Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 1 Graphing Techniques
Learning Package

Resources

- ☐ Core Concept Notes
- ☐ Discussion Questions
- ☐ Extra Practice Questions

SLS Resources

- ☐ Recordings on Core Concepts
- ☐ Quick Concept Checks

Reflection or Summary Page

H2 Mathematics (9758)
Chapter 1 Graphing Techniques
Core Concept Notes

Success Criteria:

Surface Learning	**Deep Learning**	**Transfer Learning**
☐ Determine the equations of asymptotes: horizontal, vertical, oblique (if any) of exponential, logarithmic and rational functions. ☐ Find, where applicable, the important features (**ASMILE**) of a graph. ☐ Identify restrictions on possible values of x and/or y (if any) ☐ Sketch the graph of a rational function such as: $y=\dfrac{ax+b}{cx+d}$, $y=\dfrac{ax^2+bx+c}{dx+e}$ ☐ Identify shapes of standard conic curves and state important characteristics of them: $(y-k)=a(x-h)^2$ or $(y-k)^2=a(x-h)$ $(x-h)^2+(y-k)^2=r^2$ $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$ or $\dfrac{(y-k)^2}{b^2}-\dfrac{(x-h)^2}{a^2}=1$. ☐ Explore the conic APP in the GC to graph a conic section and note the limitations of the conic APPS ☐ Use a GC to graph a given piecewise function ☐ Use a GC to graph a given modulus function ☐ Use a GC to graph a pair of parametric equations	☐ Use GC to find intersection points of graphs ☐ Draw the graph of a given function and label important characteristics such as asymptotes, turning points and intersections with the axes. ☐ Show algebraically that a function cannot lie between 2 certain values (to be determined) and use this information to draw the graph ☐ Convert conic section into standard form (using completing the square or long division) ☐ Use GC (not conic APP) to graph a conic section ☐ Draw the graph of a given piecewise function ☐ Express modulus function as a piecewise function ☐ Draw the graph of a function given in parametric form ☐ Convert the equation of a curve between parametric and Cartesian forms	☐ Sketch graphs of functions with unknowns ☐ Manipulate expressions and make inferences about intersections between graphs

Definition of Learning (John Hattie):
The process of developing sufficient surface knowledge to then move to deeper understanding such that one can appropriately transfer this learning to ***new tasks and situations****.*

§1 Features of Graphs

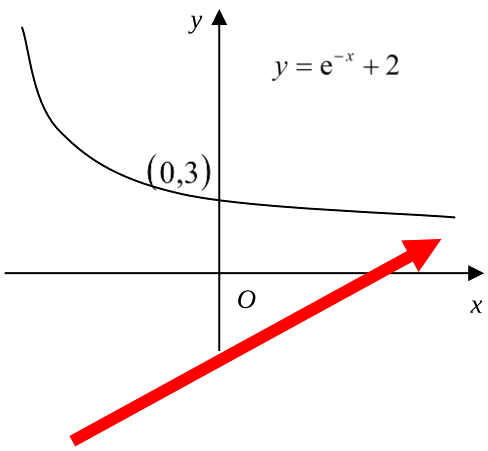
When sketching graphs, it is important to pay attention to important features. Go through the checklist and take note of each of the following whenever you sketch a graph.

1. **A**symptotes (if any)
2. **S**hape of the graph
3. **M**aximum/minimum turning points (if any)
4. **I**ntercepts with the axes (if any)
5. **L**abelling of the graph and axes
6. **E**nd points (if any)

1.1 Asymptotes

Consider the following curves:

(a) $y = e^{-x} + 2$



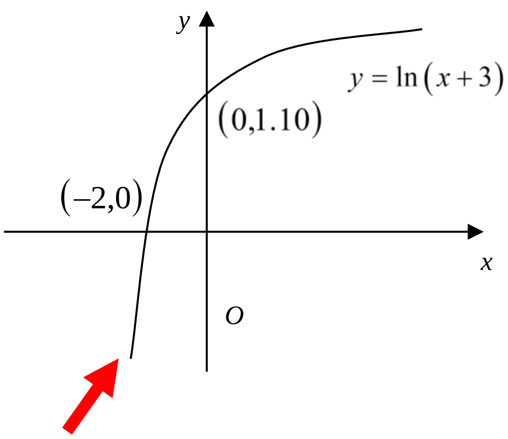
Observe that the curve seems to tend towards some value of *y* for **large values** of *x*.

Consider:
As $x \to \infty$, $e^{-x} \to 0$, $y \to 2$

In this case, the line $y = 2$ should be included in the graph.

This line is as known as a horizontal asymptote of the curve.

(b) $y = \ln(x+3)$



Observe that the curve seems to tend towards some value of *x* for **"large" negative values** of *y*.

Consider:
As $x \to -3$, $\ln(x+3) \to -\infty$, $y \to -\infty$

In this case, the line $x = -3$ should be included in the graph.

This line is as known as a vertical asymptote of the curve.

Given that the equation of a curve $y = \mathrm{f}(x)$:

Vertical asymptote	When $x \to a$, $y \to \pm\infty$ then $x = a$ is a vertical asymptote. For e.g. $x = 0$ is a vertical asymptote to the graph of $y = \ln x$.
Horizontal asymptote	When $x \to \pm\infty$, $y \to b$ then $y = b$ is a horizontal asymptote. For e.g. $y = 0$ is a horizontal asymptote to the graph of $y = \mathrm{e}^x$.
Oblique asymptote	When $x \to \pm\infty$, $y \to ax + b$ (note: $a \neq 0$) then $y = ax + b$ is an oblique asymptote.

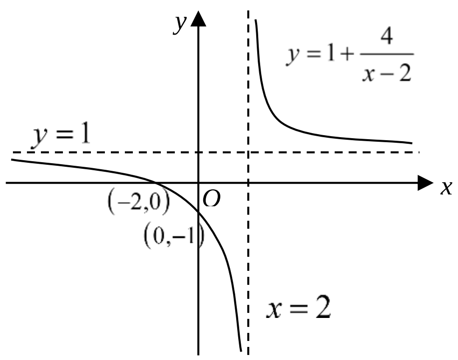
Example 1 (Determine equations of asymptotes)
State, with a reason, the equations of the asymptotes of the following graphs:

(a) $y = 1 + \dfrac{4}{x-2}$.

Determining vertical asymptote(s):

As $x \to 2$, $\dfrac{4}{x-2} \to \pm\infty$, $y \to \pm\infty$

Vertical asymptote: $x = 2$



Determining horizontal/oblique asymptote(s):

As $x \to \pm\infty$, $\dfrac{4}{x-2} \to 0$, $y \to 1$

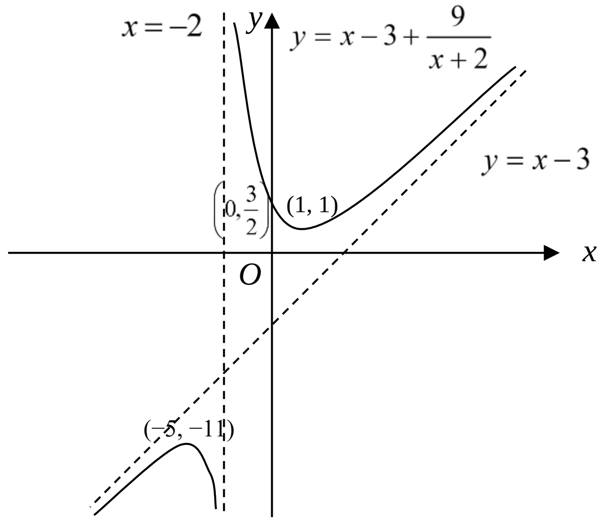
Horizontal asymptote: $y = 1$

(b) $y = x - 3 + \dfrac{9}{x+2}$.

Determining vertical asymptote(s):

As $x \to -2$, $\dfrac{9}{x+2} \to \pm\infty$, $y \to \pm\infty$

Vertical asymptote: $x = -2$



Determining horizontal/oblique asymptote(s):

As $x \to \pm\infty$, $\dfrac{9}{x+2} \to 0$, $y \to x - 3$

Oblique asymptote: $y = x - 3$

1.2 Shape, Maximum and Minimum Turning Points and Axial Intercepts

Definitions

y-intercept	Set $x = 0$ in equation and solve for the value of y. If $y = b$, then $(0, b)$ is the point where the graph crosses the y-axis. $(0, b)$ is also called the y-intercept.
x-intercept	Set $y = 0$ in equation and solve for the value of x. If $x = a$, then $(a, 0)$ is the point where the graph crosses the x-axis. $(a, 0)$ is also called the x-intercept.
Maximum turning point	Stationary point (point on curve where gradient is zero) at which the y-value is **greater** than any other y-value **in the neighbourhood on either side of the point.**
Minimum turning point	Stationary point (point on curve where gradient is zero) at which the y-value is **lower** than any other y-value **in the neighbourhood on either side of the point**.

GC Skills : TMJC Infinitum GC Video Tutorials

TMJC Mathematics Department has developed a series of GC video tutorials for the various H2 Mathematics topics.

How to access?

1. Log into your **ICON email account**,
2. Go to https://sites.google.com/tmjc.edu.sg/infinitum-gc/home or scan the QR code on the right,
3. Click on
   - Graphing Techniques
     ~ to learn about zoom features, window settings, finding key features of a graph

Do Question 1. Go through the video tutorials for Graphing Techniques or the GC keystrokes. Ensure that you are able to find the various key features (shape, maximum, minimum turning points, axial intercepts) of the curve.

Question 1 (Use of GC to obtain main features of a graph)

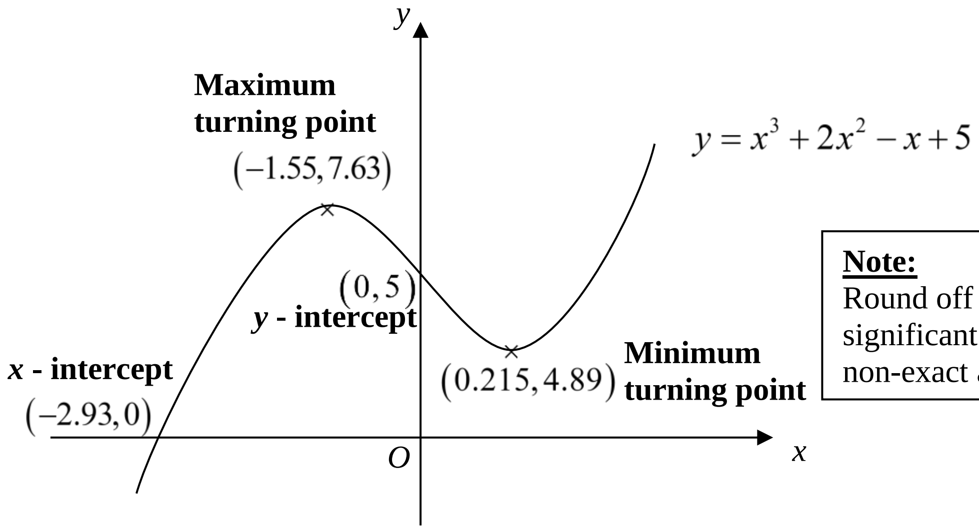
The curve C is defined by $y = x^3 + 2x^2 - x + 5$. Find the coordinates of the

(a) axial intercepts of C

(b) maximum and minimum turning points of C.

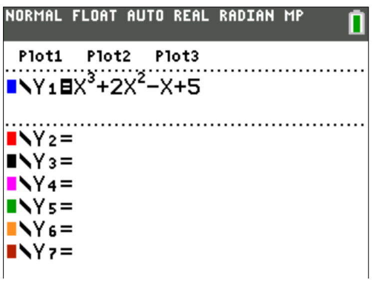
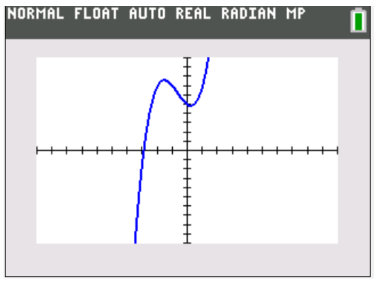
Sketch curve C.

Answer:

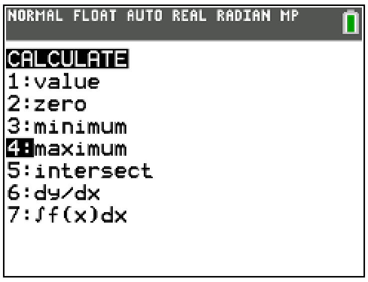
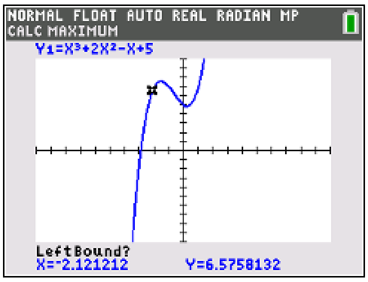
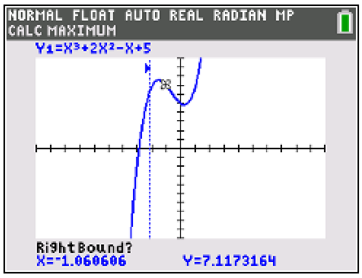
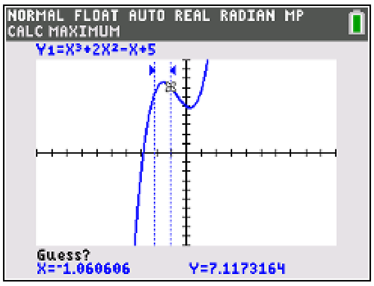
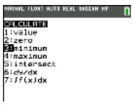
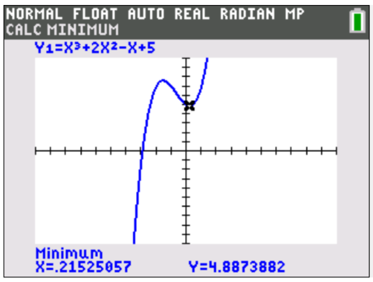


Note:
Round off to 3 significant figures for non-exact answer.

GC keystrokes to sketch curve

GC Steps	Screen on GC
• Press [y=] and key in the equation of the curve. • Press the following keys to input the given equation: [X,T,θ,n] [^] [3] [▸] [+] [2] [X,T,θ,n] [x²] [−] [X,T,θ,n] [+] [5] • To obtain the sketch of the graph, press [graph]. **Note:** To reset to standard window setting before sketching the graph, press [zoom] and select **6: ZStandard**.	 

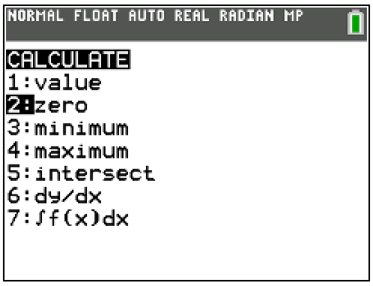
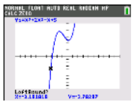
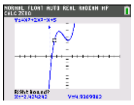
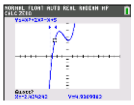
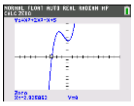
GC keystrokes to obtain turning points

GC Steps	Screen on GC
• To obtain the turning points, use the **CALCULATE** function by pressing [2nd][trace]. • To obtain the **maximum point**, Press [4] to select **4:maximum**.	
• GC will prompt for a left bound. Press [▶] or [◀] to move the cursor slightly to the left of the maximum point and press [enter]. • GC will prompt for a right bound. Press [▶] or [◀] to move the cursor slightly to the right of the maximum point and press [enter].	 
• GC will prompt for a Guess. • Press [enter]. GC will return the maximum point of $(-1.55, 7.63)$ (3 s.f.)	 
• To obtain the **minimum point**, Press [3] to select **3:minimum**. Repeat the steps stated above to obtain the minimum point.	 

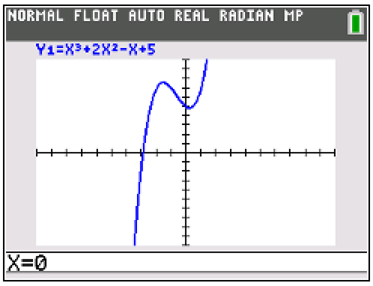
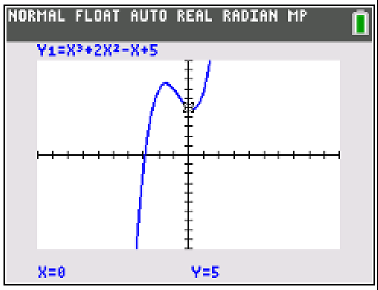
Maximum turning point: $(-1.55, 7.63)$ (to 3 s.f.)

Minimum turning point: $(0.215, 4.89)$ (to 3 s.f.)

GC keystrokes to obtain *x*-intercept(s)

GC Steps	Screen on GC
• To obtain any ***x*-intercept(s)** of the graph, use the **CALCULATE** function by pressing [2nd][trace]. • Press [2] to select **2:zero**. • GC will prompt for a left bound. Press [▶] or [◀] to move the cursor slightly to the left of the *x*-intercept and press [enter].	 
• GC will prompt for a right bound. Press [▶] or [◀] to move the cursor slightly to the right of the *x*-intercept and press [enter]. • GC will prompt for a Guess. It is not necessary to key in a value for the guess. Simply press [enter] to go to the next step.	 
• GC will return the *x*-intercept of $(-2.93, 0)$. **Note:** Non-exact answers are to be given to 3 significant figures, unless otherwise stated in the question.	

GC keystrokes to obtain *y*-intercept(s)

GC Steps	Screen on GC
• To obtain the ***y*-intercept**, press [trace][0][enter]. GC will return the *y*-intercept of (0, 5).	 

1.3 Intersections between 2 curves

GC Skills (https://sites.google.com/tmjc.edu.sg/infinitum-gc/home)
Go through the video tutorial under Graphing Techniques: Using Intersection to find intersection of graphs or the GC keystrokes printed below to find the solutions for Question 2.

Question 2 (Use of GC to obtain point of intersection between 2 graphs)
Find the coordinates of the points of intersection between the graphs of $y=6-x^2$ and $y=2x+1$, giving your answer to 3 significant figures.

Answer:

The coordinates of the points of intersection are $(1.45, 3.90)$ and $(-3.45, -5.90)$ (to 3 s.f).

<table>
<tr><th>GC Steps</th><th>Screen on GC</th></tr>
<tr><td>

- Press [y=] and key in the equations of the graphs

Note: To reset to standard window setting before sketching the graph, press [zoom] and select **6: ZStandard**.

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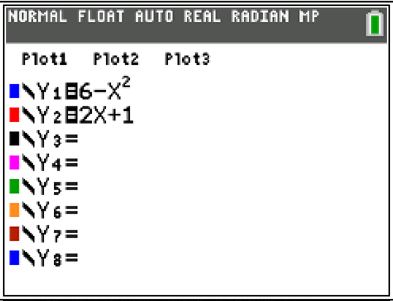

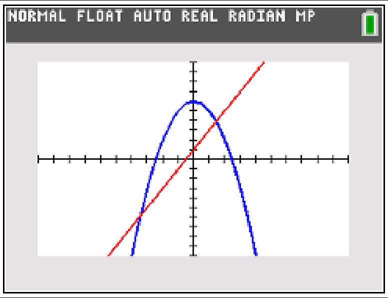

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<tr><td>

- To obtain the intersection point, use the **CALCULATE** function by pressing [2nd][trace].
- Press [5] to select **5:intersect**.
- Move the cursor to the point of intersection using [▶] or [◀] and press [enter].
- GC will prompt for First Curve. Press [enter].
- GC will prompt for a Second Curve. Press [enter].
- GC will prompt for a Guess. Press [enter]. GC will return the point of intersection: $(1.45, 3.90)$ (3 s.f.)

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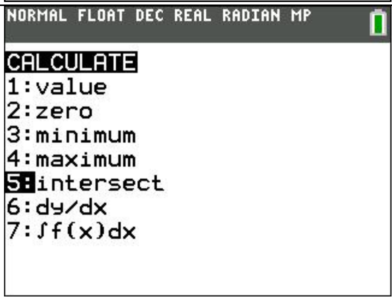

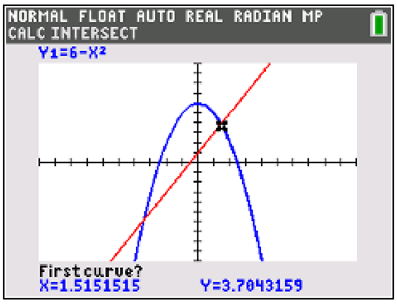

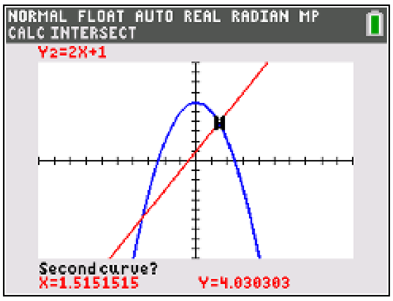

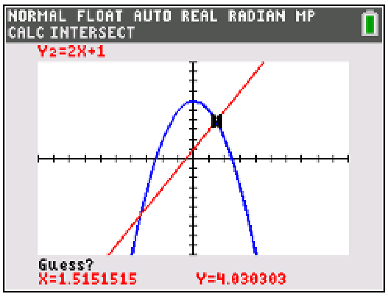

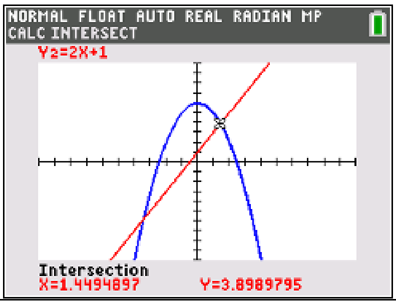

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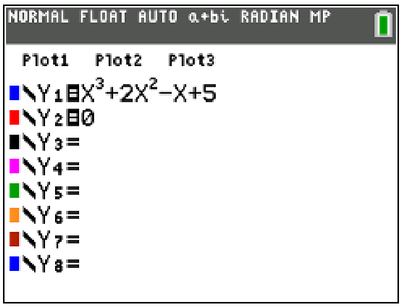
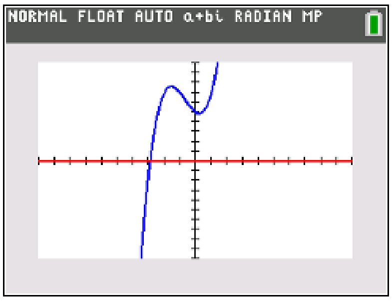
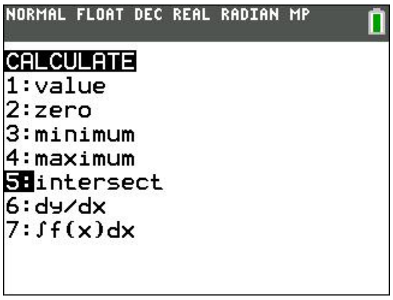
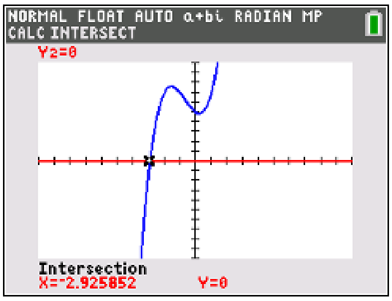
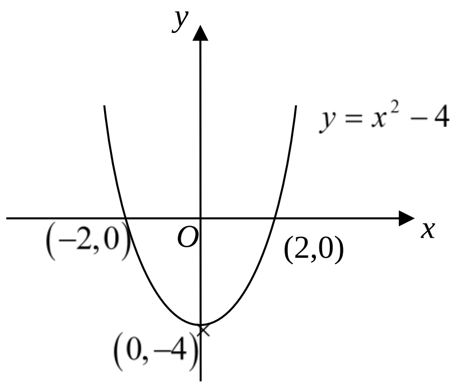
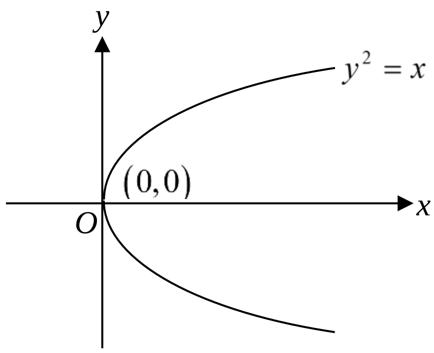
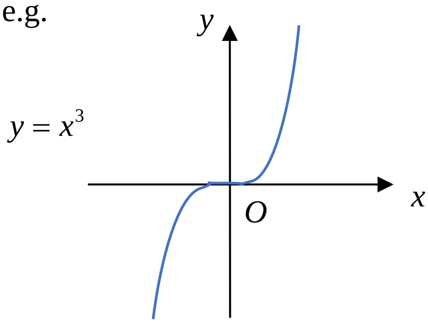
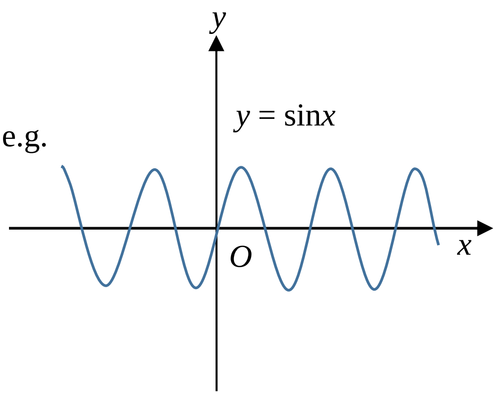
</table>

Note: Press [▲] or [▼] to toggle between graphs and press [▶] or [◀] to move the cursor along the chosen graph. • Repeat the steps to obtain the coordinates of the other point of intersection.	

Remark: The *x*-intercept of a graph can also be found by using the intersection of graphs. (Refer to Question 1(a) page 7)

GC Steps	Screen on GC
• Press [y=] and key in the equation of the curve. **Note:** To reset to standard window setting before sketching the graph, press [zoom] and select **6: ZStandard**.	 
• To obtain the intersection point, use the CALCULATE function as in previous page.	 

1.4 Symmetries

Symmetry about the y**-axis**	The equation remains unchanged when the variable x is replaced by $-x$. E.g. $y=(-x)^2-4$ $\Rightarrow y=x^2-4$ This means if (x,y) is on the curve, $(-x,y)$ is also on it.	e.g. 
Symmetry about the x**-axis**	The equation remains unchanged when the variable y is replaced by $-y$. E.g. $(-y)^2=x$ $\Rightarrow y^2=x$ This means if (x,y) is on the curve, $(x,-y)$ is also on it.	e.g. 
Symmetry about the origin	The equation remains unchanged when the variable x is replaced by $-x$, and y is replaced by $-y$ **at the same time**. E.g. $-y=(-x)^3$ $\Rightarrow -y=-x^3$ $\Rightarrow y=x^3$ E.g. $(-y)=\sin(-x)$ $\Rightarrow -y=-\sin x$ $\Rightarrow y=\sin x$ This means if (x,y) is on the curve, $(-x,-y)$ is also on it.	e.g.  e.g. 

§2 Rational Function

Consider the graph of a rational function $y=\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ where $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ are polynomials.

Examples of rational functions: $y=\frac{1}{x}$, $y=\frac{20}{x+2}$, $y=\frac{x+2}{x-2}$, $y=\frac{x^2-x+3}{x+2}$.

Recall: If the degree of the numerator $\mathrm{P}(x)$ is **less than** the degree of the denominator $\mathrm{Q}(x)$, then the fraction $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is said to be **proper**; otherwise, it is said to be **improper**.

(a) **To find vertical asymptote**

Let denominator $\mathrm{Q}(x)=0$ and obtain say, $x=k$.

As $x\to k$, $\mathrm{Q}(x)\to 0$, $y=\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\to\pm\infty$.

Thus $x=k$ is a vertical asymptote.

Example: $y=\frac{x+2}{x-2}$

Vertical asymptote: $x=2$.

(b) **To find horizontal / oblique asymptote**

We **need to ensure** $y=\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ where $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is a **proper fraction**.

If $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is not a **proper fraction**, use long division to express

$y=\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}=\mathrm{H}(x)+\frac{\mathrm{R}(x)}{\mathrm{Q}(x)}$ where $\frac{\mathrm{R}(x)}{\mathrm{Q}(x)}$ is a proper fraction.

Important Idea:

As $x\to\pm\infty$, $\frac{\mathrm{R}(x)}{\mathrm{Q}(x)}\to 0$ since $\frac{\mathrm{R}(x)}{\mathrm{Q}(x)}$ is a **proper fraction** where the degree of $\mathrm{R}(x)$ is less than the degree of $\mathrm{Q}(x)$.

Hence $y\to\mathrm{H}(x)$.

Thus $y=\mathrm{H}(x)$ is an asymptote.

Example A: $y=\frac{x+2}{x-2}=1+\frac{4}{x-2}$

Horizontal asymptote: $y=1$.

Example B: $y=2x-7+\frac{15}{x+2}$

Oblique asymptote: $y=2x-7$.

Example 2 (Sketch rational function with horizontal and vertical asymptote)

Sketch the graph of $y=\dfrac{x^2+2x}{x^2-1}$, indicating clearly the main relevant features of the graph.

Solution:

$$y=\frac{x^2+2x}{x^2-1}=1+\frac{2x+1}{(x-1)(x+1)}$$

Independent Learning on Long Division:

$$\begin{array}{r} 1 \\ x^2-1\,\big)\,\overline{x^2+2x} \\ -(x^2-1) \\ \hline 2x+1 \end{array}$$

Determining vertical asymptote(s):

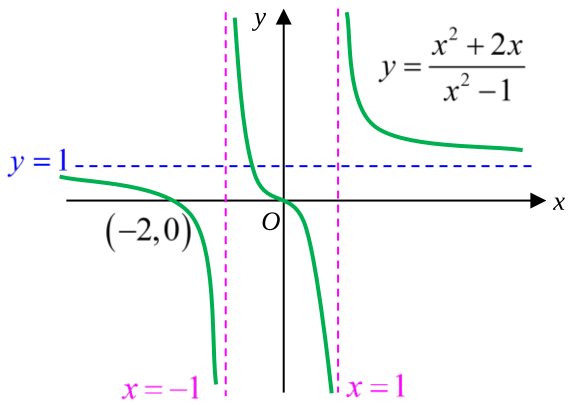
Is there any value(s) of x that leads to y approaching $\pm\infty$?

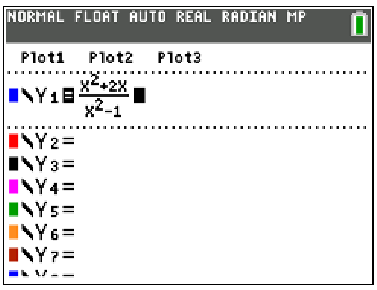
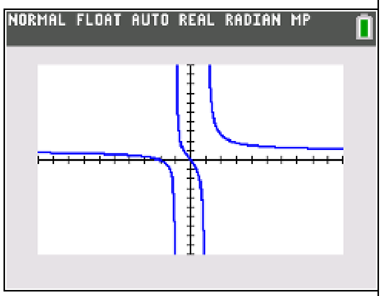
Vertical asymptotes: $x=-1$ and $x=1$

Determining horizontal/oblique asymptote(s):

As $x\to\pm\infty$, $\dfrac{2x+1}{(x-1)(x+1)}\to 0$, $y\to 1$

Horizontal asymptote: $y=1$



GC Steps	Screen on GC
• Press [Y=] [ALPHA] [X,T,Θ,n] for fraction. To key in the expression in Example 2, i.e. $y=\dfrac{x^2+2x}{x^2-1}$, press the following: [X,T,Θ,n] [x^2] [+] [2] [X,T,Θ,n] [▼] [X,T,Θ,n] [x^2] [−] [1]. • Press [zoom] and select **6: ZStandard** to view the graph in standard window settings.	 

Example 3 (Sketch rational function with oblique and vertical asymptote)

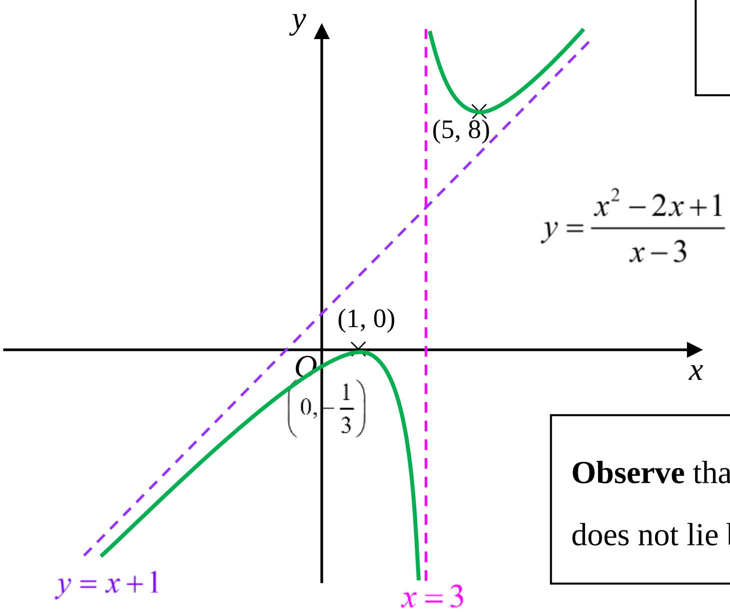
Sketch the graph of $y=\dfrac{x^2-2x+1}{x-3}$.

Solution:

$$y=\frac{x^2-2x+1}{x-3}=x+1+\frac{4}{x-3}$$

Vertical asymptote: $x=3$

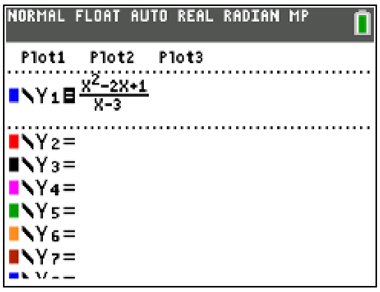
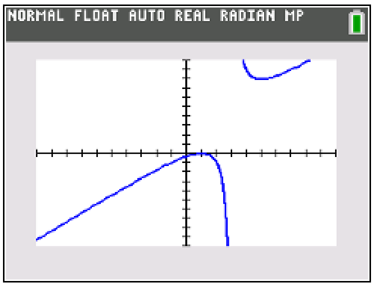
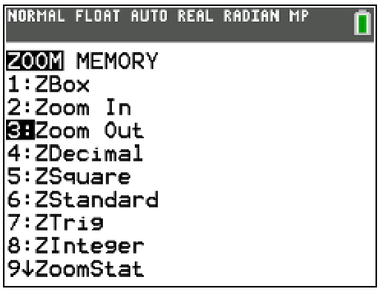
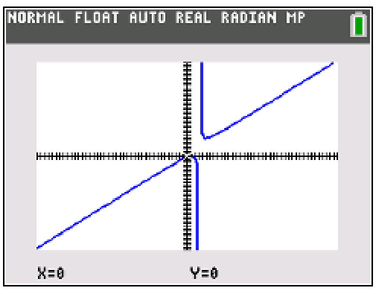
Oblique asymptote: $y=x+1$



Independent Learning on Long Division:

$$\begin{array}{r|l} & x+1 \\ x-3 & x^2-2x+1 \\ & -(x^2-3x) \\ \hline & x+1 \\ & -(x-3) \\ \hline & 4 \end{array}$$

Observe that the graph of $y=\dfrac{x^2-2x+1}{x-3}$ does not lie between $y=0$ and $y=8$.

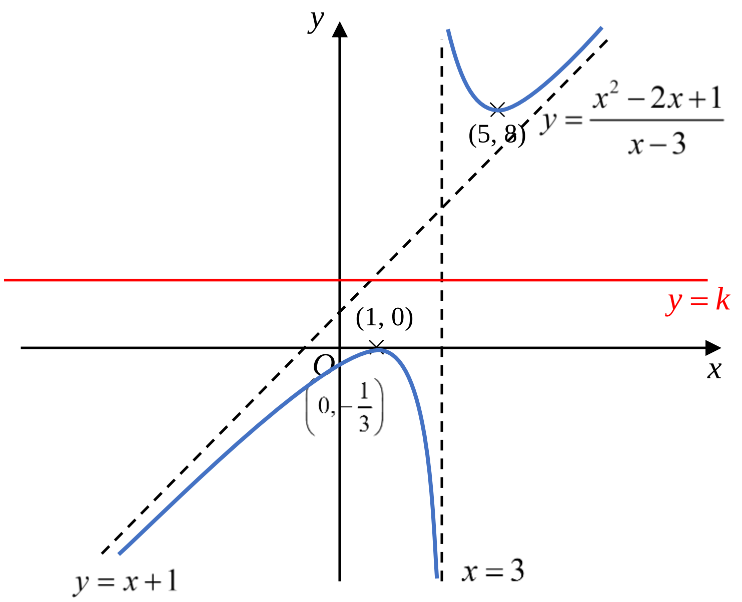
GC Steps	Screen on GC
• Press [Y=] [ALPHA] [X,T,Θ,n] • Key in the given expression as $Y_1=\dfrac{x^2-2x+1}{x-3}$ • Press [zoom] and select **6: ZStandard** to view the graph in standard window settings.	 
Note: The entire graph is not shown clearly on the screen. We can try ZoomOut or ZoomFit to have a better view of the graph. • Press [ZOOM] and select 3:Zoom Out. Press [ENTER].	 

Example 4 (Algebraic approach to determine the range of values *y* can take)

Prove, by using an **algebraic method**, that $\dfrac{x^2-2x+1}{x-3}$ cannot lie between two certain values (to be determined).

Solution:

If the graph does not lie between certain values, then the graph would have no intersection(s) with a horizontal line drawn in that region (see below).



To find intersection between the curve and horizontal line $y=k,\ k\in\mathbb{R}$,

$$k=\frac{x^2-2x+1}{x-3}$$

$$kx-3k=x^2-2x+1$$

$$x^2+(-2-k)x+(1+3k)=0$$

For a quadratic equation to have real roots, Discriminant ≥ 0 .

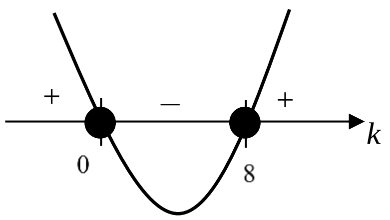
$$(-2-k)^2-4(1)(1+3k)\geq 0$$

$$k^2+4k+4-4-12k\geq 0$$

$$k^2-8k\geq 0$$

$$k(k-8)\geq 0$$

$$k\leq 0 \text{ or } k\geq 8$$



These are the values of *k* such that there are solutions to the quadratic equation.

Thus, the line $y=k$ intersects the curve $y=\dfrac{x^2-2x+1}{x-3}$ when $k\leq 0$ or $k\geq 8$

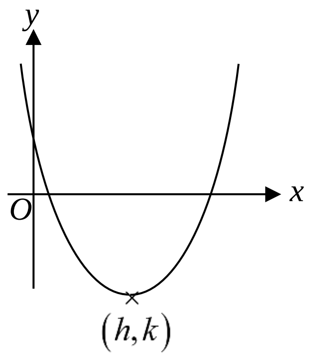
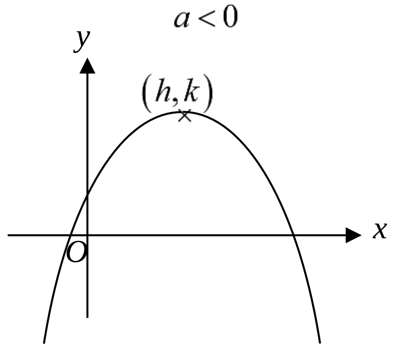
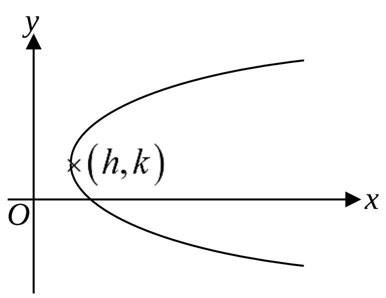
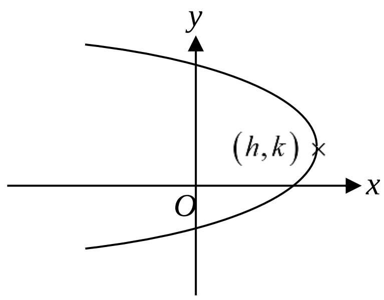
Therefore, $\dfrac{x^2-2x+1}{x-3}$ cannot lie between 0 and 8.

Do you realise that this answer is consistent with the graph sketched in Example 3?

§3 <u>Conics: Parabolas, Circles, Ellipses and Hyperbolas</u>

3.1 <u>Parabolas</u>

The **standard form** (also known as the vertex form) of an equation of a parabola is:

1	$(y-k)=a(x-h)^2$, $a \neq 0$ Line of symmetry is $x=h$ and vertex is at (h,k).	$a>0$ 	$a<0$ 
2	$(y-k)^2=a(x-h)$, $a \neq 0$ Line of symmetry is $y=k$ and vertex is at (h,k).	$a>0$ 	$a<0$ 

Note:

(1) The graph of the quadratic function $y=ax^2+bx+c$ is a parabola.

(2) If the parabola is centred at the origin $(0,\ 0)$, then the equation becomes $y=ax^2, a \neq 0$ or $y^2=ax, a \neq 0$.

(3) The equation of a parabola has no term in *xy*.

(4) **Either** the term x^2 **or** y^2 is present.

Example 5 (Sketching parabola with main features)

Sketch the following curves and state the equation of the line of symmetry (if any):

(a) $y=\frac{1}{4}(x+3)^2+2$

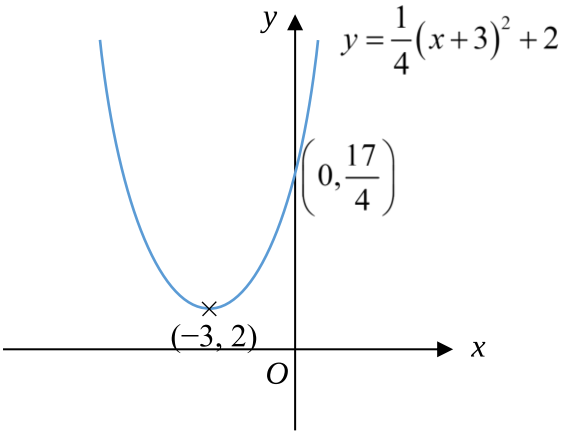
Solution:

$$y=\frac{1}{4}(x+3)^2+2$$

Vertex/minimum turning point: $(-3,\ 2)$

Axial intercept: $\left(0,\frac{17}{4}\right)$

Line of symmetry is $x=-3$.



(b) $y^2=4(1-x)$

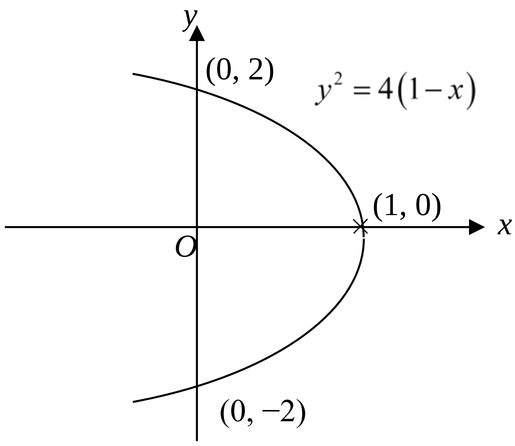
Solution:

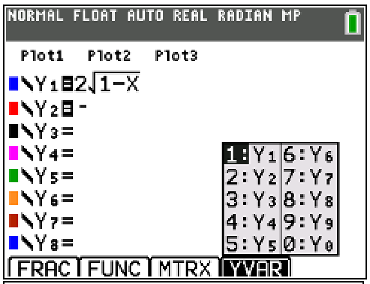
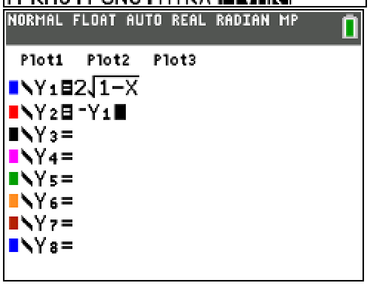
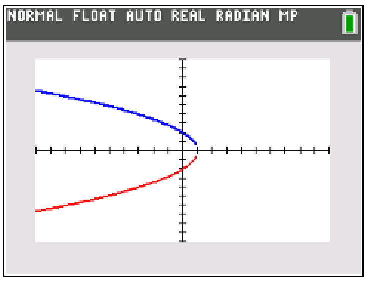
$$y^2=4(1-x)$$

Vertex: (1, 0)

Axial intercept: (1, 0), $(0,-2)$ and $(0,2)$

Line of symmetry is $y=0$.



GC Steps	Screen on GC
Note that $y^2=4(1-x)$ can be written as $y=\pm\sqrt{4(1-x)}=\pm2\sqrt{1-x}$ • Press [Y=]. Key in $Y_1=2\sqrt{1-x}$. Press [ENTER]. • To key in $y=-2\sqrt{1-x}$, we can key in $Y_2=-Y_1$. Go to Y2, press (-) , [ALPHA][TRACE] select 1:Y1. • Press [zoom] and select **6: ZStandard** to view the graph in standard window settings.	

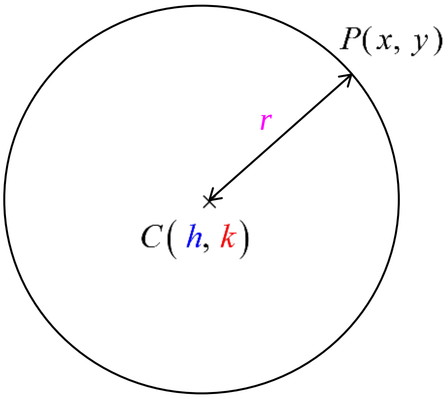
3.2 Circles

Let the length of $CP = r$.

Using Pythagoras Theorem, $r^2 = (x-h)^2 + (y-k)^2$

Therefore, the equation of a circle with centre (h, k) and radius r can take the following form:

Standard form: $(x-h)^2 + (y-k)^2 = r^2$ where $r > 0$



Note:

(1) If the circle is centred at the origin $(0, 0)$, then the equation becomes $x^2 + y^2 = r^2$.

(2) The equation of a circle has no term in xy.

(3) The coefficients of x^2 and y^2 are equal.

(4) Given an equation of the form $Ax^2 + Bx + Cy^2 + Dy + E = 0$, where $A = C$ and A, B, C, D and E are real numbers, we will be able to **complete the square** for the x and y terms separately to obtain the form $(x-h)^2 + (y-k)^2 = r^2$.

This will give the equation of a circle with centre (h, k) and radius r.

Main Features of a Circle to be labelled:

(1) Coordinates of centre of the circle.

(2) Length of radius of the circle.

(3) When the centre of the circle lies on either axis, label the axial intercepts.

Note:

Please use a **compass** to sketch a circle.

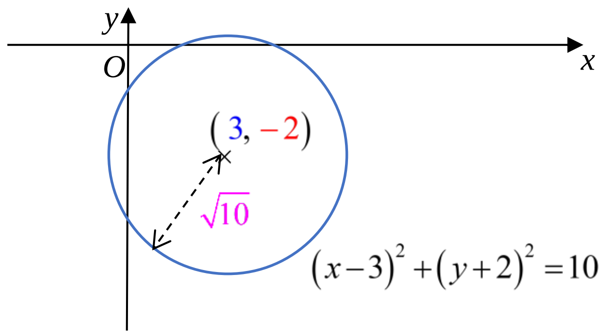
Example 6 (Sketching circle with main features)

Sketch and describe the circles geometrically and determine if they pass through the origin.

(a) $(x-3)^2+(y+2)^2=10$

Solution:

$(x-3)^2+(y-(-2))^2=\left(\sqrt{10}\right)^2$. It is a circle with centre $(3,\ -2)$ and radius $\sqrt{10}$ units.



To check if origin passes through the circle:

Substituting $(0,0)$ into the equation $(x-3)^2+(y+2)^2=10$.

$\text{LHS}=(0-3)^2+(0+2)^2=13$

$\text{RHS}=10$

Since $\text{LHS}\neq\text{RHS}$, the circle does not pass through the origin.

<table>
<tr><th>GC Steps</th><th>Screen on GC</th></tr>
<tr><td>

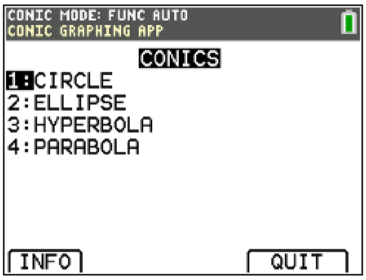
- Press [APPS]. Select 2: CONICS APP, then

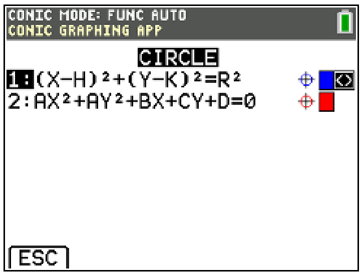
select 1: for CIRCLE.

- Press [1].

(To select the correct form)

</td><td>



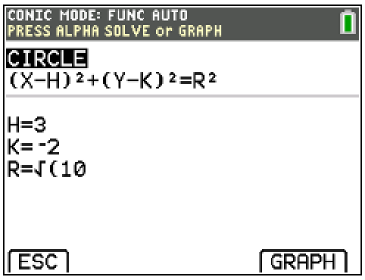


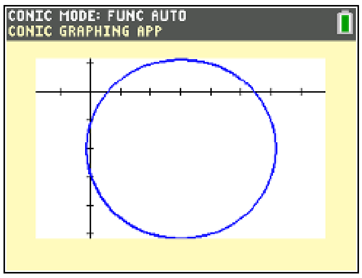
</td></tr>
<tr><td>

- Key in the values of **H**, **K** and **R** accordingly.

- Press [GRAPH].

</td><td>





</td></tr>
</table>

(b) $x^2+y^2+2x-4y=0$

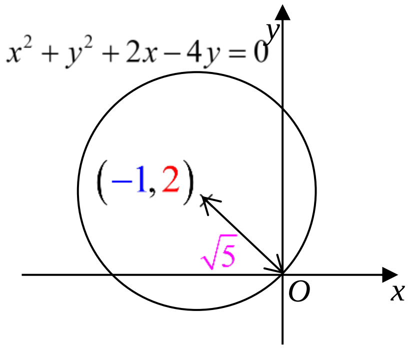
Solution:

$$x^2+y^2+2x-4y=0$$

Skill required: Completing the square.

$$(x+1)^2-1+(y-2)^2-4=0$$

$$(x-(-1))^2+(y-2)^2=(\sqrt{5})^2$$



It is a circle with centre $(-1,2)$

and radius $\sqrt{5}$ units

To check if origin passes through the circle:

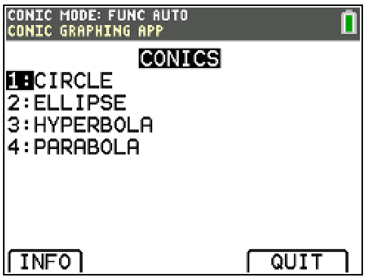
Substituting $(0,0)$ into the equation $x^2+y^2+2x-4y=0$.

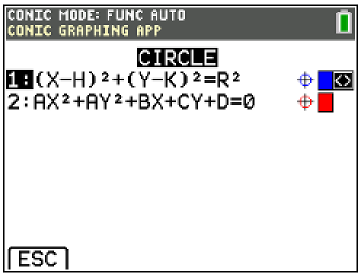
$\text{LHS}=0^2+0^2+2(0)-4(0)=0$

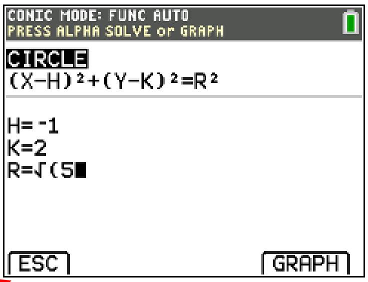
$\text{RHS}=0$

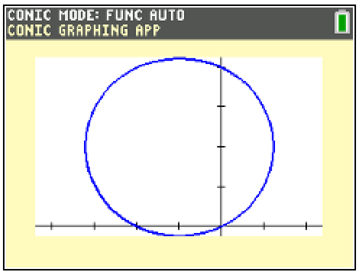
Since LHS = RHS, the circle passes through the origin.

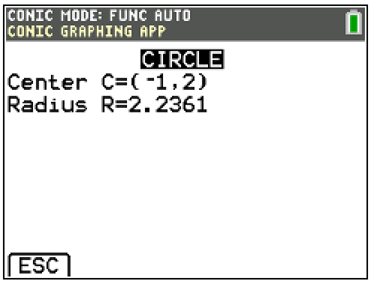
<table>
<tr><th>GC Steps</th><th>Screen on GC</th></tr>
<tr><td>• Press [apps].
Press [2] for CONICS APP.
Press [1] for CIRCLE.
Press [1] again.
(To select the correct form)

Alternatively, we can choose
2: $Ax^2+Ay^2+Bx+Cy+D=0$</td><td>


</td></tr>
<tr><td>• Key in the values of H, K and R accordingly.

• Press [graph].</td><td>


</td></tr>
<tr><td>Bonus Tips:
• Press [alpha] [enter].

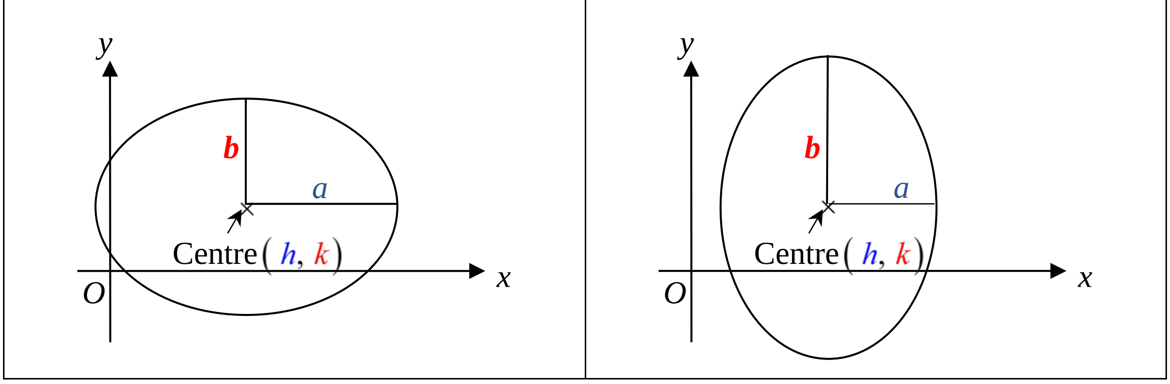
This will give the centre and radius of circle.</td><td>
</td></tr>
</table>

3.3 Ellipses

The equation of an ellipse with centre (h, k) takes the following form:

Standard form: $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ where $a>0,\ b>0$.

$0<b<a$	$0<a<b$



Note:

(1) If the ellipse is centred at the origin $(0,0)$, then the equation becomes $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

(2) The ellipse is symmetrical about the lines $x=h$ and $y=k$, where (h,k) is the centre of the ellipse.

(3) The equation of an ellipse has no term in *xy*.

(4) When $a \neq b$, the x^2 and y^2 terms have different coefficients but they have the same sign.

(5) When $a=b$, the equation becomes the equation of a circle of radius *a* and centre (h,k). Hence, **a circle is a special ellipse**.

Main Features of an Ellipse to be labelled:

(1) Coordinates of centre of the ellipse.

(2) Length of both horizontal/vertical semi-major/semi-minor axis of the ellipse.

(3) When the centre of the ellipse lies on either axis, label the axial intercepts.

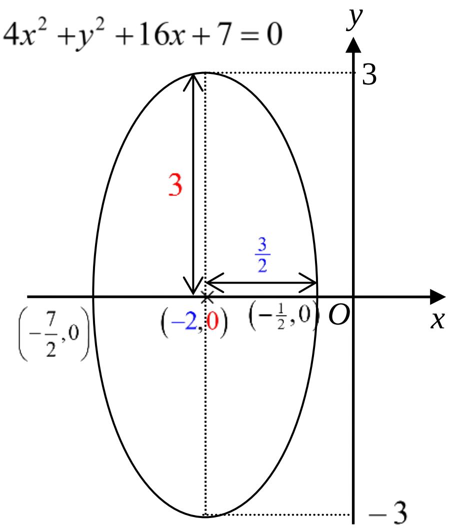
Example 7 (Sketching ellipse with main features)

Sketch and describe geometrically the graph of $4x^2+y^2+16x+7=0$. State the line(s) of symmetry.

Solution:

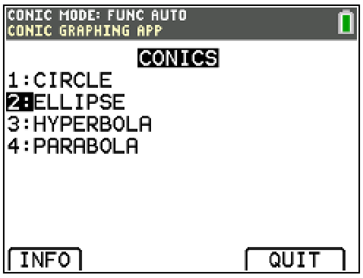
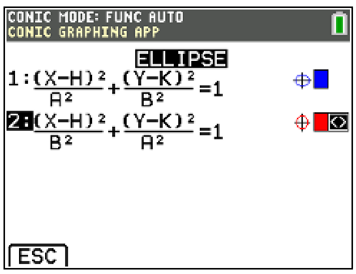
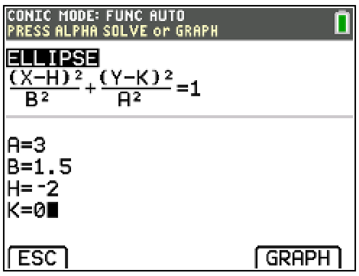
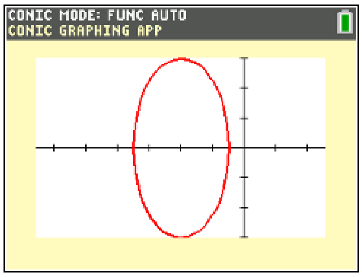
$$4x^2+y^2+16x+7=0$$
$$4\left(x^2+4x\right)+y^2+7=0$$
$$4\left[(x+2)^2-4\right]+y^2=-7$$
$$4(x+2)^2+y^2=9$$
$$\frac{(x+2)^2}{\frac{9}{4}}+\frac{y^2}{9}=1$$
$$\frac{(x-(-2))^2}{\left(\frac{3}{2}\right)^2}+\frac{(y-0)^2}{3^2}=1$$

Skill required:
Completing the square.



It is an ellipse, centre at $(-2,0)$ with vertical **semi**-major axis of length 3 units and horizontal **semi**-minor axis of length $\frac{3}{2}$ units.

The lines of symmetry are $x=-2$ and $y=0$.

GC Steps	Screen on GC
• Press [apps]. Press [2] for CONICS APP. Press [2] for ELLIPSE. Press [2]. (To select the correct form)	 
• Key in the values of **H**, **K** and **R** accordingly. • Press [graph].	 

Bonus Tips:

• Press [alpha] [enter]

This will give the centre of the ellipse

3.4 Hyperbolas

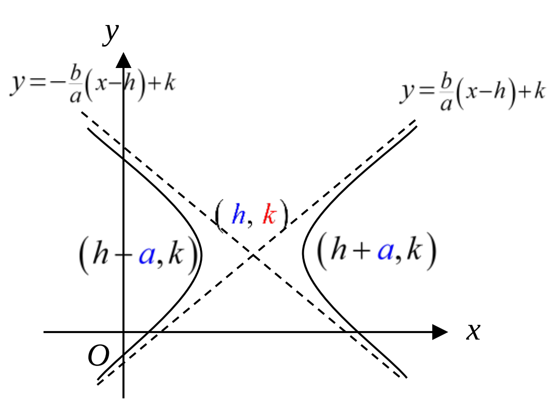
The equation of a hyperbola with centre (h, k) takes the following form:

Standard form:

1. $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$, where $a>0$, $b>0$.
2. $\dfrac{(y-k)^2}{b^2}-\dfrac{(x-h)^2}{a^2}=1$, where $a>0$, $b>0$

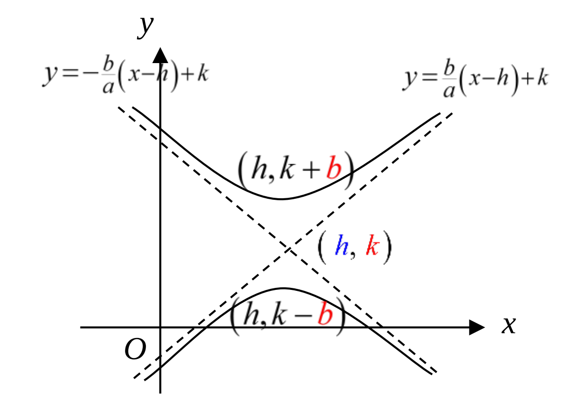
*Note: (h,k) is also the **intersection point** of the two oblique asymptotes of the hyperbola.

$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$



Vertices are $(h-a,k)$ and $(h+a,k)$

$\dfrac{(y-k)^2}{b^2}-\dfrac{(x-h)^2}{a^2}=1$



Vertices are $(h,k-b)$ and $(h,k+b)$

$y=k\pm\dfrac{b}{a}(x-h)$ are the oblique asymptotes.

Proof (Not in Syllabus)

Consider $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

Rewriting, we obtain $y=\pm\dfrac{b}{a}x\sqrt{1-\dfrac{a^2}{x^2}}$.

As $x\to\pm\infty$, $\dfrac{a^2}{x^2}\to 0$. Hence, $y\to\pm\dfrac{b}{a}x$.

Consequently, we get $y=\pm\dfrac{b}{a}x$ as the oblique asymptotes. (Similarly for $\dfrac{y^2}{b^2}-\dfrac{x^2}{a^2}=1$.)

Note:

(1) If the hyperbola is centred at the origin $(0,0)$, then the equation becomes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ and $y=\pm\frac{b}{a}x$ are the oblique asymptotes.

(2) The hyperbola is symmetrical about the line $x=h$ and $y=k$, where (h,k) is the centre of the hyperbola.

(3) The equation of a hyperbola has no term in *xy*.

(4) The x^2 and y^2 terms have different sign.

Main Features of a Hyperbola to be labelled:

(1) Equations of the oblique asymptotes of the hyperbola.

(2) Coordinates of centre of the hyperbola (the intersection of the oblique asymptotes – can obtain from GC).

(3) Coordinates of the vertices of the hyperbola (can obtain from GC).

Example 8 (Sketching hyperbola with main features)

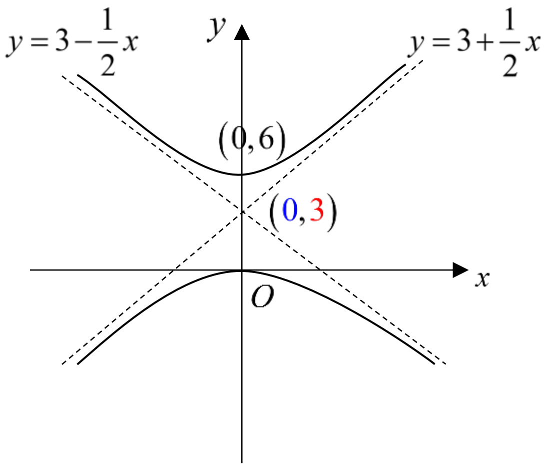
Sketch the curve with equation $(y-3)^2-\frac{x^2}{4}=9$.

Solution:

$$(y-3)^2-\frac{x^2}{4}=9$$

$$\frac{(y-3)^2}{9}-\frac{x^2}{36}=1$$

$$\frac{(y-3)^2}{3^2}-\frac{(x-0)^2}{6^2}=1$$



Axial Intercepts: $(0,0)$ and $(0,6)$

Oblique Asymptotes: $y=3\pm\frac{1}{2}x$

Finding the asymptotes,

consider $\frac{(y-3)^2}{3^2}-\frac{x^2}{6^2}=0$

$$(y-3)^2=\frac{3^2}{6^2}x^2$$

$$y-3=\pm\frac{1}{2}x$$

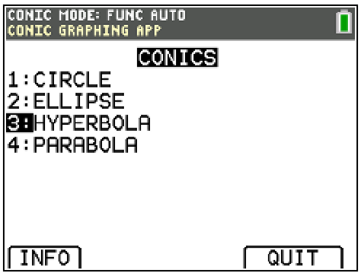
$\therefore y=3\pm\frac{1}{2}x$ are the oblique asymptotes

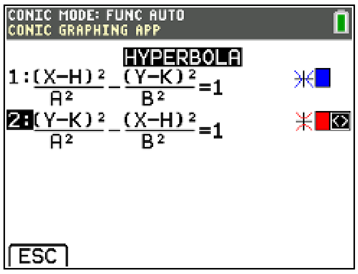
<table>
<tr><th>GC Steps</th><th>Screen on GC</th></tr>
<tr><td>

- Press [apps].

Press [2] for CONICS APP.
Press [2] for HYPERBOLA.
Press [3].
(To select the correct form)

</td><td>

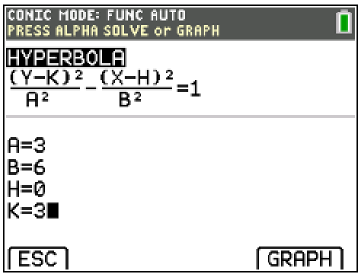


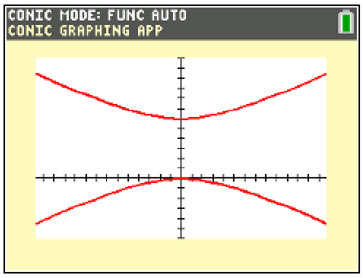


</td></tr>
<tr><td>

- Key in the values of **A**, **B**, **H** and **K** accordingly.
- Press [graph].

</td><td>





</td></tr>
</table>

Bonus Tips:

- Press [alpha] [enter].

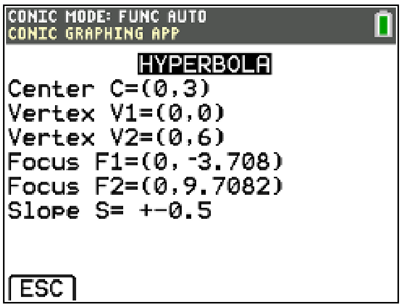
This will give

1) 2 vertices
2) Centre (intersection point of the 2 oblique asymptotes)
3) Slope (gradient of the 2 oblique asymptotes)

With 2) and 3), we can also work out the equations of the asymptotes using the equation of a straight line:

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \pm\frac{1}{2}(x - 0)$$
$$y - 3 = \pm\frac{1}{2}x$$
$$y = \pm\frac{1}{2}x + 3$$

$\therefore y = 3 \pm \frac{1}{2}x$ are the oblique asymptotes .



Limitations of Conics App

The use of the Conics App is useful in helping us find the centre and radius of a circle, and the vertices and asymptotes of a hyperbola. Other than these, there are some limitations to the App, in which case the conventional way of using the graphing calculator to sketch a curve is advised:

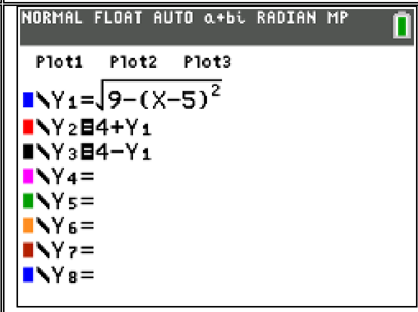
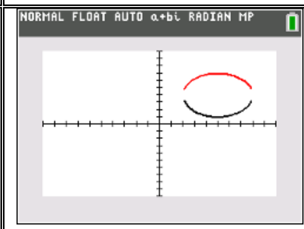
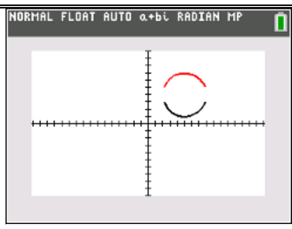
(i) We cannot find the *x*-intercepts and *y*-intercepts when we use this application to draw circles, ellipses and hyperbolas.

(ii) This application does not allow us to sketch additional graphs on the same diagram. Thus we would not be able to find the intersection points of the curves. (This can be done when you draw the graphs the conventional way in the [Y=] screen)

Sketching a conics graph the conventional way

Example

Sketch the graph of $(x-5)^2+(y-4)^2=3^2$.

Step 1: Recall that this is the standard from of the equation of a circle. First, convert the equation to the form of $y=\mathrm{f}(x)$. (make *y* the subject)	$(x-5)^2+(y-4)^2=3^2$ $(y-4)^2=3^2-(x-5)^2$ $y=4\pm\sqrt{3^2-(x-5)^2}$
Step 2: Sketch the curve. Press [Y=]. Key in $\sqrt{9-(x-5)^2}$ for Y_1 and un-highlight at the equal sign so that Y_1 will not be plotted. Key in [ALPHA] [TRACE][1] to input Y_1 at Y_2 and Y_3.	
Step 3: Sketch the curve. The default graph (or [ZOOM][6] (ZStandard)) does not look like a circle, due to the different scale of the axes. Use [ZOOM][5] (ZSquare) to standardize the scale on both axes.	 

Note: The "gap" in the circle is a limitation resulting from the screen resolution.

§4 <u>Piecewise Functions</u>

Generally, a piecewise function is a function that is defined by different expressions on mutually exclusive (non-overlapping) intervals.

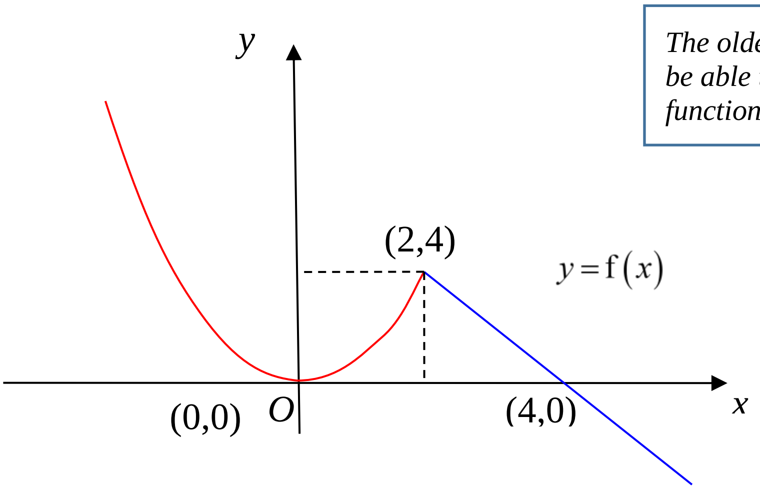
Example 9 (Sketching piecewise function with end points)

(a) The function $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x)=\begin{cases} x^2, & x\le 2 \\ 8-2x, & x>2 \end{cases}$$

Sketch the graph of $y=\mathrm{f}(x)$.

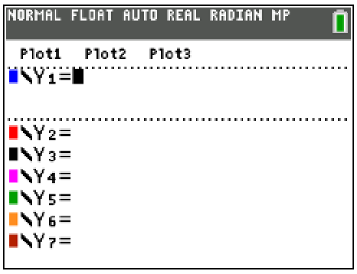
Solution:

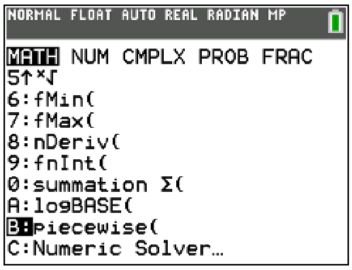


The older model of GC will not be able to use the B:piecewise function.

<table>
<tr><th>GC Steps</th><th>Screen on GC</th></tr>
<tr><td>

- Press [y=].
- Press [math] and scroll up by pressing [▲] twice to go to **B:piecewise(** function. Then press [enter].

</td><td>




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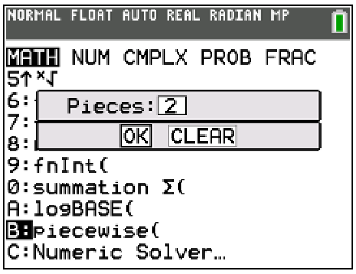
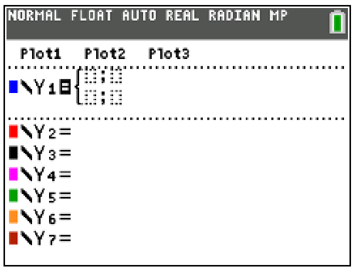
<tr><td>

- Press [◄] once to choose 2 pieces. Then press [enter] twice to confirm.

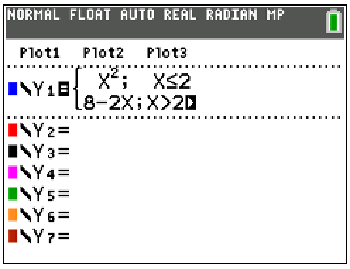
</td><td>


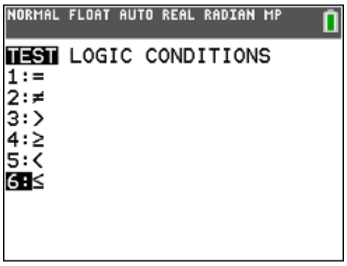


</td></tr>
</table>

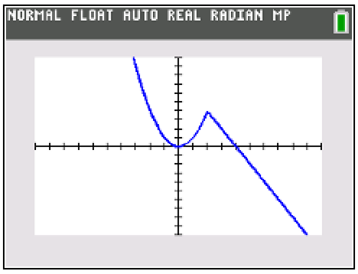
- Key in the given equations accordingly.
- Press [▶] to go to the right side of the '**;**' sign and key in the restricted range of values of *x* for each of these two equations.

Note:
To use inequality signs in your GC, press [2nd][math].

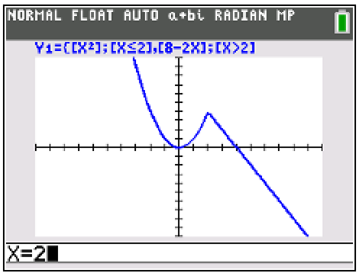


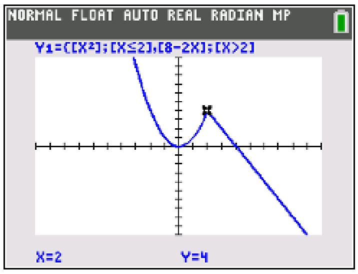


- Press [zoom] and select **6: ZStandard** to view the graph in standard window settings after you have keyed in the equations in Y_1.



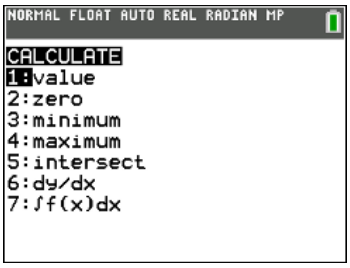
- You may use the **TRACE** function to obtain the coordinates of the point at $x = 2$.
- Press [trace].
- Press [2] followed by [enter] for the GC to return the value of *y*, given $x = 2$.





<u>Alternatively,</u>

- You may use the **CALCULATE** function to obtain the coordinates of the point where the 2 graphs meet at $x = 2$.
- Press [2nd][trace] to go to **CALCULATE** function.
- Press [1] to choose **1:value**.
- Press [2] followed by [enter] for the GC to return the value of *y*, given $x = 2$.



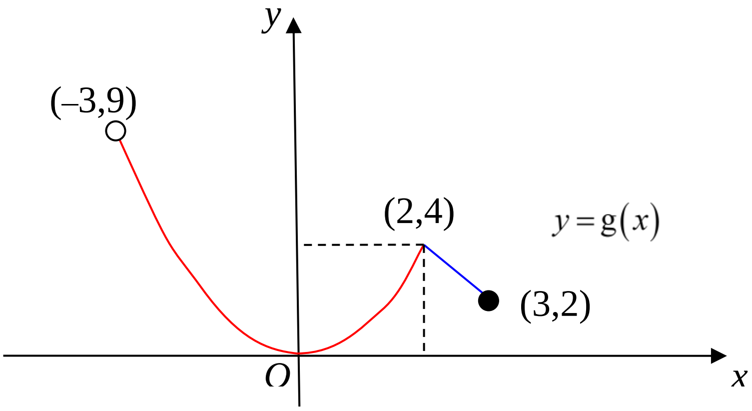
(b) Sketch the graph of $y = \mathrm{g}(x)$ where $\mathrm{g}(x) = \begin{cases} x^2, & -3 < x \le 2, \\ 8 - 2x, & 2 < x \le 3. \end{cases}$

Solution:

When $x = -3, y = (-3)^2 = 9 \Rightarrow (-3, 9)$ is excluded $\Rightarrow$ open circle

When $x = 2, y = 2^2 = 4 \Rightarrow (2, 4)$ is included
When $x = 2, y = 8 - 2(2) = 2 \Rightarrow (2, 4)$ is excluded
} continuous

When $x = 3, y = 8 - 2(3) = 2 \Rightarrow (3, 2)$ is included $\Rightarrow$ closed circle



§5 Modulus Functions

The modulus of x, written as $|x|$ is defined by

$$|x| = \begin{cases} x & , \text{ if } x \geq 0 \\ -x & , \text{ if } x < 0 \end{cases}$$

For example, $|3| = 3$ and $|-2| = -(-2) = 2$.

Example 10 (Sketching modulus function)

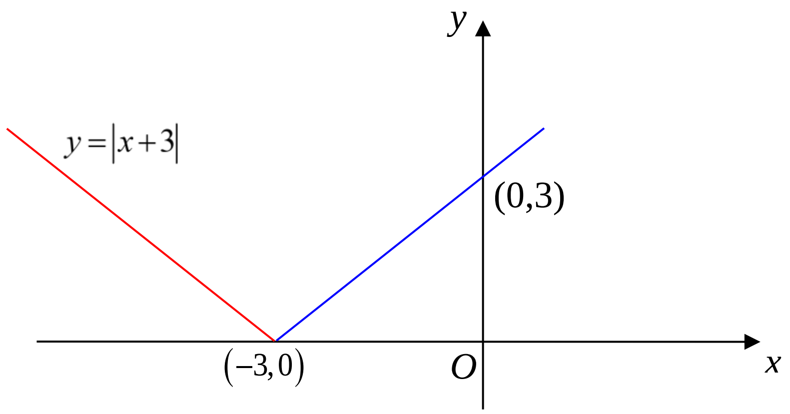
(a) Sketch $y = |x+3|$.

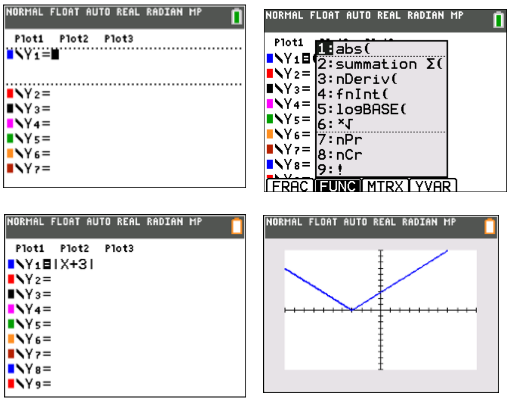
Solution:

$$|x+3| = \begin{cases} x+3 & , \text{ if } x+3 \geq 0 \\ -(x+3) & , \text{ if } x+3 < 0 \end{cases}$$

$$\Rightarrow |x+3| = \begin{cases} x+3 & , \text{ if } x \geq -3 \\ -(x+3) & , \text{ if } x < -3 \end{cases}$$

Unpacking the equations of each piece of function.



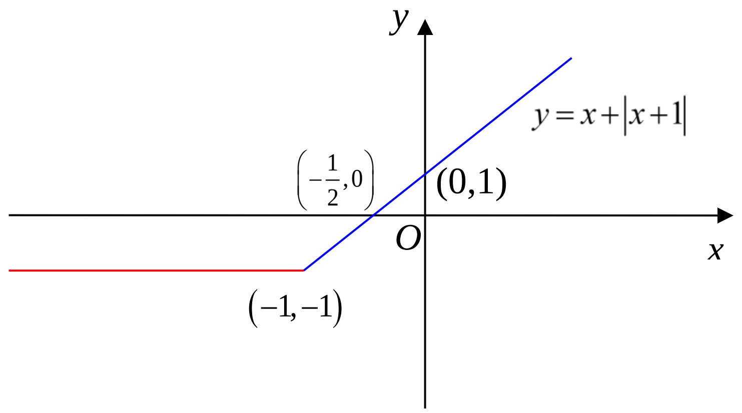
GC Steps	Screen on GC
• Press [y=]. • Press [ALPHA][WINDOW] Select **1:abs(** • Key in the given equations accordingly. • Press [zoom] and select **6: ZStandard** to view the graph in standard window settings. after you have keyed in the equations in Y_1.	

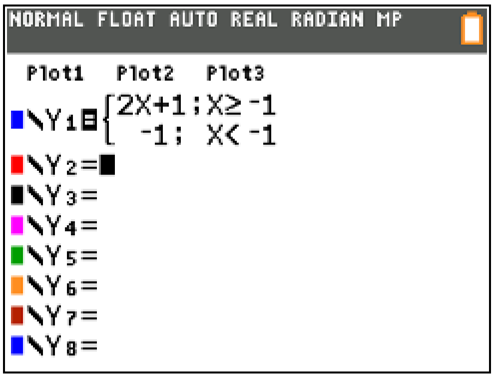
(b) Express $y = x + |x+1|$ as a piecewise function. Hence, sketch $y = x + |x+1|$.

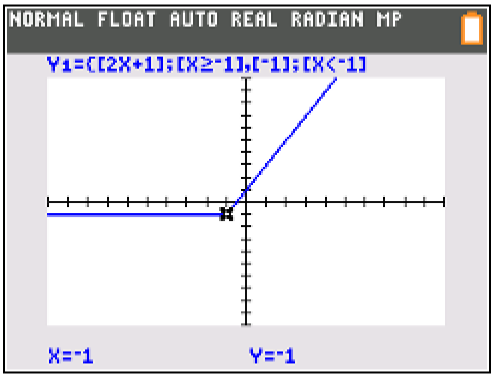
Solution:

$$x + |x+1| = \begin{cases} x + (x+1) & , \text{ if } x+1 \geq 0 \\ x - (x+1) & , \text{ if } x+1 < 0 \end{cases}$$

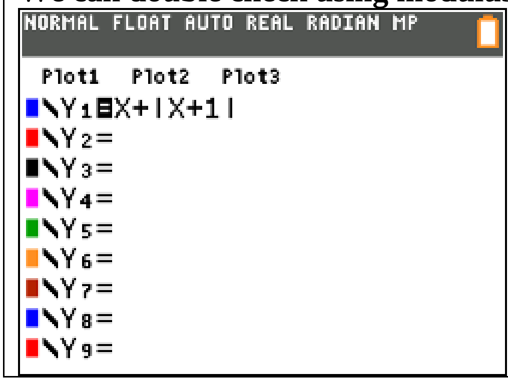
$$\Rightarrow x + |x+1| = \begin{cases} 2x+1 & , \text{ if } x \geq -1 \\ -1 & , \text{ if } x < -1 \end{cases}$$

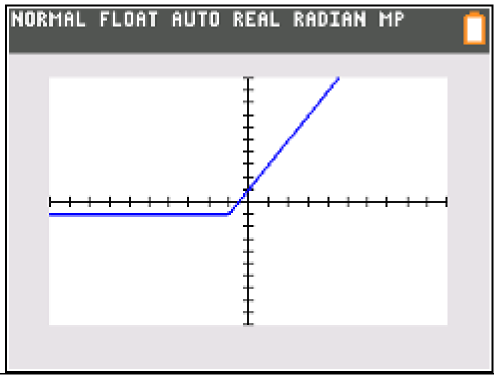






We can double check using modulus function:

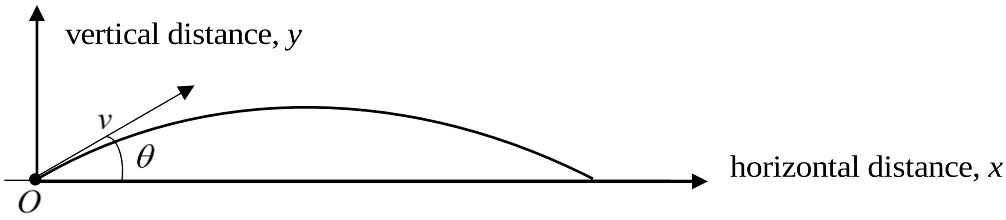




§6 <u>Parametric Equations</u>

In many real life applications, the variables x and y are dependent of a third variable, for example, time.

Consider the projectile motion of an object:



In this example, the horizontal and vertical displacement travelled by the object is clearly dependent on the time elapsed after it is being projected. Thus, we would express the relationships as follows:

Horizontal displacement: $x=\left(v\cos\theta\right)t$

Vertical displacement: $y=\left(v\sin\theta\right)t-\frac{1}{2}gt^2$

where v is the velocity the object is being projected, θ is the angle of projection and g is the free-fall acceleration.

The above pair of equations is called the parametric equations of the location of the object at time t and t is called a parameter.

Suppose $v=3\sqrt{2},\ \theta=\frac{\pi}{4},\ g=9.81$.

Horizontal distance: $x=3t$

Vertical distance: $y=3t-\frac{1}{2}\left(9.81\right)t^2$

By eliminating t, we get $y=x-\frac{1}{18}\left(9.81\right)x^2$. This equation is known as the Cartesian equation of the path of the object, relating the horizontal and vertical displacements.

Other Examples of parametric curves:

(i) $x = \cos t,\ y = \sin t$; **(ii)** $x = 2t,\ y = \frac{1}{t}$; **(iii)** $x=t^3,\ y=t^2-t$.

By <u>eliminating t</u> from the equations, the <u>cartesian equations</u> of the above parametric equations are:

(i) $x^2+y^2=1$ **(ii)** $y=\frac{2}{x}$ **(iii)** $y=x^{\frac{2}{3}}-x^{\frac{1}{3}}$

Example 11 (Sketching parametric equations with change in window settings)

Sketch the curve defined by the parametric equations $x = 1 + t$, $y = 4t - t^2$, $t \in \mathbb{R}$.

Solution:

To find axial intercept:

When $x = 0$, $t = -1$, $y = -5$.

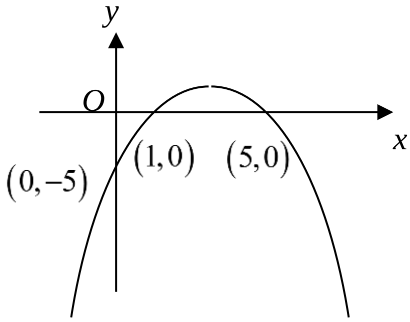
y–intercepts: $(0, -5)$

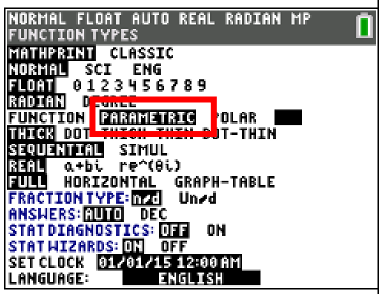
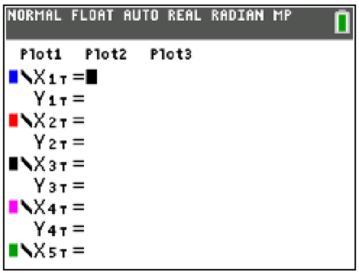
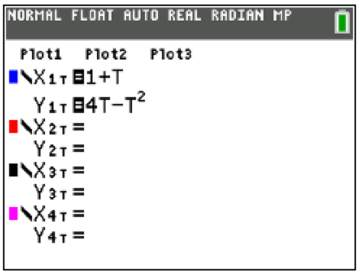
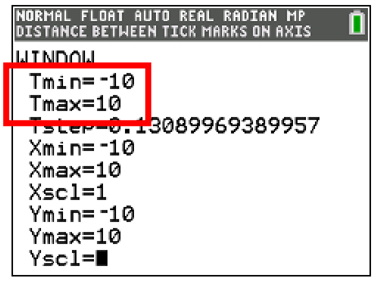
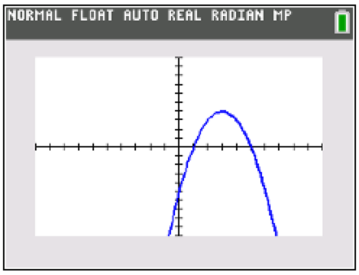
When $y = 0$, $t = 0$ or 4

When $t = 0$, $x = 1$

When $t = 4$, $x = 5$

x–intercepts: $(1, 0)$ and $(5, 0)$



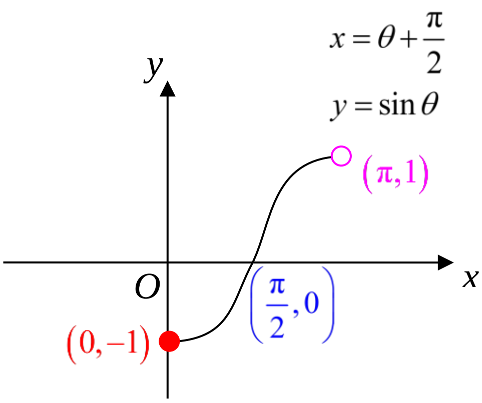
GC Steps	Screen on GC
To sketch curve with parametric equations using GC, you have to change the **mode** to PARAMETRIC! • Press [mode] and scroll down to "FUNCTION PARAMETRIC POLAR SEQ". Choose **PARAMETRIC** by pressing [enter] after scrolling the cursor over it.	
• Press [y=] to key in the parametric equations. You may have to clear any default equations appeared on the screen. • Press [X,T,θ,n] for the use of variable T. Key in the equations for X_{1T} and Y_{1T} accordingly.	 
• Press [window] to set the range of values of t given in the question. (i.e. $t \in \mathbb{R}$) *Set Tmin = –10 and Tmax = 10* **By default**, it is *Tmin* = 0 *and* *Tmax* = 2π **Note:** if you press ZStandard, it will **return to default settings**. • Press [graph].	 

Example 12 (Sketching parametric equations with end points)

Sketch the curve defined by the parametric equations $x=\theta+\frac{\pi}{2},\ y=\sin\theta,\ -\frac{\pi}{2}\le\theta<\frac{\pi}{2}$.

Solution:

To find coordinates of axial intercepts:



When $x=0,\ \theta=-\frac{\pi}{2} \Rightarrow y=\sin\left(-\frac{\pi}{2}\right)=-1$

y–intercept: $(0,-1)$.

When $y=0,\ \theta=\sin^{-1}(0)=0 \Rightarrow x=0+\frac{\pi}{2}=\frac{\pi}{2}$.

x–intercept: $\left(\frac{\pi}{2},0\right)$.

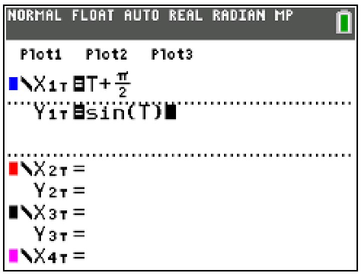
To find coordinates of end points:

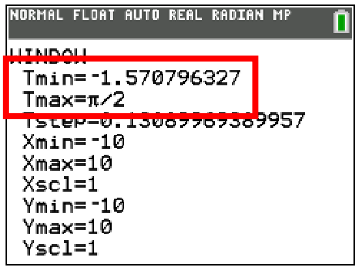
When $\theta=-\frac{\pi}{2},\ x=0,\ y=-1$ as found above. $(0,-1)$ is included $\Rightarrow$ closed circle

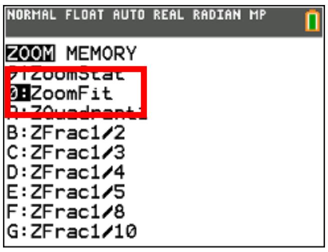
When $\theta=\frac{\pi}{2},\ x=\frac{\pi}{2}+\frac{\pi}{2}=\pi,\ y=\sin\left(\frac{\pi}{2}\right)=1$. $(\pi,1)$ is excluded $\Rightarrow$ open circle

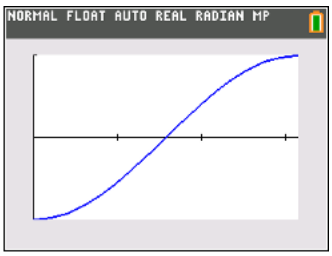
End-points: $(0,-1)$ and $(\pi,1)$.

<table>
<tr><th>GC Steps</th><th>Screen on GC</th></tr>
<tr><td>
• Press [mode] to key in the parametric equations.
• Press [X,T,θ,n] for the use of variable θ (it will appear as T on the GC).
• Press [window] to set the range of values of θ given in the question. (i.e. $-\frac{\pi}{2}\le\theta<\frac{\pi}{2}$)
By default, it is $Tmin=0$ *and* $Tmax=2\pi$

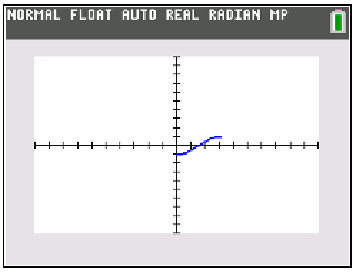
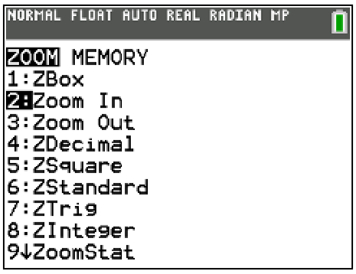
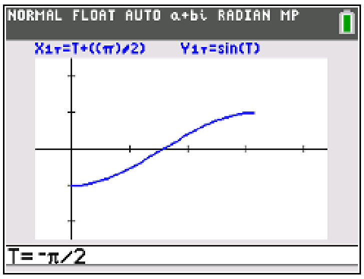
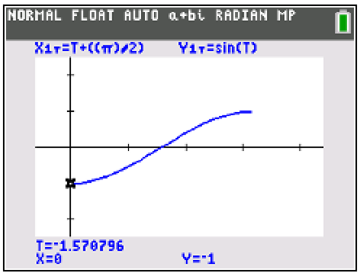
Note: if you press ZStandard, it will **<u>return to default settings</u>**.
</td><td>




</td></tr>
<tr><td>
• Press [ZOOM] and select option **0:** Zoomfit to fit the window according to the range of values of θ
</td><td>




</td></tr>
</table>

Alternatively • Press [graph]. **Note:** You can consider using Zoom In to have a better view of the curve.	 
Useful feature: • You may use the **TRACE** function to obtain the coordinates of the point at $\theta=-\frac{\pi}{2}$. • Press [trace]. • Press [(-)][2nd][^][÷][2] followed by [enter] for the GC to return the value of x and y, given $\theta=-\frac{\pi}{2}$.	 

Example 13 (Sketching parametric equations with asymptotes)

Sketch the curve defined by the parametric equations $x = t^2 + t, y = t - \frac{1}{t}$ where $t > 0$.

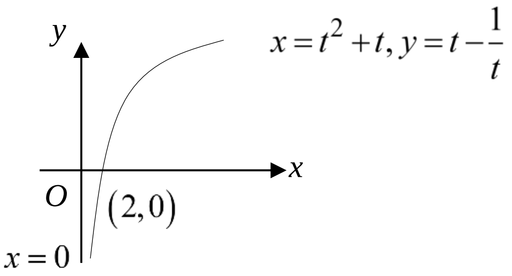
Solution:

To find equation of vertical asymptote:

Let $y \to \pm\infty$, then $t \to 0$.

Thus, $x \to 0$.

Vertical asymptote: $x = 0$

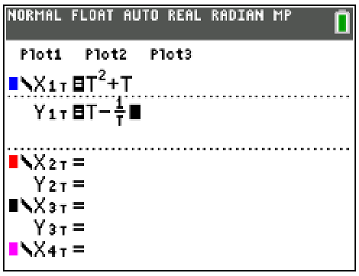
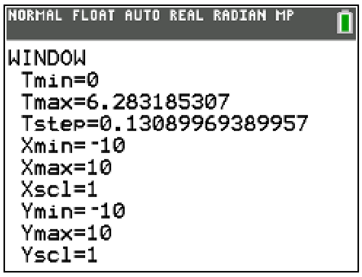
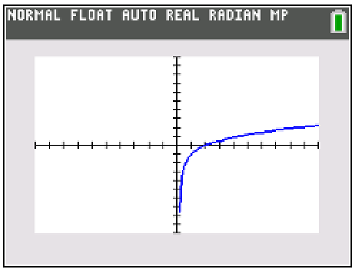


To find axial intercept:

(Note: $x \neq 0$)

When $y = 0,\ t = 1,\ x = 2$

Axial Intercepts: $(2, 0)$

GC Steps	Screen on GC
• Press [y=] to key in the parametric equations. • Press [X,T,θ,n] for the use of variable θ (it will appear as T on the GC). • Press [window] to set the range of values of t given in the question. (i.e. $t > 0$) *Ensure* $Tmin = 0$ *and Tmax could be any positive value* **By default**, it is $Tmin = 0$ *and* $Tmax\ = 2\pi$. **Note:** if you press ZStandard, it will **return to default settings**. • Press [graph].	  

Example 14 (Conversion of parametric equation to cartesian equation)
Find the Cartesian equation of the curve whose parametric equations are
$x = 1+t$, $y = 4t - t^2$, $t \in \mathbb{R}$

Solution:

To convert the parametric equations to Cartesian form, we eliminate t.

$x = 1+t \Rightarrow t = x-1$

Substitute $t = x-1$ into $y = 4t - t^2$

$y = 4(x-1) - (x-1)^2$

$y = (x-1)(4-x+1)$

$y = (x-1)(-x+5)$

> Easiest way to eliminate *t* is to make *t* the subject (for whichever is easier)

Example 15 (Conversion of parametric equation involving trigonometric functions to cartesian equation)
Find the Cartesian equation of the curve whose parametric equations are $x = 3\cos\theta$, $y = 3\sin\theta$.

Solution:
To convert the parametric equations to Cartesian form, we eliminate θ.

$x = 3\cos\theta \Rightarrow \cos\theta = \frac{x}{3}$

$y = 3\sin\theta \Rightarrow \sin\theta = \frac{y}{3}$

Since $\sin^2\theta + \cos^2\theta = 1$, $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$$x^2 + y^2 = 9$$

[Note: The equation represents a circle with centre $(0,0)$ and radius 3 units.]

> To eliminate θ, we make use of trigonometric identities.

Note:

(1) Every relationship between *x* and *y* of the form $y = \mathrm{f}(x)$ may be expressed parametrically and the representation is **not unique**.

E.g. $y = (x-1)(x-3)$ may be expressed as $\begin{cases} x = t \\ y = (t-1)(t-3) \end{cases}$ or $\begin{cases} x = t+1 \\ y = t(t-2) \end{cases}$.

(2) For parametric equations containing trigonometric terms, it would be very helpful to use the appropriate trigonometric identities to remove the parameter.

Trigonometric Identities

(i) $\cos^2 x + \sin^2 x = 1$

(ii) $1 + \tan^2 x = \sec^2 x$

(iii) $\cot^2 x + 1 = \operatorname{cosec}^2 x$

H2 Mathematics (9758)
Chapter 1 Graphing Techniques
Discussion Questions

Level 1

1 The curve C_1 and C_2 have equations $y = \mathrm{e}^{-x} + 1$ and $y = \ln(x+2)$ respectively.

(i) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes.

(ii) Use your calculator to determine the coordinates of the intersection point(s).

2 Write down the equations of asymptotes of the following graphs:

(a) $y = \dfrac{2}{1-x}$ **(b)** $y = 2 - \dfrac{7}{x+3}$ **(c)** $y = x - 1 - \dfrac{1}{x-2}$

Hence, sketch the graphs on separate diagrams, indicating equations of asymptotes and coordinates of axial intercepts and turning points (if any).

3 Sketch, on separate diagrams, the following graphs:

(a) $(x-3)^2 + (y+4)^2 = 25$ **(b)** $x^2 + \dfrac{y^2}{4} = 1$

(c) $\dfrac{y^2}{4} - \dfrac{x^2}{16} = 1$ **(d)** $\dfrac{(x-2)^2}{4} - y^2 = 1$

indicating clearly the main relevant features of the graph.

4 Sketch, on separate diagrams, the following graphs:

(a) $y = \begin{cases} x, & \text{if } 0 \le x \le 2, \\ 2, & \text{if } x > 2. \end{cases}$

(b) $y = |x-3| + |x+2|$

5 **(i)** Sketch the curve C defined by the parametric equations $x = \dfrac{1}{t^2}, y = 2t$ where t is a non-zero real parameter.

(ii) Find the Cartesian equation of the curve C.

Level 2

6 H2 Specimen Paper 2006/1/9 Modified

Consider the curve $y=\dfrac{3x-6}{x(x+6)}$.

(i) State the coordinates of any points of intersection with the axes.

(ii) State the equations of the asymptotes.

(iii) Prove, using an algebraic method, that $\dfrac{3x-6}{x(x+6)}$ cannot lie between two certain values (to be determined).

(iv) Draw a sketch of the curve, $y=\dfrac{3x-6}{x(x+6)}$, indicating clearly the main relevant features of the curve.

7 N2009/1/6

The curve C_1 has equation $y=\dfrac{x-2}{x+2}$. The curve C_2 has equation $\dfrac{x^2}{6}+\dfrac{y^2}{3}=1$.

(i) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

(ii) Show algebraically that the *x*-coordinates of the points of intersection of C_1 and C_2 satisfy the equation $2(x-2)^2=(x+2)^2(6-x^2)$. [2]

(iii) Use your calculator to find these *x*-coordinates. [2]

8 2013/NJC Promo/12(b)(modified)

The curve C_3 has equation $y=\dfrac{x-1}{x+1}$. The curve C_4 has equation $\dfrac{x^2}{20}-\dfrac{y^2}{5}=1$.

Sketch C_3 and C_4 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

Hence find the number of solutions to the equation $x^2-\dfrac{4(x-1)^2}{(x+1)^2}=20$. [2]

9 2018/MI/Promo/8

The curve *C* has equation $9x^2+18x+4y^2-8y=23$.

(i) By completing the square, show that the equation of *C* can be expressed as $\dfrac{(x+1)^2}{2^2}+\dfrac{(y-1)^2}{3^2}=1$. [2]

(ii) Sketch *C*, stating the coordinates of any points of intersection with the axes. [3]

10 Sketch, on the **same** diagram, $x^2+y^2-4y=0$ and $x^2-4(y-2)^2=4$, giving the equations of asymptotes and other relevant features. [5]

11 Sketch the graph of $y=\text{f}(x)$ where

(a) $\text{f}(x)=\begin{cases} 2x+1, & -2<x\le 1, \\ -x^2+2x+2, & x>1. \end{cases}$

(b) $\text{f}(x)=\begin{cases} 3x-11, & \text{for } x\in\mathbb{R}, x\le 4, \\ 4-x, & \text{for } x\in\mathbb{R}, x>4. \end{cases}$

12 **(a)** A curve *C* has parametric equations $x=2+\cos\theta,\ \ y=1+\sin\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.

(i) State the range of values of *x* and *y*.

(ii) Find a Cartesian equation of curve *C*. Hence sketch *C*.

(b) Find the Cartesian equation of the curve whose parametric equations are $x=2\tan\theta,\ \ y=3\cos\theta.$

<u>Level 3</u>

13 **2018/DHS Promo/Q4 (Modified)**

The curve *C* has equation

$$y=\frac{x^2+2x+1}{x-p},\ \ x\neq p,$$

where *p* is a real constant. It is given that the line $y=x+3$ is an asymptote of *C*. Show that $p=1$. [2]

(i) Sketch *C*. [3]

(ii) By adding another graph, deduce that for all positive β, the equation

$$(\beta+1)x^2+2x+1=\beta x^3$$

has exactly one real root. [2]

14 **2013/PJC Prelim/2/5**

The curve *C* has equation $y=\frac{x^2+ax+b}{c-x}$. The vertical asymptote of *C* is $x=-2$, and the coordinates of the turning points are $(-4,2)$ and $(0,-6)$.

(i) Find the values of *a*, *b* and *c*. [3]

(ii) Sketch *C*, stating the equations of the asymptotes. [2]

(iii) By drawing an appropriate graph on the sketch of *C*, find the range of values of *k* $(k>0)$ such that the equation $(x+4)^2+\left(\frac{x^2+ax+b}{c-x}\right)^2=k^2$ has no real roots. [2]

15 2013/NJC Promo/12(a)(modified)

The curve C_1 has parametric equations

$$x = t^2 + t, \quad y = 4t - t^2, \quad -1 \le t \le 1.$$

(i) Sketch C_1, labelling the coordinates of the end-points and the axial intercepts (if any) of this curve. [2]

(ii) The curve C_2 is defined parametrically by the equations

$$x = t^2 + t, \quad y = 4t - t^2, \quad t \in \mathbb{R}.$$

Find a Cartesian equation of C_2. [2]

Answers:

	Answers
1(ii)	Using GC, the intersection point is $(1.44, 1.24)$ to 3 s.f
2(a)	Asymptotes: $x = 1, y = 0$
2(b)	Asymptotes: $x = -3, y = 2$
2(c)	Asymptotes: $x = 2, y = x - 1$
5(ii)	$xy^2 = 4$
6(i)	$(2, 0)$
6(ii)	Asymptotes: $x = 0, x = -6$, $y = 0$
6(iii)	$\dfrac{3x-6}{x(x+6)}$ cannot lie between $\dfrac{1}{6}$ and $\dfrac{3}{2}$.
7(iii)	$x = -0.515$ and $x = 2.45$
8	Number of solutions = 2
12(a)(i)	$2 \le x \le 3$ and $0 \le y \le 2$
12 (a)(ii)	$(y-1)^2 + (x-2)^2 = 1$, $2 \le x \le 3$
14(i)	$a = 6,\ b = 12,\ c = -2$
14(iii)	$0 < k < 2$
15(ii)	$(x+y)^2 = 5(4x - y)$ Alternate form: $y = \pm 5\sqrt{x + \frac{1}{4}} - \frac{5}{2} - x$; $x = 10 \pm 5\sqrt{4 - y} - y$

H2 Mathematics (9758)
Chapter 1 Graphing Techniques
Extra Practice Questions

1 Sketch the graph of $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \begin{cases} 7 - x^2, & \text{for } 0 < x \leq 2, \\ 2x - 1, & \text{for } 2 < x \leq 4. \end{cases}$$

2 **2017/PJC Promo/11(i)**

The curve C has parametric equations

$$x = \mathrm{e}^t + 2t, \qquad y = \mathrm{e}^t + t\,.$$

Sketch C for $-3 \leq t \leq 2$. [1]

3 **2017/RVHS Promo/12(i)(ii)**

A curve C has parametric equations

$$x = 2\cos t \text{ and } y = \sin t \text{ for } 0 \leq t < 2\pi\,.$$

(i) Find the cartesian equation of C. [2]

(ii) Sketch C. [2]

4 **2016/SAJC Promo/2(i)**

Sketch the curve given by the equation $2x^2 - 12x + y^2 + 2y + 17 = 0$, showing clearly the main features of the graph. [3]

5 2018/AJC Promo/4

The curve C_1 has equation $y = \dfrac{x^2 - 5x + 10}{x - 2}$, $x \neq 2$. The curve C_2 has equation $x^2 - y^2 = 4$.

(i) Sketch C_1 and C_2 on the same diagram, indicating any points of intersection with the axes and the coordinates of any stationary points. Equations of asymptotes should be clearly labelled. [5]

(ii) Show algebraically that the x-coordinate of the point of intersection of C_1 and C_2 satisfies the equation $(x-2)^2(x^2-4) = (x^2-5x+10)^2$. [1]

(iii) Use your calculator to find this x-coordinate. [1]

6 2007/HCI Prelim/1/13 (Modified)

The curve C has equation $y = \dfrac{x^2 - 4x + 4}{x + a}$.

It is given that C has a vertical asymptote $x = -1$.

(i) Determine the value of a. [1]

(ii) Find the equation of the other asymptote of C. [1]

(iii) Prove, using an algebraic method, that C cannot lie between two values (to be determined). [4]

(iv) Draw a sketch of C, showing clearly any axial intercepts, asymptotes and stationary points. [3]

(v) Deduce the number of real roots of the equation

$$(4 - x^2)(x+1)^2 = (x^2 - 4x + 4)^2.$$ [2]

7 2009/SRJC Prelim/1/7

The curve C has equation $y = \dfrac{x^2 + b}{x + a}$, where $a > 0$ and $b > 0$.

(i) State the coordinates of the intersection(s) of the curve C with the axes in terms of a and b. [2]

(ii) Find the equation(s) of the asymptote(s). [2]

(iii) Draw a sketch of the curve C, labeling the equation(s) of its asymptote(s) and coordinates of any intersection with the axes. [2]

(iv) Hence find the range of values of k, where k is a positive constant, for which the equation $x^2 + b = (x + a)(kx - a)$ has no real root. [2]

8 2016/MJC Promo/3

In this question, a, b, c and d are non-zero constants.

The curve C has equation $y=\dfrac{ax^2+bx-7}{x-c}$ with oblique asymptote $y=dx+2$ and vertical asymptote intersecting at $(4, 6)$.

(i) Find the values of a, b, c and d. [3]

(ii) Sketch C. [3]

(iii) Hence find the set of values of k such that the equation $\dfrac{ax^2+bx-7}{x-c}=k$ has no real roots. [2]

9 2018/VJC Promo/Q6

A curve C has equation $y^2-9(x-1)^2=9$.

(i) Sketch C, indicating the relevant features on your diagram. [You are not required to give the coordinates of its intersections with any of the two axes.] [4]

(ii) Find the set of values of k such that the graph of $y=k(x-1)$ intersects C at two points. [2]

(iii) The curve with equation $\dfrac{(x-1)^2}{(2r)^2}+\dfrac{y^2}{r^2}=1$, where r is a positive constant, intersects C at 4 distinct points. Write down an inequality satisfied by r. [1]

Answer Key
3(i) $y^2+\left(\dfrac{x}{2}\right)^2=1$
5(iii) 3.66
6(i) $a=1$ **(ii)** $y=x-5$ **(v)** 2 real roots
7(i) $\left(0,\dfrac{b}{a}\right)$ **(ii)** $x=-a$; $y=x-a$ are equations of the asymptotes.
8(i) $a=1, b=-2, c=4, d=1$ **(iii)** $\{k\in\mathbb{R}:4<k<8\}$
9(iii) $r>3$