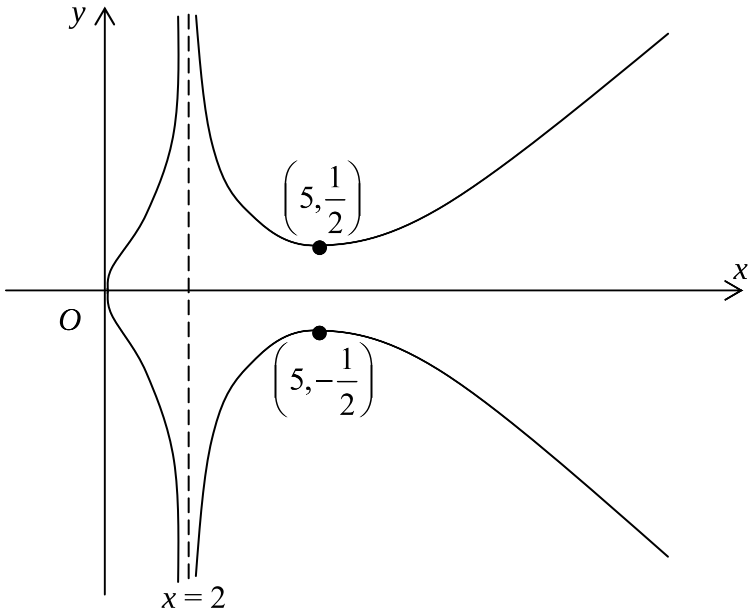
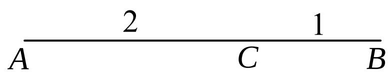
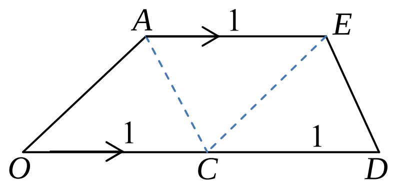
2014 Year 6 Prelim Examination Paper 1
Suggested Solution

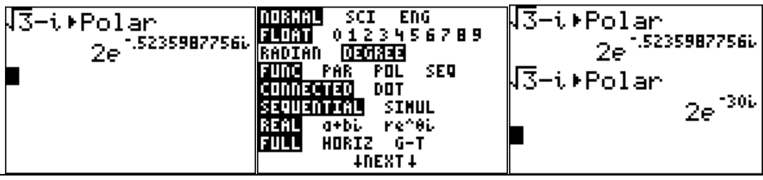
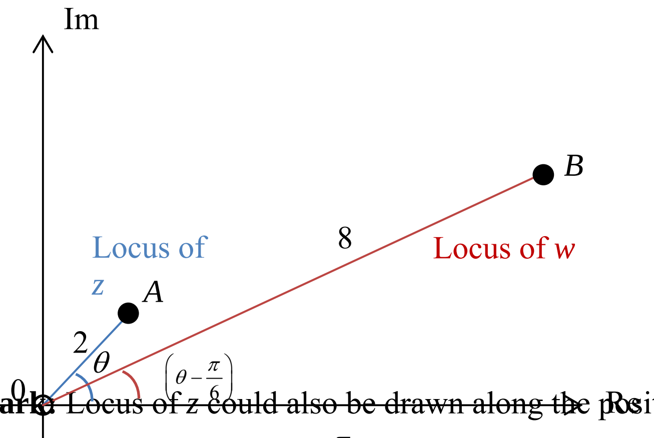
Qn	Suggested Solution
1(i)	m : weight of mackerel in kg s : weight of salmon in kg t : weight of tuna in kg $m + s + t = 800$ $7m + 21s + 39t = 20300$ $5m + 23s + 49t = 23900$ Using GC, $m = 200,\ s = 250,\ t = 350$. Therefore the fisherman has 250 kg of salmon.
(ii)	m : weight of mackerel in kg s : weight of salmon in kg t : weight of tuna in kg $m + s + t = 600$ $7m + 21s + 39t = 20300$ $5m + 23s + 49t = 23900$ Using GC, $m = -460,\ s = 990,\ t = 70$. Since the weight of all fishes must be non-negative, the fisherman's claim is not possible. Or Since the weight of salmon and tuna is more than 600kg, the fisherman's claim is not possible.

Qn	Suggested Solution
2(a)	
2(b)	$\frac{(x-1)^2}{4}+(y-2)^2=1$ Making y the subject of formula: $y=2\pm\sqrt{1-\frac{(x-1)^2}{4}}$ Let the volume of solid generated when the curve $y=2+\sqrt{1-\frac{(x-1)^2}{4}}$ is rotated about x-axis from $x=2$ to $x=3$ be V_1. $V_1=\int_2^3 \pi y^2\ \mathrm{d}x$ $=\int_2^3 \pi\left(2+\sqrt{1-\frac{(x-1)^2}{4}}\right)^2\ \mathrm{d}x$ = 21.593 Let the volume of solid generated when the curve $y=2-\sqrt{1-\frac{(x-1)^2}{4}}$ is rotated about x-axis from $x=2$ to $x=3$ be V_2. $V_2=\int_2^3 \pi y^2\ \mathrm{d}x$ $=\int_2^3 \pi\left(2-\sqrt{1-\frac{(x-1)^2}{4}}\right)^2\ \mathrm{d}x$ = 6.1573 Volume of required solid $= V_1 - V_2$ = 15.4 (to 3sf)

Qn	Suggested Solution
3(i)	$e^{y}=1+3x+2x^{2}$ Differentiate with respect to x, $e^{y}\left(\dfrac{dy}{dx}\right)=3+4x$ Differentiate again with respect to x, $e^{y}\left(\dfrac{dy}{dx}\right)^{2}+e^{y}\dfrac{d^{2}y}{dx^{2}}=4$ $\left(\dfrac{dy}{dx}\right)^{2}+\dfrac{d^{2}y}{dx^{2}}=4e^{-y}$ (shown)
3(ii)	$\left(\dfrac{dy}{dx}\right)^{2}+\dfrac{d^{2}y}{dx^{2}}=4e^{-y}$ Differentiating again with respect to x, $2\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^{2}y}{dx^{2}}\right)+\dfrac{d^{3}y}{dx^{3}}=-4e^{-y}\dfrac{dy}{dx}$ $\therefore$ when $x=0,\ y=0,\ \dfrac{dy}{dx}=3,\ \dfrac{d^{2}y}{dx^{2}}=-5,$ $e^{0}(3)^{3}+3e^{0}(3)(-5)+e^{0}\dfrac{d^{3}y}{dx^{3}}=0$ $\dfrac{d^{3}y}{dx^{3}}=18$ $\therefore y=0+3x-\dfrac{5}{2!}x^{2}+\dfrac{18}{3!}x^{3}\cdots=3x-\dfrac{5}{2}x^{2}+3x^{3}\cdots$
3(iii)	$1+3x+2x^{2}=1.0302$ $x=-1.51$ (reject) or $x=0.01$ $y=\ln\left(1+3x+2x^{2}\right)=3x-\dfrac{5}{2}x^{2}+3x^{3}+\cdots$ Using $x=0.01$, $\ln\left(1+3(0.01)+2(0.01)^{2}\right)\approx 3(0.01)-\dfrac{5}{2}(0.01)^{2}+3(0.01)^{3}$ $\ln(1.0302)\approx 0.0298$ (4 d.p.)

Qn	Suggested Solution
4(i)	

Qn	Suggested Solution
	$\overrightarrow{OC}=\dfrac{\mathbf{a}+2\mathbf{b}}{3}=\dfrac{\begin{pmatrix}p\\1\\-3\end{pmatrix}+2\begin{pmatrix}1\\0\\0\end{pmatrix}}{3}=\dfrac{1}{3}\begin{pmatrix}p+2\\1\\-3\end{pmatrix}$ Area of triangle *OAC* $=\dfrac{1}{2}\left\lvert\overrightarrow{OA}\times\overrightarrow{OC}\right\rvert$ $=\dfrac{1}{2}\left\lvert\begin{pmatrix}p\\1\\-3\end{pmatrix}\times\dfrac{1}{3}\begin{pmatrix}p+2\\1\\-3\end{pmatrix}\right\rvert$ $=\dfrac{1}{6}\left\lvert\begin{pmatrix}0\\-6-3p+3p\\p-2-p\end{pmatrix}\right\rvert=\dfrac{2}{6}\left\lvert\begin{pmatrix}0\\-3\\-1\end{pmatrix}\right\rvert$ $=\dfrac{\sqrt{10}}{3}$
4(ii)	 Triangle *OAD*, triangle *ADE* and triangle *OAC* have the same height and base and thus they have the same area. Area of trapezium *OAED* $=3\left(\dfrac{\sqrt{10}}{3}\right)=\sqrt{10}$
4(iii)	$\cos 135^\circ=\dfrac{\begin{pmatrix}p\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\sqrt{p^2+10}\sqrt{1}}$ $-\dfrac{1}{\sqrt{2}}=\dfrac{p}{\sqrt{p^2+10}}$ $p^2+10=2p^2$ $p^2=10$ $p=-\sqrt{10}$ (reject $p=\sqrt{10}$ since $\mathbf{a}\cdot\mathbf{b}<0$)

Qn	Suggested Solution
5(i)	$\sqrt{3}-\mathrm{i}=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$ $w=2\left(\sqrt{3}-\mathrm{i}\right)z$ $=2\left(2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}\right)r\mathrm{e}^{\mathrm{i}\theta}$ $=4r\mathrm{e}^{\mathrm{i}\left(\theta-\frac{\pi}{6}\right)}$ $\lvert w\rvert=4r$ $\arg w=\theta-\frac{\pi}{6}\quad\left(\because \frac{\pi}{6}<\theta\le\frac{\pi}{2}\right)$ **Useful screenshots:** 
5(ii)	 **Remark:** Locus of z could also be drawn along the positive Im-axis as values of θ include $\frac{\pi}{2}$.
5(iii)	$\left\lvert\frac{w^2}{2z^*}\right\rvert=\frac{\lvert w\rvert^2}{2\lvert z\rvert}=\frac{16r^2}{2r}=8r$ Since $0<r\le 2$, $\therefore 0<\left\lvert\frac{w^2}{2z^*}\right\rvert\le 16.$

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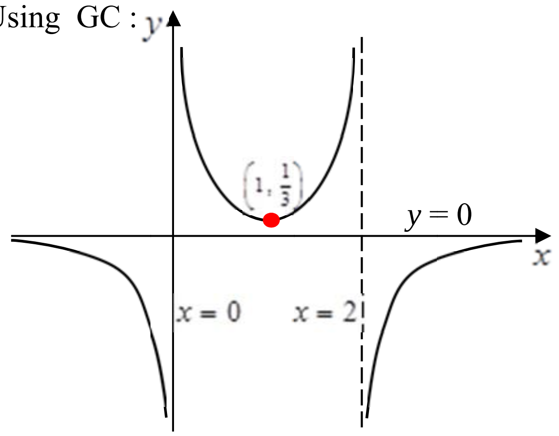
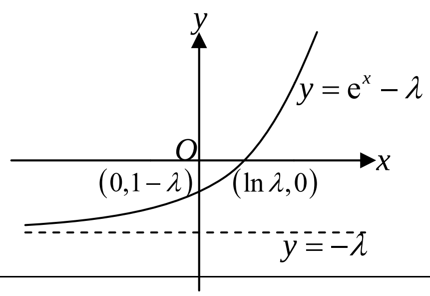
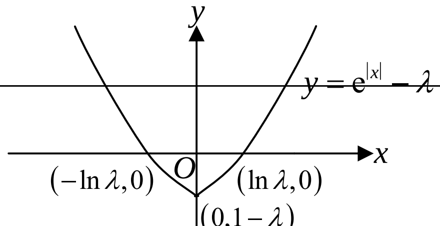
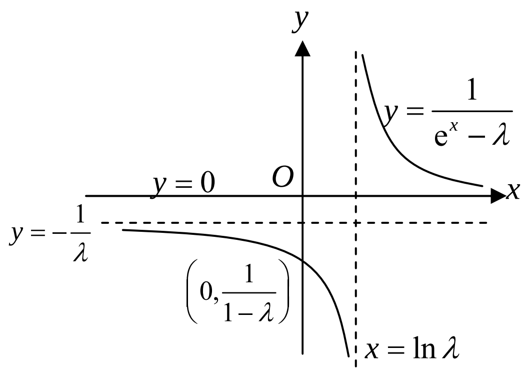
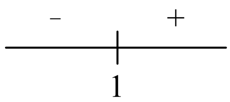
6	$x\frac{\mathrm{d}y}{\mathrm{d}x}+y-3(xy)^2=0$ ....(1) Given $u=xy$: $\frac{\mathrm{d}u}{\mathrm{d}x}=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$

	Substitute into (1): $\dfrac{\mathrm{d}u}{\mathrm{d}x}-3u^2=0$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}=3u^2$ $\displaystyle\int \frac{1}{u^2}\,du=\int 3\,dx$ $\Rightarrow\ -\dfrac{1}{u}=3x+C$ (C arbitrary constant) or $u=-\dfrac{1}{3x+C}$ $\therefore y=-\dfrac{1}{x(3x+C)}$ (*)
6(i)	Given $\left(1\,,\ \dfrac{1}{3}\right)$ $\dfrac{1}{3}=-\dfrac{1}{1(3+C)}\Rightarrow C=-6$ Hence, $y=-\dfrac{1}{3x(x-2)}$ Using GC : 
6(ii)	y has no turning point when $C=0$, i.e. particular solution is $y=-\dfrac{1}{3x^2}$

Qn	Suggested Solution
7(i)	 As $x\to-\infty,\mathrm{e}^x\to0$ $\Rightarrow\ y\to-\lambda$ hence, horizontal asym is $y=-\lambda$
7(ii)	

Qn	Suggested Solution
7(iii)	
7(iv)	$(e^x-e)(e^{\|x\|}-e)=1$ $\Rightarrow e^{\|x\|}-e=\frac{1}{e^x-e}$ i.e. $\lambda=e$ Since $\frac{1}{1-e}>1-e$, from the graphs of $y=e^{\|x\|}-e$ and $y=\frac{1}{e^x-e}$ will intersect 3 times. Thus there will be 3 solutions for $(e^x-e)(e^{\|x\|}-e)=1$.

Qn	Suggested Solution
8(a)	$\int x\sec^2(x+a)\,dx$ $=x\tan(x+a)-\int\tan(x+a)\,dx$ $=x\tan(x+a)-\ln\left\|\sec(x+a)\right\|+C$ **OR**: $x\tan(x+a)+\ln\left\|\cos(x+a)\right\|+C$
8(b)	$\int\frac{x-1}{x^2-2x+2}\,dx=\frac{1}{2}\int\frac{2x-2}{x^2-2x+2}\,dx$ $=\frac{1}{2}\ln(x^2-2x+2)+C$

Qn	Suggested Solution
8(b) (i)	$\int_1^2 \frac{x-4}{x^2-2x+2}\,\mathrm{d}x$ $=\int_1^2 \frac{x-1}{x^2-2x+2}\,\mathrm{d}x-\int_1^2 \frac{3}{x^2-2x+2}\,\mathrm{d}x$ $=\int_1^2 \frac{x-1}{x^2-2x+2}\,\mathrm{d}x-\int_1^2 \frac{3}{(x-1)^2+1}\,\mathrm{d}x$ $=\frac{1}{2}\left[\ln\left(x^2-2x+2\right)\right]_1^2-3\left[\tan^{-1}(x-1)\right]_1^2$ $=\frac{1}{2}\left[\ln 2-\ln 1\right]-3\left[\tan^{-1}1-\tan^{-1}0\right]$ $=\frac{1}{2}\ln 2-\frac{3\pi}{4}$
8(b) (ii)	Note that $\frac{x-1}{x^2-2x+2}=\frac{x-1}{\left(x-1\right)^2+1}$:  $\int_{2-p}^{p}\left\|\frac{x-1}{x^2-2x+2}\right\|\mathrm{d}x$ $=-\int_{2-p}^{1}\frac{x-1}{\left(x-1\right)^2+1}\mathrm{d}x+\int_1^p \frac{x-1}{\left(x-1\right)^2+1}\,\mathrm{d}x$ $=2\int_1^p \frac{x-1}{\left(x-1\right)^2+1}\,\mathrm{d}x$ (by symmetry) $=2\left[\frac{1}{2}\ln(x^2-2x+2)\right]_1^p=\ln\left(p^2-2p+2\right)$

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>9(a)</td><td>

$y_n - y_{n-1}$

$= \log_k x_n + k - (\log_k x_{n-1} + k)$

$= \log_k x_n - \log_k x_{n-1}$

$= \log_k \dfrac{x_n}{x_{n-1}}$

$= \log_k r$ (a constant, where r is the common ratio)

Since the difference between any two consecutive terms is a constant, $\{y_n\}$ is an arithmetic sequence.

</td></tr>
<tr><td>9b(i)</td><td>

$1.01(20000 - x) = 20000$

$1.01x = 0.01(20000)$

$x = 198.02$

</td></tr>
<tr><td>9b(ii)</td><td>

<table>
<tr><td>No. of payments</td><td>Amount owed after each payment in the middle of the month</td></tr>
<tr><td>1</td><td>$20\,000 - x$</td></tr>
<tr><td>2</td><td>$1.01(20\,000 - x) - x$
$= 1.01(20\,000) - 1.01x - x$</td></tr>
<tr><td>3</td><td>$1.01[1.01(20\,000) - 1.01x - x] - x$
$= 1.01^2(20\,000) - 1.01^2 x - 1.01x - x$</td></tr>
<tr><td>…</td><td></td></tr>
<tr><td>n</td><td>$1.01^{n-1}(20\,000) - x(1.01^{n-1} + 1.01^{n-2} + ... + 1.01 + 1)$
$= 1.01^{n-1}(20\,000) - x\dfrac{1.01^n - 1}{1.01 - 1}$</td></tr>
</table>

For the loan to be paid in full after the n^{th} payment,

$1.01^{n-1}(20\,000) - x\dfrac{1.01^n - 1}{1.01 - 1} = 0$

$1.01^{n-1}(20\,000) = x\dfrac{1.01^n - 1}{0.01}$

$x = \dfrac{200(1.01^{n-1})}{1.01^n - 1}$ (shown)

<u>Alternatively</u>

<table>
<tr><td>No. of payments</td><td>Amount owed at the end of each month</td></tr>
<tr><td>1</td><td>$1.01(20\,000 - x)$</td></tr>
</table>

</td></tr>
</table>

Qn	Suggested Solution

2	$1.01[1.01(20\,000-x)-x]$ $=1.01^2(20\,000)-1.01^2x-1.01x$
…	
n–1	$1.01^{n-1}(20\,000)-x(1.01^{n-1}+1.01^{n-2}+...+1.01)$ $=1.01^{n-1}(20\,000)-x\left[\dfrac{1.01(1.01^{n-1}-1)}{1.01-1}\right]$

For the loan to be paid in full after the n^{th} payment, then

$$1.01^{n-1}(20\,000)-x\frac{1.01(1.01^{n-1}-1)}{1.01-1}-x=0$$

$$1.01^{n-1}(20\,000)=x\frac{1.01(1.01^{n-1}-1)}{0.01}+x$$

$$x\frac{1.01(1.01^{n-1}-1)+0.01}{0.01}=1.01^{n-1}(20\,000)$$

$$x\frac{1.01^n-1}{0.01}=1.01^{n-1}(20\,000)$$

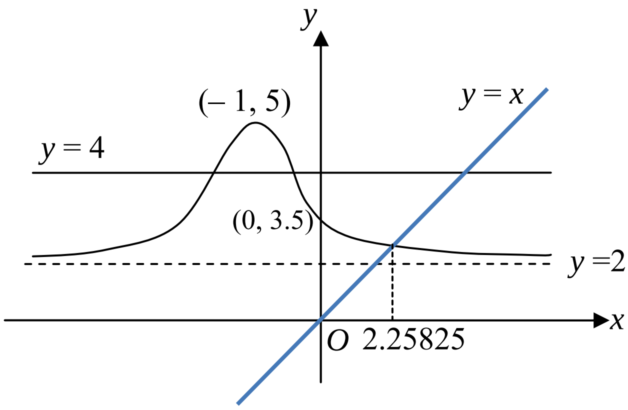
$$x=\frac{200(1.01^{n-1})}{1.01^n-1} \qquad \text{(shown)}$$

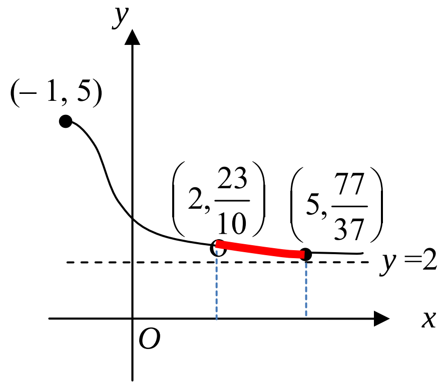
For the loan to be fully paid in 3 years (n = 36 months),

$$x=\frac{200(1.01^{36-1})}{1.01^{36}-1}$$

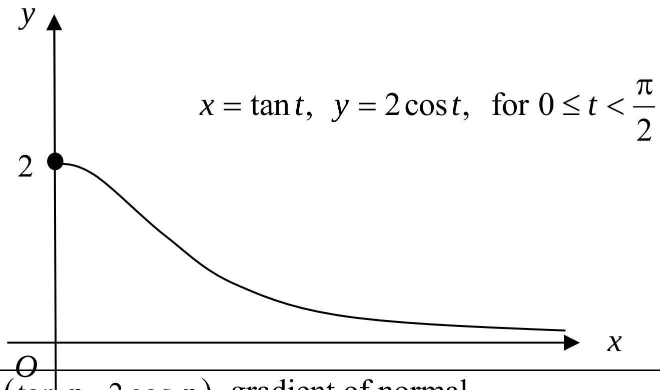
$x\approx 657.709$

Hence, for Thomas to fully pay up the loan in exactly 3 years, he should be paying a monthly amount of \$657.71

Qn	Suggested Solution
10(i)	$\mathrm{f}: x \mapsto 2+\dfrac{3}{x^2+2x+2},\ x \in \mathbb{R}$.  Since the horizontal line $y = 4$ passes through the graph of f at two distinct points, f is not one-one, hence f^{-1} does not exist. Note: Horizontal line $y = a$, where $a \in (2,5)$.
10(ii)	For f^{-1} to exist, domain of f is restricted to $x \geq -1$. The smallest value of k is -1.
10(iii)	To find f^{-1}, $y = 2+\dfrac{3}{x^2+2x+2}$ $y-2 = \dfrac{3}{x^2+2x+2}$ $y-2 = \dfrac{3}{(x+1)^2+1}$ $(x+1)^2 = \dfrac{3}{y-2}-1$ $x = -1+\sqrt{\dfrac{3}{y-2}-1}$ (reject $x = -1-\sqrt{\dfrac{3}{y-2}-1}$ $\because x \geq -1$) Therefore, $\mathrm{f}^{-1}(x) = -1+\sqrt{\dfrac{3}{x-2}-1}$ $D_{\mathrm{f}^{-1}} = (2,5]$
10(iv)	$D_\mathrm{f} = [-1, \infty)$ $R_\mathrm{f} = (2, 5]$ Since $R_\mathrm{f} \subseteq D_\mathrm{f}$, f^2 exists.

Qn	Suggested Solution
	 For range of f^2, $[-1, \infty) \to (2, 5] \to [\frac{77}{37}, \frac{23}{10})$. When $x = 2$, $y = 2 + \frac{3}{(2)^2 + 2(2) + 2} = \frac{23}{10}$. When $x = 5$, $y = 2 + \frac{3}{(5)^2 + 2(5) + 2} = \frac{77}{37}$.
10(v)	$y = \frac{3}{4x^2 + 4x + 2} \xrightarrow{\text{Replace } x \text{ by } \frac{x}{2}} y = \frac{3}{4\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) + 2}$ $y = \frac{3}{x^2 + 2x + 2}$ $y = \frac{3}{x^2 + 2x + 2} \xrightarrow{\text{Replace } y \text{ by } y\text{-2}} y = 2 + \frac{3}{x^2 + 2x + 2}$ 1. Scale by a factor of 2 parallel to the *x*-axis. 2. Translate 2 units in the positive *y*-direction.
10(vi)	$2 + \frac{3}{9x^2 + 6x + 2} < 3x$ $2 + \frac{3}{(3x)^2 + 2(3x) + 2} < 3x$ Replace x by $3x$ to the inequality $2 + \frac{3}{x^2 + 2x + 2} < x$ The range is $3x > 2.25825$ $x > 0.753$ (3 s.f.)

Qn	Suggested Solution
11(i)	$x = \tan t, \quad y = 2\cos t, \quad \text{for } 0 \le t < \frac{\pi}{2}$ $\frac{dx}{dt} = \sec^2 t, \quad \frac{dy}{dt} = -2\sin t \quad \Rightarrow \frac{dy}{dx} = \frac{-2\sin t}{\sec^2 t} = -2\sin t \cos^2 t$

Qn	Suggested Solution
	As $t \to 0,\ \frac{\mathrm{d}y}{\mathrm{d}x} \to 0.$ The tangent becomes parallel to the x-axis/tangent is a horizontal line. $x = \tan 0 = 0,\ y = 2\cos 0 = 2$ 
11(ii)	At $P(\tan p,\ 2\cos p)$, gradient of normal $= -\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = -\frac{1}{(-2\sin p\cos^2 p)} = \frac{1}{2\sin p\cos^2 p},$ **<u>Method 1</u>** Since normal passes through origin, equation of normal : $y = \left(\frac{1}{2\sin p\cos^2 p}\right)x \ \ ..... (1)$ Since normal intersects curve also at P, substitute $x = \tan p, y = 2\cos p$ into eqn (1) $2\cos p = \frac{1}{2\sin p\cos^2 p}(\tan p)$ $= \frac{1}{2\cos^3 p}$ $\cos^4 p = \frac{1}{4}$ $\cos p = \pm\frac{1}{\sqrt{2}}$ $\therefore p = \frac{\pi}{4} \ \left(\because 0 < p < \frac{\pi}{2}\right)$
	Equation of normal is

Qn	Suggested Solution

$$y=\frac{x}{2\sin\frac{\pi}{4}\cos^2\frac{\pi}{4}}$$

$$y=\frac{x}{2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^2} \quad \text{....... (2)}$$

$$\therefore y=x\sqrt{2} \text{ (shown)}$$

<u>Method 2</u>

Equation of normal :

$$y-2\cos p=\frac{1}{2\sin p\cos^2 p}(x-\tan p) \quad \text{..... (1)}$$

Since the normal passes through origin (0,0), substitute $x=0, y=0$ into eqn (1)

$$0-2\cos p=\frac{1}{2\sin p\cos^2 p}(0-\tan p)$$

$$-4\sin p\cos^3 p=\frac{-\sin p}{\cos p}$$

$$\sin p(4\cos^4 p-1)=0$$

$$\sin p=0 \text{ or } \cos p=\pm\frac{1}{\sqrt{2}}$$

$$\therefore p=\frac{\pi}{4} \quad \left(\because 0<p<\frac{\pi}{2}\right)$$

Equation of normal which passes through origin is

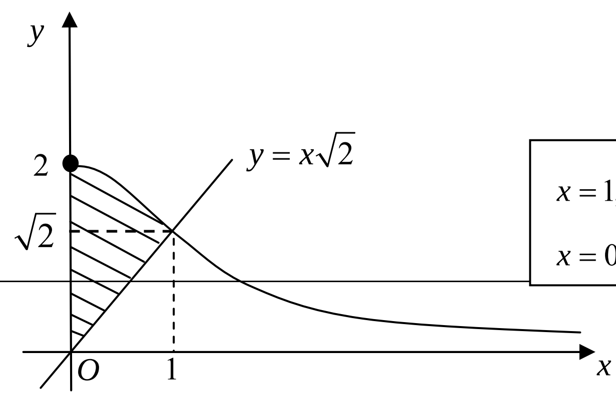
$$y-2\cos\frac{\pi}{4}=\frac{1}{2\sin\frac{\pi}{4}\cos^2\frac{\pi}{4}}\left(x-\tan\frac{\pi}{4}\right)$$

$$y-2\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^2}(x-1) \quad \text{....... (2)}$$

$$y-\sqrt{2}=\sqrt{2}(x-1)$$

$$\therefore y=x\sqrt{2} \text{ (shown)}$$

11(iii)



$x=1, t=\frac{\pi}{4}$

$x=0, t=0$

Qn	Suggested Solution
	When $p=\frac{\pi}{4}, x=\tan\frac{\pi}{4}=1, y=2\cos\frac{\pi}{4}=\sqrt{2}$ **Method 1 (with respect to *x*-axis)** Required area $=\int_0^1 y\ \mathrm{d}x-\frac{1}{2}bh$ or $\left(\int_0^1 x\sqrt{2}\ \mathrm{d}x\right)$ $=\int_0^{\frac{\pi}{4}} 2\cos t\ \left(\sec^2 t\right)\mathrm{d}t-\frac{1}{2}(1)\left(\sqrt{2}\right)$ or $\left[\frac{x^2}{2}\sqrt{2}\right]_0^1$ $=2\int_0^{\frac{\pi}{4}}\sec t\ \mathrm{d}t-\frac{\sqrt{2}}{2}$ $=2\left[\ln\lvert\sec t+\tan t\rvert\right]_0^{\frac{\pi}{4}}-\frac{\sqrt{2}}{2}$ $=2\ln\left(\frac{1}{\cos\frac{\pi}{4}}+\tan\frac{\pi}{4}\right)-\frac{\sqrt{2}}{2}$ $=2\ln\left(\sqrt{2}+1\right)-\frac{\sqrt{2}}{2}$ unit^2 **Method 2 (with respect to *y*-axis)** Required area $y=2, t=0$ $y=\sqrt{2}, t=\frac{\pi}{4}$ $=\int_{\sqrt{2}}^{2} x\ \mathrm{d}y+\frac{1}{2}bh$ $\left(\int_0^{\sqrt{2}}\frac{y}{\sqrt{2}}\mathrm{d}y\right)$ $=\int_{\frac{\pi}{4}}^{0}\tan t\ \left(-2\sin t\right)\mathrm{d}t+\frac{1}{2}\left(\sqrt{2}\right)(1)$ or $\frac{1}{\sqrt{2}}\left[\frac{y^2}{2}\right]_0^{\sqrt{2}}$ $=2\int_0^{\frac{\pi}{4}}\frac{\sin^2 t}{\cos t}\ \mathrm{d}t+\frac{\sqrt{2}}{2}$ $=2\int_0^{\frac{\pi}{4}}\frac{1-\cos^2 t}{\cos t}\ \mathrm{d}t+\frac{\sqrt{2}}{2}$ $=2\int_0^{\frac{\pi}{4}}\left(\sec t-\cos t\right)\ \mathrm{d}t+\frac{\sqrt{2}}{2}$ $=2\left[\ln\lvert\sec t+\tan t\rvert-\sin t\right]_0^{\frac{\pi}{4}}+\frac{\sqrt{2}}{2}$

Qn	Suggested Solution
	$= 2\ln\left(\frac{1}{\cos\frac{\pi}{4}} + \tan\frac{\pi}{4} - \sin\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}$ $= 2\ln\left(\sqrt{2} + 1 - \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}$ $= 2\ln\left(\sqrt{2}+1\right) - \frac{\sqrt{2}}{2}$ unit^2 Note : Generally $\int \sec t \ \mathrm{d}t = \ln\lvert\sec t + \tan t\rvert$. But in this question where the limits are $0 \le t \le \frac{\pi}{4}$, $\int_0^{\frac{\pi}{4}} \sec t \ \mathrm{d}t = \ln\left(\sec t + \tan t\right)$ is acceptable.