**1** A graphic calculator is **not** to be used in answering this question.

(a) Solve

$$\frac{7x-4}{x^2+2x-3} \geq 2 .$$ [3]

(b) Hence solve $$\frac{\left(|2x|+1\right)\left(|x|-2\right)}{x^2+|2x|-3} \leq 0 .$$ [2]

**2** In the triangle $ABC$, $AB=1$, $BC=\frac{1}{2}$ and angle $ABC=2\theta$ radians. Given that angle $ABC$ is a sufficiently small angle, show that

$$AC=\left(\frac{1}{4}+2\theta^2\right)^{\frac{1}{2}} .$$

Hence, express $AC$ in the form $a+b\theta^2+c\theta^4$, where $a$, $b$ and $c$ are constants to be determined. [5]

**3** (a) The sum of the first $n$ terms of a sequence is given by $S_n=2n^2+3n$. Prove that the given sequence is an arithmetic progression. Find the least value of $n$ such that the sum of the first $n$ terms exceeds 2015. [4]

(b) In an extreme cake-making competition, Kimberley is tasked to make a multi-layered cake in which each layer is in the shape of a cylinder, starting with the base layer of height 10 cm. For each subsequent layer above, it must be smaller than the previous layer. In particular, the height of any layer above the base must be $(5k)\%$ of the height of the previous layer, where $k$ is a positive integer. The layers are joined together using whipped cream.

(i) Show that the height of an $n$-layer cake is given by

$$\frac{200}{20-k}\left[1-\left(\frac{k}{20}\right)^n\right]\text{cm.}$$ [1]

(ii) It is found that a cake with height exceeding 1.2 m will become structurally unstable. Using $k$ = 19, find the maximum number of layers that Kimberley's cake can have before it becomes structurally unstable. [2]

(iii) State a possible assumption used in part (ii). [1]

**4** (a) Show, by using the substitution $x = \tan u$,

$$\int \frac{\left(\tan^{-1} x\right) e^{\left(\tan^{-1} x\right)^2}}{1+x^2}\, \mathrm{d}x = \int u e^{u^2}\, \mathrm{d}u\,.$$

Hence find the given integral. [3]

(b) The region $R$ is bounded by the curve $y = \sqrt{\dfrac{\left(\tan^{-1} x\right) e^{\left(\tan^{-1} x\right)^2}}{1+x^2}}$, the line $y = \sqrt{\dfrac{\pi}{8}}\, \mathrm{e}^{\frac{\pi^2}{32}}$ and the $y$-axis.

(i) Sketch the curve and identify the region $R$. [1]

(ii) Find the exact volume generated when region $R$ is rotated $2\pi$ radians about the $x$-axis. [4]

**5** (a) Given that $u_{n+1} = u_n - \dfrac{n^2+n-1}{(n+1)!}$ and $u_0 = 1$, prove by induction that

$$u_n = \frac{n+1}{n!} \text{ for } n \geq 0.$$ [5]

(b) Using the recurrence relation in part (a), find

$$\sum_{r=1}^{N} \frac{(r+1)^2 + r}{(r+2)!}.$$ [4]

**6** (i) For two non-zero and non-parallel vectors **a** and **b**, give the geometrical meaning of $|\mathbf{a} \times \mathbf{b}|$. [1]

(ii) The position vectors of the points $A$ and $B$ are **a** and **b** respectively. Another point $C$ on $OB$ produced is such that $OB : BC = 1 : 2$ and the point $D$ on line $AC$ is such that $AD : DC = 2 : 1$. The point $M$ is the point of intersection between $AB$ and $OD$.

Find the following vectors in terms of **a** and/or **b**

(a) $\overrightarrow{OD}$; [2]

(b) $\overrightarrow{OM}$. [3]

(iii) It is given further that $|\mathbf{b}| = 2$ and the area of triangle $OAC$ is 12 square units, find the shortest distance from $A$ to $OC$. [3]

**7** A graphic calculator is **not** to be used in answering this question.

(i) Find the roots of the equation $z^2 = -8\text{i}$ in the form $a + b\text{i}$. [4]

(ii) Hence, sketch on a single Argand diagram, the roots of $w^4 = -64$. [3]

(iii) Find the roots of the equation $z^2 + (2 + 2\text{i})z + 4\text{i} = 0$. [3]

**8** Given that $y = \sin^{-1}[\ln(x+1)]$, show that

$$\cos y \frac{\text{d}y}{\text{d}x} = \frac{1}{x+1} \quad \text{and} \quad \cos y \frac{\text{d}^2 y}{\text{d}x^2} - \sin y \left(\frac{\text{d}y}{\text{d}x}\right)^2 = -\frac{1}{(x+1)^2}.$$ [2]

(i) By further differentiation, find the Maclaurin series for $y$, up to and including the term in $x^3$. [3]

(ii) Find the set of values of $x$ for which the value of $y$ is within $\pm 0.1$ of the value found by its Maclaurin series. [3]

(iii) Deduce the series expansion for $\dfrac{1}{(x+1)\sqrt{1-(\ln(x+1))^2}}$ up to and including the term in $x^2$. [2]

**9** A plane $\Pi_1$ contains points $A$, $B$ and $C$ with coordinates $(2, 0, 3)$, $(-1, -1, 4)$ and $(5, 1, 0)$ respectively.

(i) Show that the vector $\mathbf{i} - 3\mathbf{j}$ is perpendicular to the plane $\Pi_1$. [3]

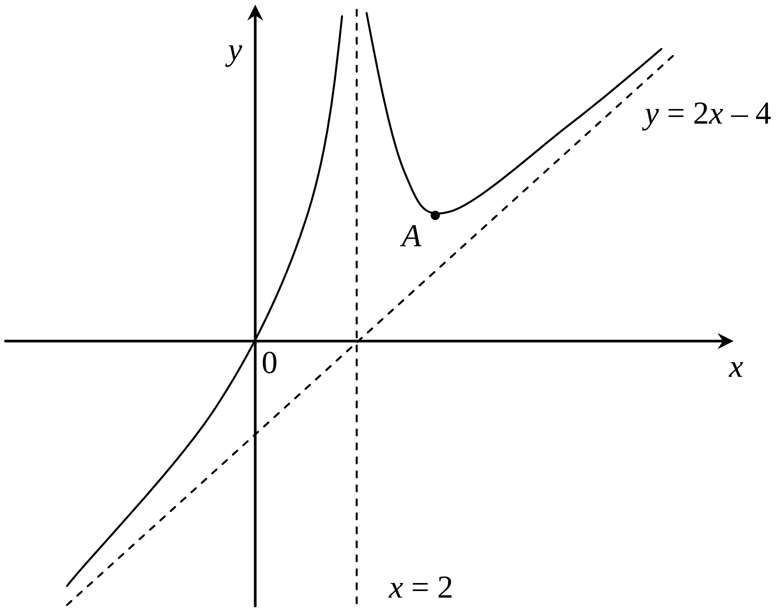
(ii) Given a point $M$ with position vector $\mathbf{k}$, find the position vector of the foot of perpendicular from $M$ to plane $\Pi_1$. Hence, find the distance of $M$ to the plane $\Pi_1$ in the form $\sqrt{\dfrac{a}{b}}$, where $a$ and $b$ are positive integers to be determined. [4]

Another plane $\Pi_2$ contains the point $M$ and has equation $ax - y + bz = 5$.

(iii) Given that plane $\Pi_2$ is perpendicular to the $y$-$z$ plane, find the acute angle between $\Pi_1$ and $\Pi_2$. [4]

**10** (a) A curve $y = \mathrm{f}(x)$ is transformed by a stretch with scale factor 2 parallel to the $y$-axis, followed by a reflection about the $x$-axis. The equation of the resultant curve is $y = \ln\left[\frac{1}{(x+a)^2}\right]$, where $x > -a$ and $a > 1$. Find $\mathrm{f}(x)$ and sketch $y = \mathrm{f}(x)$. [4]

(b) The diagram shows the graph of $y = \mathrm{g}(x)$. The graph passes through the origin and has a turning point $A(3,\ 4)$. The asymptotes of the graph are $x = 2$ and $y = 2x - 4$.




On separate diagram, sketch the following graphs indicating the points corresponding to the axial intercepts, turning point and asymptotes where necessary.

(i) $y^2 = \mathrm{g}(x)$ [2]

(ii) $y = \mathrm{g}'(x)$ [3]

(iii) $y = \dfrac{1}{\mathrm{g}(x)}$ [3]

**11** The curve $C$ is defined by

$$x = \frac{1}{t},\ y = \ln t^2,$$

for $-2 \le t \le 2,\ t \ne 0$.

Sketch $C$. [2]

Find the equation of the normal at point $P$ where $t = p$. [3]

Find the coordinates of the point on $C$ where the tangent does not intersect the normal at $P$. [2]

Use $p = \frac{1}{2}$ in the following.

Find the exact area bounded by the curve $C$, the $x$-axis and the line $x = 2$. [3]

By using cross product or otherwise, find the distance between the non-intersecting tangent and normal. [3]

**– End of Paper –**