
- Solution curves for the general solution to a differential equation with arbitrary constants can be sketched by substituting suitable values into the **constant**.
- Should there be a context to the question, do remember to
  1. Label axes correctly with the relevant variables
  2. Note that some variables do not take on negative values (e.g. time, population)

Particular Solution

- If the arbitrary constant can be evaluated, a **particular solution** is obtained.

Applications of Differential Equations

Forming Differential Equation

Example:
If there is a rate of increase and rate of decrease, then

$$\frac{dx}{dt} = (\text{rate of increase}) - (\text{rate of decrease})$$

Chapter 10: Complex Numbers

<table>
<tr><th>Algebraic form (Cartesian form)</th><th>Trigonometric form (Polar form)</th><th>Exponential form (Polar form)</th></tr>
<tr><td>$z = x + y\mathrm{i}$</td><td>$z = r(\cos\theta + \mathrm{i}\sin\theta)$</td><td>$z = r\mathrm{e}^{\mathrm{i}\theta}$, where $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ and $\left|\mathrm{e}^{\mathrm{i}\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$</td></tr>
<tr><td>$|z| = \sqrt{x^2 + y^2}$</td><td colspan="2">$r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arg z \in (-\pi, \pi]$

To find θ :
Mtd 1: Use GC: MATH > CMPLX > 4: angle(,divide by π then convert to fraction to express argument in terms of π

Mtd 2: Apply ABS Approach
A: Plot the point representing z on an Argand diagram
B: Find Basic ∡, $\tan^{-1}\left(\frac{|y|}{|x|}\right)$.
S: Determine which quadrant z lies in and hence find θ.
(However still useful to use GC to check if the argument is correct)</td></tr>
<tr><td>Useful for addition and subtraction</td><td>Useful for converting complex number into algebraic form given its modulus and argument (∴ useful for converting exponential form to algebraic form)</td><td>Useful for multiplication, division and taking powers
• $z_1 z_2 = r_1\mathrm{e}^{\mathrm{i}\theta_1} r_2\mathrm{e}^{\mathrm{i}\theta_2} = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$
• $\frac{z_1}{z_2} = \frac{r_1\mathrm{e}^{\mathrm{i}\theta_1}}{r_2\mathrm{e}^{\mathrm{i}\theta_2}} = \frac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)}$
• $z^n = r^n\mathrm{e}^{\mathrm{i}n\theta}$</td></tr>
</table>

(i) Properties of conjugate

- $zz^* = |z|^2$
- $(z^*)^* = z$
- $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$
- $(z_1 z_2)^* = z_1^* z_2^*$ and $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$
- $(z^n)^* = (z^*)^n$

• $z + z^* = 2\,\mathrm{Re}(z)$ • $z - z^* = 2\mathrm{i}\,\mathrm{Im}(z)$	**Useful result:** If $z = \mathrm{e}^{\mathrm{i}\theta}$, $z + z^* = \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} = 2\cos\theta$ $z - z^* = \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} = 2\mathrm{i}\sin\theta$

(ii) Properties of modulus and argument

To find a complex number z, it suffices to find its modulus r and argument θ. Make use of the properties of the modulus and argument to do so.	• $\lvert z_1 z_2\rvert = \lvert z_1\rvert \lvert z_2\rvert$ • $\left\lvert \dfrac{z_1}{z_2} \right\rvert = \dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}$ • If $k \in \mathbb{R}$, $\lvert kz\rvert = \lvert k\rvert \lvert z\rvert$ • $\lvert z^k\rvert = \lvert z\rvert^k$ • $\left\lvert \dfrac{1}{z} \right\rvert = \dfrac{1}{\lvert z\rvert}$ • $\lvert z^*\rvert = \lvert z\rvert$	• $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ • $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ • $\arg(kz) = \begin{cases} \arg z, & \text{if } k \in \mathbb{R}^+ \\ \pi + \arg z, & \text{if } k \in \mathbb{R}^- \end{cases}$ <u>Note</u>: we do not consider $k = 0$ $\because \arg(0)$ is undefined • $\arg(z^k) = k \arg z$ • $\arg\left(\dfrac{1}{z}\right) = -\arg z$ • $\arg(z^*) = -\arg z$
• $z^n = r^n \mathrm{e}^{\mathrm{i}n\theta} = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$		(Note: $+2\pi$ or -2π so that $\arg z \in (-\pi, \pi]$)

Argument of Real Numbers and Purely Imaginary Numbers

Argument of a
- real number: $k\pi, k \in \mathbb{Z}$
- positive real number: $0 + 2k\pi, k \in \mathbb{Z}$
- negative real number: $\pi + 2k\pi, k \in \mathbb{Z}$
- purely imaginary number: $\frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

Finding Roots of Polynomial Equations

Case 1: If the polynomial has unknown coefficients, substitute the given root into the equation. Solve for the unknown coefficients by comparing real and imaginary parts.

Case 2: Check if the polynomial equation *has real coefficients*. If so, apply the Conjugate Root Theorem (*complex roots occur in conjugate pairs*): Given a complex root α, its conjugate α^* is another root.

Useful result 1: (Factor Theorem)

If α is a root of a polynomial equation $\mathrm{P}(z) = 0$, then $(z - \alpha)$ is a factor of the polynomial $\mathrm{P}(z)$, e.g. if '1' is a root of the polynomial equation $\mathrm{P}(z) = 0$, then $(z - 1)$ is a factor of $\mathrm{P}(z)$; if '$1 + 2\mathrm{i}$' is a root of the polynomial equation $\mathrm{P}(z) = 0$, then $\left[z - (1 + 2\mathrm{i})\right]$ is a factor of $\mathrm{P}(z)$.

Useful result 2: (Sum and Product of Roots)

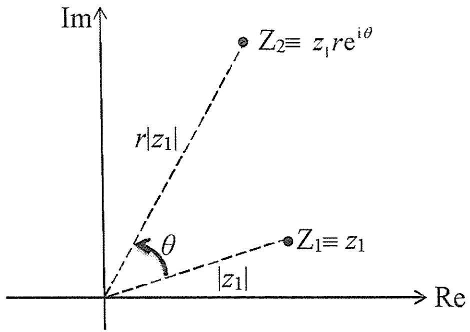
If α and β are roots $\Rightarrow (x - \alpha)$ and $(x - \beta)$ are factors $\Rightarrow (x - \alpha)(x - \beta)$ is a quadratic factor

$$= x^2 - \underbrace{(\alpha + \beta)}_{\text{negative sum of roots}}x + \underbrace{\alpha\beta}_{\text{product of roots}}$$

For $az^2 + bz + c = 0$, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Geometrical Interpretation of Arithmetic Operations

- Multiplying a complex number z by
    - i: rotation *anti-clockwise* by 90° about origin
    - $-\mathrm{i}$: rotation *clockwise* by 90° about origin
    - -1: rotation by 180°
    - $\mathrm{e}^{\mathrm{i}\theta}$: rotation by θ about origin ($\theta > 0$: anti-clockwise; $\theta < 0$: clockwise)
    - a real number k: $\begin{cases} \text{scaling by factor } k & \text{if } k > 0 \\ \text{scaling by factor } k \text{ followed by reflection in the origin} & \text{if } k < 0 \end{cases}$
    - $r\mathrm{e}^{\mathrm{i}\theta}$: rotation by θ about origin followed by scaling by factor r ($\theta > 0$: anti-clockwise; $\theta < 0$: clockwise)
    - w: rotation by $\arg(w)$ about origin and scaling by factor $|w|$



- Taking conjugate of a complex number z: *reflection* in the real axis
- Adding/subtracting of two complex numbers: apply parallelogram/triangle law of vector addition

Section B: Statistics

Chapter 1: Permutation and Combinations

1. Basic Principles

(i) Addition Principle (to be used when we are considering mutually exclusive cases)
(ii) Multiplication Principle (to be used when there is a sequence of events for the outcome we want)
(iii) Complementation Principle (to be used when there are too many cases for direct counting)

2. Some Techniques

(i) Box method (to be used when we want to consider arrangements/permutations)
(ii) Grouping technique (to be used when we want to have a group of objects/people together)
(iii) Slotting technique/insertion method (to be used when we want to separate objects/people)

Step 1: Exclude the objects that we want to separate and arrange the remaining objects.
Step 2: Insert the objects that we want to separate (count the number of ways to do so that will fulfil the given condition(s)).
Step 3: Arrange the inserted objects.

3. Some Common scenarios

(i) Objects Problems

- No. of ways of choosing r objects from n distinct objects = ${}^nC_r$
- No. of ways of arranging n distinct objects (in a straight line) = $n!$
- No. of ways of arranging n objects (in a straight line) with n_1 identical and n_2 identical $=\dfrac{n!}{n_1!n_2!}$
- No. of ways of choosing and arranging r objects from n distinct objects (without replacement)

$= {}^nP_r = \left({}^nC_r\right)r!$

- No. of ways of arranging n distinct objects in a circle = $(n-1)!$
- No. of ways of arranging n distinct objects in a circle with marked positions = $(n-1)! \times n = n!$
- No. of ways of **choosing** and **arranging** r objects from n **distinct** objects (with replacement) $= \underbrace{n \times n \times \cdots \times n}_{r \text{ times}} = n^r$

(iii) People Problems

- No. of ways of forming a team with r people out of $n = {}^nC_r$
- No. of ways of forming a team with r people out of n with a particular person included $= {}^{n-1}C_{r-1}$
- No. of ways of forming two groups of size r and m from n people (where $r \neq m$) = ${}^nC_r \times {}^{n-r}C_m$
- No. of ways of forming two groups of people of size r each from n people ($2r = n$) $= \dfrac{{}^nC_r \times {}^{n-r}C_r}{2!}$
- No. of ways of forming k groups of people of size r from n people ($kr = n$) $= \dfrac{{}^nC_r \times {}^{n-r}C_r \times {}^{n-2r}C_r \times \ldots \times {}^rC_r}{k!}$
- No. of ways of arranging n people (in a straight line) = $n!$
- No. of ways of arranging n people (in a straight line) with r particular people together $= \underbrace{(n-r+1)!}_{\text{arrange the group of } r \text{ people and remaining } n-r \text{ people}} \cdot \underbrace{r!}_{\text{arrange people in the group of } r \text{ people}}$

- No. of ways of arranging n guys and m girls with guys separated ($m+1 \geq n$) (in a straight line)
  $= \underbrace{m!}_{\text{arrange girls}} \cdot \underbrace{\left({}^{m+1}C_n\right)}_{\text{insert the guys whom we want to separate}} \cdot \underbrace{n!}_{\text{arrange the guys whom we inserted}}$
- No. of ways of arranging n people in a circle $=(n-1)!$
- No. of ways of arranging n people in a circle with labelled seats $=(n-1)! \times n = n!$
- No. of ways of arranging n people in a circle with r particular people together and seats are not labelled $= \underbrace{(n-r)!}_{\text{arrange the group of } r \text{ people and remaining } n\text{-}r \text{ people}} \cdot \underbrace{r!}_{\text{arrange people in the group of } r \text{ people}}$
- No. of ways of arranging n people in a circle with r particular people together and seats are labelled
  $= \underbrace{(n-r)!}_{\text{arrange the group of } r \text{ people and remaining } n\text{-}r \text{ people}} \cdot \underbrace{r!}_{\text{arrange people in the group of } r \text{ people}} \cdot \underbrace{n}_{\text{label the } n \text{ seats}}$
- No. of ways of arranging n (n is even) people in a circle with 2 particular people on opposite sides and seats are not labelled $=(n-2)!$
- No. of ways of arranging n (n is even) people in a circle with 2 particular people on opposite sides and seats are labelled $=(n-2)! \times n$
- No. of ways of arranging people in a circle of n chairs in which r are empty and seats are not labelled
  $= \frac{(n-1)!}{r!}$
- No. of ways of arranging people in a circle of n chairs in which r are empty and seats are labelled
  $= \frac{(n-1)!}{r!} \times n = \frac{n!}{r!}$

Chapter 2: Probability

1. Concepts

(i) Definition: If the sample space S contains a *finite* number of *equally likely* outcomes, then the probability that an event A will occur is given by

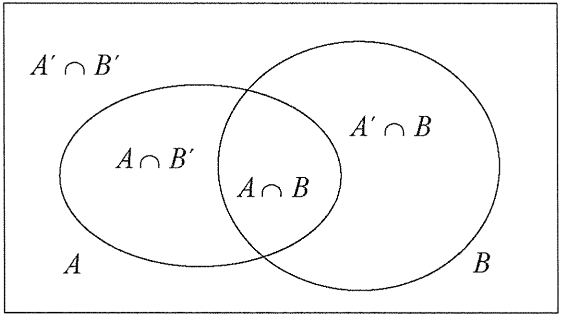
$$\mathrm{P}(A)=\frac{n(A)}{n(S)}=\frac{\text{number of outcomes corresponding to event } A}{\text{total number of possible outcomes in the sample space } S}$$

(ii) Probability is a number between 0 and 1 (inclusive).

2. Techniques

(i) Permutations and Combinations
(ii) Tree diagrams (*useful when events are sequential or conditional on earlier events*)
(iii) Venn diagrams (*useful for visualizing a complex question*)

Sample Venn diagram:



3. Formulae

(i) $\mathrm{P}(A') = 1 - \mathrm{P}(A)$

(ii) Addition Formula

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ [Note: This means P(A or B or both)]

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ if A and B are mutually exclusive [as $\mathrm{P}(A \cap B) = 0$]

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - \left[P(A \cap B) + P(A \cap C) + P(B \cap C)\right] + P(A \cap B \cap C)$$

(iii) Conditional Probability

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

(iv) Mutually Exclusive

If A and B are mutually exclusive, $\mathrm{P}(A \cap B) = 0$.

(v) Independent Events

If A and B are independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$

Also, $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ and $\mathrm{P}(B \mid A) = \mathrm{P}(B)$ if A and B are independent

(vi) P(A or B, **but not both**) $= \mathrm{P}(A) + \mathrm{P}(B) - 2\mathrm{P}(A \cap B)$

Chapter 3: Discrete Random Variable

Let X have the following properties:

1. It is a discrete variable and can take only values $x_1, x_2, \ldots, x_n$.
2. The probability that X assumes $x_1, x_2, \ldots, x_n$ are $p_1, p_2, \ldots, p_n$ respectively.

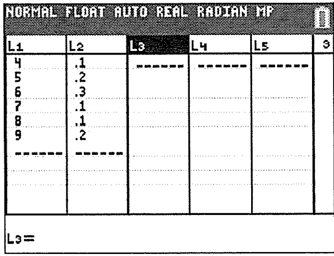
Then X is a **discrete random variable** if $\sum_{i=1}^{n} p_i = 1$.

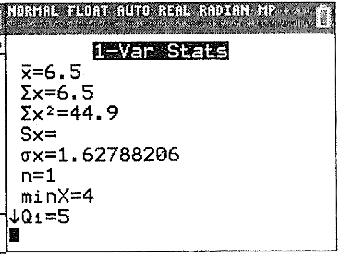
Expectation	Variance
$\mathrm{E}(X) = \mu = \sum_{i=1}^{n} x_i \mathrm{P}(X = x_i)$	$\mathrm{Var}(X) = \sigma^2 = \mathrm{E}(X-\mu)^2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ Standard deviation of $X = \sigma = \sqrt{\mathrm{Var}(X)}$, $\sigma > 0$
If a and b are constants, then • $\mathrm{E}(a) = a$, • $\mathrm{E}(aX) = a\mathrm{E}(X)$ and • $\mathrm{E}(aX+b) = a\mathrm{E}(X)+b$.	If a and b are constant, then • $\mathrm{Var}(a) = 0$, • $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$ and • $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$.
If X and Y are discrete random variables, $\mathrm{E}(aX \pm bY) = a\mathrm{E}(X) \pm b\mathrm{E}(Y)$ for constants a and b.	If X and Y are **independent** discrete random variables, then $\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$.
$\mathrm{E}(X_1 + \cdots + X_n) = n\mathrm{E}(X)$	$\mathrm{Var}(X_1 + \cdots + X_n) = n\mathrm{Var}(X)$
$\mathrm{E}(\bar{X}) = \mathrm{E}(X)$, $\bar{X} = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$	$\mathrm{Var}(\bar{X}) = \dfrac{\mathrm{Var}(X)}{n}$
If g(X) is a function of the discrete random variable of X, then $\mathrm{E}(\mathrm{g}(X)) = \sum_{i=1}^{n} \mathrm{g}(x_i)\mathrm{P}(X = x_i)$.	

Note: Questions involving X_1 and X_2, eg $\mathrm{P}\left(X_1 + X_2 = 6\right)$, will require one to break down into cases.

Using GC to compute E(X) and Var(X)

- Key in x_i and p_i into L_1 and L_2 respectively.
- Use 1-Var Stats (List:L_1, FreqList:L_2) to
- compute E(X), E(X^2) and standard deviation of X. [$\sum x$, $\sum x^2$ and σ_x respectively]





Chapter 4: Binomial Distribution $X \sim \mathrm{B}(n, p)$

Reminder:
When solving questions involving probability distributions, it is useful to remember the DICE framework:

D: Define the random variable

I: Identify the distribution of the random variable

C: Compute the probability

E: Evaluate if the answer makes sense

1. Overview

X is the number of successes occurring in n independent trials (p is probability of success for each trial)

$$X \sim \mathrm{B}(n, p) \quad X \text{ takes integer values } 0,1,2,\ldots, n$$

2. Concepts

(i) <u>Outcomes</u> of the n trials <u>are independent</u>.
(ii) <u>Probability</u> of success is <u>constant</u> for each trial.

Important: These 2 conditions are often used for exam questions on assumptions because they are usually not found in the question; always answer in context of the question.

(iii) 2 possible outcomes that are mutually exclusive; "*success*" and "*failures*".

(iv) The experiment has a **<u>fixed number</u>**, n, of repeated trials, where $n \in \mathbb{Z}^+$.

3. Techniques

(i) To find $\mathrm{P}(X = k)$, we use the GC keystroke Binompdf(n, p, k).
(ii) To find $\mathrm{P}(X \leq k)$, we use the GC keystroke Binomcdf(n, p, k). (*Take note of the inequality sign.*)

You need to change the expression to " $\leq$" so that you can calculate the probability using G.C.

	Given integers k, a and b,
(i)	$P(X < k) = P(X \le k-1)$
(ii)	$P(X > k) = 1 - P(X \le k)$
(iii)	$P(X \ge k) = 1 - P(X \le k-1)$
(iv)	$P(a \le X < b) = P(X \le b-1) - P(X \le a-1)$
(v)	$P(a < X < b) = P(X \le b-1) - P(X \le a)$
(vi)	$P(a \le X \le b) = P(X \le b) - P(X \le a-1)$

(iii) Find unknown probability, unknown k, n, p, or mode

(i)	Given $X \sim B(n, p)$, find $P(X \le k) = ?$	Use binomcdf
(ii)	Given $X \sim B(n, p)$, find $P(X \le ?) = P$	Use table $\because k \in \mathbb{Z}^+$
(iii)	Given $X \sim B(?, p)$, find $P(X \le k) = P$	Use table $\because n \in \mathbb{Z}^+$
(iv)	Given $X \sim B(n, ?)$, find $P(X \le k) = P$	Use graph $\because 0 < p < 1$
(v)	Given $X \sim B(n, p)$, find mode of X	Use table $\because x \in \mathbb{Z}^+$

(*Whether you use graph or table to find the answer depends on the nature of the unknown you are finding; generally using a table if the unknown is an integer, and graph, if the unknown is a real number*).

4. Formulae

$X \sim B(n, p)$

(i) mean $= \mu = E(X) = np$ **[in MF26]**

(ii) Variance $= \sigma^2 = \text{Var}(X) = np(1-p)$ **[in MF26]**

(iii) $P(X = x) = \binom{n}{x}(p)^x(1-p)^{n-x}$, $x = 0, 1, 2, \ldots, n$ **[in MF26]**

Chapter 5: Normal Distribution $X \sim \mathrm{N}(\mu, \sigma^2)$

1. Overview

It is different from Binomial Distribution in that it is a *continuous probability distribution.* (The normal random variable is a real number and takes values from negative infinity to positive infinity.) *Always* draw a diagram to *visualize* and *capitalize* on the *symmetry* of a normal distribution.

2. Concepts

(i)	The probability density function of a normally distributed variable is symmetrical about μ.
(ii)	The probability distribution is bell-shaped and has the following proportions: ±1 sd: 68.3% , ±2 sd: 95.4% , ±3 sd: 99.7% . Use GC and the standard normal distribution $Z \sim \mathrm{N}(0, 1)$ to find the probabilities, e.g. $\mathrm{P}(-1 \le Z \le 1) = 0.6827$.
(iii)	$\mathrm{P}(a < X < b)$ is the area under the probability density function between a and b.
(iv)	Since the area under the probability density function gives the probability, $\mathrm{P}(X = a) = 0$ and thus: $\mathrm{P}(a < X < b) = \mathrm{P}(a \le X < b) = \mathrm{P}(a < X \le b) = \mathrm{P}(a \le X \le b)$.
(v)	Standardizing a normal random variable $X \sim \mathrm{N}(\mu, \sigma^2)$: $Z = \dfrac{X-\mu}{\sigma}$ Hence $\mathrm{P}(X < a) = \mathrm{P}(Z < \dfrac{a-\mu}{\sigma})$ where $Z \sim \mathrm{N}(0, 1)$

(vi)	If $X \sim N(\mu, \sigma^2)$, then $nX \sim N(n\mu, n^2\sigma^2)$, and $X_1 + X_2 + ... + X_n \sim N(n\mu, n\sigma^2)$. Here, $X_i \sim N(\mu, \sigma^2)$ are **independent** and identically distributed (i.i.d.). To distinguish between nX (n times the value of one observation X) or $X_1 + X_2 + X_3 + \ldots + X_n$ (the sum of n independent observations), read the question carefully to look out for keywords.
(vii)	If $X_i \sim N(\mu, \sigma^2)$ for $i = 1, 2, 3 \ldots n$, and $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.
(viii)	If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are **independent** , then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ $X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$ $aX + bY \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

3. Techniques

(i) To find $P(X < a)$, we use the keystroke Normalcdf(–E99, a, μ, σ).

(ii) To find $P(a < X < b)$, we use the keystroke Normalcdf(a, b, μ, σ).

(iii) To find $P(X > a)$, we use the keystroke Normalcdf(a, E99, μ, σ).

(iv) Given probability p, known parameters μ and σ^2,

- invNorm(p, μ, σ, **LEFT**) returns the value x such that $P(X < x) = p$, where $X \sim N(\mu, \sigma^2)$.
- invNorm(p, μ, σ, **RIGHT**) returns the value x such that $P(X > x) = p$, where $X \sim N(\mu, \sigma^2)$.
- invNorm(p, μ, σ, **CENTER**) returns the value x such that $P(\mu - x < X < \mu + x) = p$, where $X \sim N(\mu, \sigma^2)$.

When you key into G.C., you are keying standard deviation, σ!

To use invNorm + Center, both the lower and upper bound must be **symmetrical about the mean**.

(v) We will standardize a normal distribution, if there are unknown parameters (i.e. μ and/or σ is unknown). We standardize a normal distribution by performing the operation $Z = \dfrac{X - \mu}{\sigma}$.

(vi) Very often concepts on probability, conditional probability and Binomial distribution can be infused into a question with Normal Distribution.

4. Checklist

(i) How to check whether a normal distribution $N(\mu, \sigma^2)$ is a suitable probability model for a given context?

Ans: $P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997$

(ii) Know how to standardize.

(iii) Be conscious of the symmetry of the normal distribution; you may need it when solving some problems.

E.g. (1) Given $P(X \le 80) = P(X \ge 100)$, find μ.

(2) Given $X \sim N(\mu, 10^2)$ and $P(|X - \mu| \le a) = 0.9$, find a.

Chapter 6: Sampling & Estimation Theory

Sampling Distribution (Central Limit Theorem):

If n is large and X is not normally distributed, then by Central Limit Theorem,

$$\overline{X} = \frac{\sum X}{n} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right) \text{ approximately} \quad \text{and} \quad \sum X = X_1 + \ldots + X_n \sim \mathrm{N}(n\mu, n\sigma^2) \text{ approximately.}$$

Note:
- CLT can be applied when $n \geq 30$. For symmetric distribution, n can be less than 30 to apply CLT.
- $X_1, X_2, \ldots, X_n$ are **independent** and identically distributed.

Estimation Theory:

If a random sample of size n is taken,

(i) the unbiased estimate of the population mean, μ, is the sample mean $\overline{x}$ where

$$\overline{x} = \frac{\Sigma x}{n} \quad \text{or} \quad \overline{x} = \frac{\Sigma(x-a)}{n} + a, \quad \text{where } a \text{ is the assumed mean;}$$

(ii) the unbiased estimate of the population variance σ^2 is

$$s^2 = \frac{n}{n-1}\left(\frac{\sum(x-\overline{x})^2}{n}\right) = \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right) \text{(in MF 26)}$$

$$= \frac{1}{n-1}\left(\sum(x-a)^2 - \frac{\left(\sum(x-a)\right)^2}{n}\right)$$

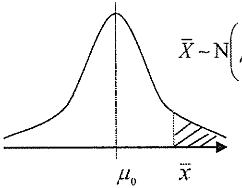
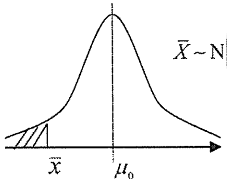
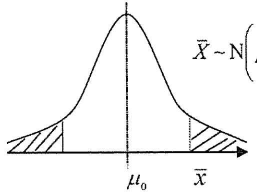
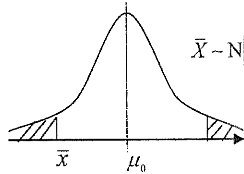
$$= \frac{n}{n-1}(\text{sample variance})$$

Chapter 7: Hypothesis Testing

The 5 Steps in Hypothesis Testing: T^2C^3 Approach

Steps	What to do	Remarks
(i) To test H_0 and H_1	Define the **null and alternative hypotheses** and **any symbols used**	**READ CAREFULLY** for the correct H_1. Wrong choice will lead to almost ZERO mark for the question (usually 4 to 5 marks). Read the question for μ_0, $\bar{x}$, n or σ (or s) & α. (Do not confuse μ_0 and $\bar{x}$.)
(ii) Test statistic	State the **test** to be conducted	***Z*-test** Need to check if there is a need to apply CLT
(iii) Critical region	State the **rejection criteria** p-value $\leq \alpha\%$	
(iv) Calculations	Obtain critical value or ***p*-value** from GC, based on information from sample	
(v) Conclusion	Make **conclusion** (i.e. sufficient/insufficient evidence to reject H_0).	If p–value $\leq \alpha/100$, **reject H_0, sufficient evidence** to reject H_0 in favour of H_1 at the $\alpha\%$ level of significance. If p–value $> \alpha/100$, **do not reject H_0, insufficient evidence** to reject H_0 in favour of H_1 at the $\alpha\%$ level of significance.

When there is unknown in μ_0, $\bar{x}$, n or σ (or s):

$H_0: \mu=\mu_0$ $H_1: \mu>\mu_0$	$H_0: \mu=\mu_0$ $H_1: \mu<\mu_0$	$H_0: \mu=\mu_0$ $H_1: \mu\neq\mu_0$	
 $\bar{X}\sim N\left(\mu_0,\frac{\sigma^2}{n}\right)$ Critical Region: $z_c \ge z_\alpha$ p-value = $P(\bar{X}\ge\bar{x})$	 $\bar{X}\sim N\left(\mu_0,\frac{\sigma^2}{n}\right)$ Critical Region: $z_c \le -z_\alpha$ p-value = $P(\bar{X}\le\bar{x})$	**Case a)** If $\bar{x}>\mu_0$  $\bar{X}\sim N\left(\mu_0,\frac{\sigma^2}{n}\right)$	**Case b)** If $\bar{x}<\mu_0$  $\bar{X}\sim N\left(\mu_0,\frac{\sigma^2}{n}\right)$
		Critical Region: $\lvert z_c\rvert \ge z_{\alpha/2}$	
		p-value = $2\cdot P(\bar{X}\ge\bar{x})$	p-value = $2\cdot P(\bar{X}\le\bar{x})$
For z-test : $\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\sim N(0,1)$. Suppose H_0 is rejected at α% level of significance.		Note: need to make use of '**invNorm**' to find z_α	
Critical Region Approach			
$z_{calc}\ge z_\alpha$ $\Rightarrow \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\ge z_\alpha$	$z_{calc}\le -z_\alpha$ $\Rightarrow \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\le -z_\alpha$	$\lvert z_{calc}\rvert \ge z_{\alpha/2}$ $\Rightarrow \left\lvert\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right\rvert\ge z_{\alpha/2}$	
p-value Approach			
$p\text{-value}\le\frac{\alpha}{100}$ $P(\bar{X}\ge\bar{x})\le\frac{\alpha}{100}$ $P\left(Z\ge\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)\le\frac{\alpha}{100}$ $\frac{\bar{x}-\mu_0}{\sigma-\sqrt{n}}\ge z_\alpha$	$p\text{-value}\le\frac{\alpha}{100}$ $P(\bar{X}\le\bar{x})\le\frac{\alpha}{100}$ $P\left(Z\le\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)\le\frac{\alpha}{100}$ $\frac{\bar{x}-\mu_0}{\sigma-\sqrt{n}}\le -z_\alpha$	$p\text{-value}\le\frac{\alpha}{100}$ $2P(\bar{X}\ge\bar{x})\le\frac{\alpha}{100}$ $P\left(Z\ge\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)\le\frac{1}{2}\left(\frac{\alpha}{100}\right)$ $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\ge z_{\alpha/2}$	$p\text{-value}\le\frac{\alpha}{100}$ $2P(\bar{X}\le\bar{x})\le\frac{\alpha}{100}$ $P\left(Z\le\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)\le\frac{1}{2}\left(\frac{\alpha}{100}\right)$ $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\le -z_{\alpha/2}$

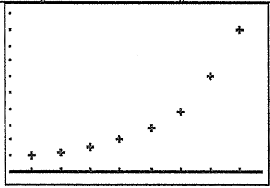
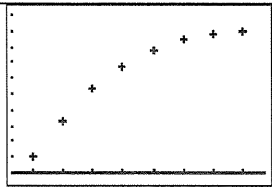
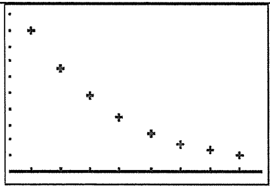
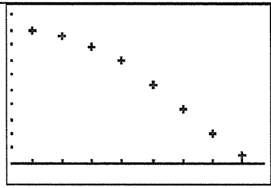
Chapter 8: Correlation and Linear Regression

1. Scatter Diagram

For scatter diagram, you need to

- identify the independent (controlled) variable (horizontal axis) and the dependent variable (vertical axis),
- label the axes,
- draw to scale,
- use a cross ‘×’ to mark the points

Suggested description of the trend of the scatter plot:

 As x increases, y increases at an increasing rate	 As x increases, y increases at a decreasing rate
 As x increases, y decreases at a decreasing rate	 As x increases, y decreases at an increasing rate

2. The Product Moment Correlation Coefficient, r

(i) The product moment correlation coefficient indicates the ***linear*** *degree of scatter* and is a measure of the ***linear*** *correlation* between the variables x and y.

(ii) r indicates both the *strength* and *direction* of the linear relationship, where $-1 \leq r \leq 1$.

- $r = 0$
  implies that there is no linear correlation between x and y. However, there could be a non-linear correlation between x and y.
- $r = 1$ $(r = -1)$
  implies that there is a perfectly positive (negative) linear correlation between x and y. In this case, all the sample points lie exactly on a straight line with a positive (negative) gradient
- Positive (Negative) values of r
  implies that there is a positive (negative) linear correlation between x and y, that is, y increases (decreases) as x increases.
- The value of r *should always be interpreted together* with a scatter diagram. The value of r alone is *insufficient* to make any conclusion about the linear correlation between two variables.

3. Regression Line

For the least squares regression line of y on x, $y = a + bx$ (Use GC. Formula given in MF26)

(i) b is the gradient of the regression line of y on x and is known as the **coefficient of regression** of y on x.
(ii) Interpret b as the **expected** corresponding increase (or decrease) of y for every unit increase of x.
(iii) The regression line *always passes* through the point $(\bar{x}, \bar{y})$ and is the line of best fit based on the least squares criterion.
(iv) We use the least-squares regression line of y on x to estimate y given x. We cannot use the regression line to find directly the value of a missing actual data point.

For the least squares regression line of x on y, $x = c + dy$ (Switch the x and y variable in the formula in MF26)

(i) The least-squares regression line of y on x and the least-squares regression line of x on y both pass through the point $(\bar{x}, \bar{y})$.

(ii) The least-squares regression line of y on x and the least-squares regression line of x on y are different lines if $r \neq \pm 1$. They only coincide when $r = \pm 1$, that is when there is *perfect* linear correlation between x and y.

(iii) To estimate the value of y given a value of x, we should use the least-squares regression line of y on x '$y = a + bx$' unless instructed otherwise by the question. Similarly, to estimate a value of x given a value of y, we should in general use the least-squares regression line of x on y '$x = c + dy$'.

Note: If x is the **independent** variable, we use the least-squares regression line of y on x '$y = a + bx$' regardless whether we are estimating y given x or x given y.

(iv) To plot the least squares regression line of x on y, we must rewrite the equation $x = cy + d$ in the form $y = \frac{1}{c}x - \frac{d}{c}$.

Remark: The estimates obtained from the least squares regression lines are reliable if the given values are within the data range **and** r is close to 1 or -1.
Estimates obtained from extrapolation are not reliable.

4. Difference Between Correlation and Regression

Correlation is used to decide if there is a relationship between two variables and to determine how strong such a relationship is. Regression is used to find a model for the relationship and to estimate values of one variable given values of the other variable.

5. Transformation of Data Points From Non-Linear to Linear

(i) The transformation will usually be given in the question.

(ii) To compare two or more proposed models and determine which model is a better one:
- Use the scatter diagram to determine which model is applicable.
- If more than one model is applicable, compute the product moment correlation coefficient for each model.
- The model with the value of $|r|$ closest to 1 will, in general, be the better model.
- A scatter diagram involving the transformed variable will ascertain the goodness of fit of these models.

Note that the regression line is a straight line with reference to the transformed variables.

6. Outliers

(i) An outlier is an observation that lies outside the overall pattern of the data.
(ii) A scatter diagram can also give us visual evidence of outliers or suspicious observations.
(iii) An outlier can sometimes have a significant effect on a regression analysis.
(iv) We need to identify outliers and remove them from the analysis when appropriate.
(v) The outlier could be due to measurement or recording error.

7. Remarks on Accuracy

Do note that we will usually adopt the same number of significant figures as given in the question (data values of x or y) for our answers if the question does not specify the required accuracy.