Tutorial 8D: Vectors IV (Planes & 3D Geometry Problems)

Basic Mastery Questions

1. Find the **vector equations in parametric form**, **vector equations in scalar product form and the Cartesian equations** of the planes containing
   (i) the point (0, 1, 1) and the two vectors $\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $-\mathbf{i}+\mathbf{k}$;
   (ii) the points (1, 0, 1), (1, 2, 1) and (1, 1, 0);
   (iii) the point $(-1, 2, 3)$ and the line with equation $\mathbf{r}=-2\mathbf{i}-\mathbf{k}+\lambda(-\mathbf{i}+7\mathbf{j}+2\mathbf{k})$;
   (iv) the lines $\mathbf{r}=\mathbf{k}+s(\mathbf{i}-3\mathbf{j})$ and $\mathbf{r}=\mathbf{k}+t(2\mathbf{j}+5\mathbf{k})$.

(i) $\pi: \underset{\sim}{r}=\begin{pmatrix}0\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix}$ // where $\lambda,\mu\in\mathbb{R}$ — vector eqn in parametric form

$\underset{\sim}{n}=\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}2\\-2\\2\end{pmatrix}=2\begin{pmatrix}1\\-1\\1\end{pmatrix}$

$\therefore$ vector equation in scalar product form : $\underset{\sim}{r}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$

ie. $\underset{\sim}{r}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$ //

cartesian eqn : $x-y+z=0$ //

(ii) $\begin{pmatrix}1\\2\\1\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\therefore$ vector eqn in parametric form : $\underset{\sim}{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+s\begin{pmatrix}0\\1\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\-1\end{pmatrix}$ //, $s,t\in\mathbb{R}$

$\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix}\Rightarrow$ vector eqn in scalar product form : $\underset{\sim}{r}\cdot\begin{pmatrix}-1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\0\end{pmatrix}$

ie. $\underset{\sim}{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=1$ //.

Cartesian eqn : $-x=-1$ or $x=1$ //.

(iii) $\begin{pmatrix}-2\\0\\-1\end{pmatrix}-\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}-1\\-2\\-4\end{pmatrix}=(-1)\begin{pmatrix}1\\2\\4\end{pmatrix}$. parametric form : $\underset{\sim}{r}=\begin{pmatrix}-1\\2\\3\end{pmatrix}+s\begin{pmatrix}-1\\7\\2\end{pmatrix}+t\begin{pmatrix}1\\2\\4\end{pmatrix}$ //, $s,t\in\mathbb{R}$

$\begin{pmatrix}-1\\7\\2\end{pmatrix}\times\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}24\\6\\-9\end{pmatrix}=3\begin{pmatrix}8\\2\\-3\end{pmatrix}\Rightarrow$ scalar prod form : $\underset{\sim}{r}\cdot\begin{pmatrix}8\\2\\-3\end{pmatrix}=\begin{pmatrix}-1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}8\\2\\-3\end{pmatrix}$

ie. $\underset{\sim}{r}\cdot\begin{pmatrix}8\\2\\-3\end{pmatrix}=-13$ //.

cartesian eqn : $8x+2y-3z=-13$ //.

(iv) parametric form : $\underset{\sim}{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}0\\2\\5\end{pmatrix}$ //, $\lambda,\mu\in\mathbb{R}$

$\begin{pmatrix}1\\-3\\0\end{pmatrix}\times\begin{pmatrix}0\\2\\5\end{pmatrix}=\begin{pmatrix}-15\\-5\\2\end{pmatrix}\Rightarrow$ scalar prod. form : $\underset{\sim}{r}\cdot\begin{pmatrix}-15\\-5\\2\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-15\\-5\\2\end{pmatrix}$

ie. $\underset{\sim}{r}\cdot\begin{pmatrix}-15\\-5\\2\end{pmatrix}=2$ //.

$\therefore$ cartesian eqn : $-15x-5y+2z=2$ //

2. A line ℓ has equation $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \alpha(2\mathbf{i} - \mathbf{j} + \mathbf{k})$. Find the **vector equation** of the plane containing the point $A(1, 3, 1)$, perpendicular to the plane OXZ and parallel to ℓ.

$\ell: \ \underset{\sim}{r} = \begin{pmatrix}0\\1\\2\end{pmatrix} + \alpha\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$\therefore$ Vector eqn of the plane: $\underset{\sim}{r} = \begin{pmatrix}1\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\0\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\1\end{pmatrix}$ //.

where $\lambda, \mu \in \mathbb{R}$

3. **N89/II/15(partial)** The plane Π has equation $3x + 2y - z + 1 = 0$ and the line ℓ has Cartesian equation $\dfrac{x-4}{-1} = \dfrac{y+3}{2} = \dfrac{z-7}{1}$. Show that ℓ lies in Π.

Method 1

$\ell: \ \underset{\sim}{r} = \begin{pmatrix}4\\-3\\7\end{pmatrix} + \alpha\begin{pmatrix}-1\\2\\1\end{pmatrix}$, $\Pi: \ \underset{\sim}{r} \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} = -1$

①: $\begin{pmatrix}4\\-3\\7\end{pmatrix} \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} = 12 - 6 - 7 = -1$. $\therefore (4, -3, 7)$ is on Π.

②: $\begin{pmatrix}-1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} = -3 + 4 - 1 = 0$ ($\perp$) $\Rightarrow \begin{pmatrix}-1\\2\\1\end{pmatrix}$ is // to Π.

Implying from ① & ②: ℓ is in Π //. (shown)

Method 2

If ℓ lies in Π, then $\mathbf{r} = \begin{pmatrix}4\\-3\\7\end{pmatrix} + \alpha\begin{pmatrix}-1\\2\\1\end{pmatrix}$ must satisfy $\mathbf{r} \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} = -1$ for all $\alpha \in \mathbb{R}$.

Substituting $\mathbf{r} = \begin{pmatrix}4\\-3\\7\end{pmatrix} + \alpha\begin{pmatrix}-1\\2\\1\end{pmatrix}$ into $\mathbf{r} \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} = -1$:

$$\left[\begin{pmatrix}4\\-3\\7\end{pmatrix} + \alpha\begin{pmatrix}-1\\2\\1\end{pmatrix}\right] \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}4-\alpha\\-3+2\alpha\\7+\alpha\end{pmatrix} \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix}$$

$= 12 - 3\alpha - 6 + 4\alpha - 7 - \alpha = -1$ which is true for all $\alpha \in \mathbb{R}$.

$\therefore \ell$ lies in Π (shown).

4. IJC/2018/CT/10 In a particular experiment, Scott shoots a laser beam from point A with coordinates $(9, 1, -5)$ towards a plane Π with equation $5x + y - 8z = -4$. The laser beam travels in a line L with equation $\frac{x-4}{5} = \frac{3-y}{2} = z + 6$.

Find

(i) the acute angle between L and Π. [3]

(ii) the coordinates of the point where the laser beam meets the plane and deduce the shortest distance from A to Π. [5]

Immediately after the laser beam meets the plane, it is being reflected as line M such that the angle between L and Π equals to the angle between M and Π. Find the equation of the line M. [5]

Q4	Solution
(i)	Equation of line L: $\frac{x-4}{5} = \frac{3-y}{2} = z + 6$ $\Rightarrow \mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$ Let θ be the acute angle between L and Π. $\sin\theta = \frac{\left\| \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix} \right\|}{\sqrt{5^2+2^2+1^2} \cdot \sqrt{5^2+1^2+8^2}} = \frac{15}{\sqrt{2700}}$ $\therefore \theta = \sin^{-1}\left(\frac{15}{\sqrt{2700}}\right) = 16.8^o$ (to 1 d.p.)

(ii)	Let B be the point of intersection between L and Π $\overrightarrow{OB}=\begin{pmatrix}4\\3\\-6\end{pmatrix}+\lambda\begin{pmatrix}5\\-2\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\quad\cdots(1)$ $\left[\begin{pmatrix}4\\3\\-6\end{pmatrix}+\lambda\begin{pmatrix}5\\-2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}5\\1\\-8\end{pmatrix}=-4$ $71+15\lambda=-4$ $\lambda=-5$ $\therefore\overrightarrow{OB}=\begin{pmatrix}4\\3\\-6\end{pmatrix}-5\begin{pmatrix}5\\-2\\1\end{pmatrix}=\begin{pmatrix}-21\\13\\-11\end{pmatrix}$ i.e. coordinates of B is $(-21,13,-11)$. $\overrightarrow{AB}=\begin{pmatrix}-21\\13\\-11\end{pmatrix}-\begin{pmatrix}9\\1\\-5\end{pmatrix}=\begin{pmatrix}-30\\12\\-6\end{pmatrix}$ Shortest distance $=\left\|\overrightarrow{AB}\right\|\sin\theta$ $=\sqrt{30^2+12^2+6^2}\left(\frac{15}{\sqrt{2700}}\right)$ $=15\times\sqrt{\frac{2}{5}}$ $=3\sqrt{10}$ $=9.49\ \text{(3.s.f)}$ **<u>Alternative:</u>**

	Shortest distance $= \dfrac{\left\lvert \overrightarrow{AB} \bullet \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix} \right\rvert}{\sqrt{90}} = \dfrac{\left\lvert 6 \begin{pmatrix} -5 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix} \right\rvert}{\sqrt{90}}$ $= \dfrac{\lvert 6(-25+2+8) \rvert}{\sqrt{90}}$ $= \dfrac{90}{\sqrt{90}}$ $= 3\sqrt{10}$
(iii)	Let F be the foot of perpendicular from A to Π $\overrightarrow{OF} = \begin{pmatrix} 9 \\ 1 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix}$ for some $\mu \in \mathbb{R}$ $\cdots(1)$ $\left[\begin{pmatrix} 9 \\ 1 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix} \right] \cdot \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix} = -4$ $86 + 90\mu = -4$ $\mu = -1$ $\therefore \overrightarrow{OF} = \begin{pmatrix} 9 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$

Let A' be the reflection of A in Π

$$\overrightarrow{OF} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$$

$$\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA}$$

$$\overrightarrow{OA'} = 2\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 9 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 11 \end{pmatrix}$$

OR $\overrightarrow{BF} = \dfrac{\overrightarrow{BA} + \overrightarrow{BA'}}{2}$

$$\overrightarrow{BA'} = 2\overrightarrow{BF} - \overrightarrow{BA}$$

$$= 2\left[\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 21 \\ 13 \\ -11 \end{pmatrix}\right] + \begin{pmatrix} -30 \\ 12 \\ -6 \end{pmatrix}$$

$$= 2\begin{pmatrix} 10 \\ -7 \\ 11 \end{pmatrix}$$

$$\mathbf{d} = \overrightarrow{A'B} = \begin{pmatrix} -21 \\ 13 \\ -11 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 11 \end{pmatrix} = \begin{pmatrix} -20 \\ 14 \\ -22 \end{pmatrix} = 2\begin{pmatrix} -10 \\ 7 \\ -11 \end{pmatrix}$$

$\therefore$ Equation of line M:

$$\mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 11 \end{pmatrix} + \alpha\begin{pmatrix} -10 \\ 7 \\ -11 \end{pmatrix},\ \alpha \in \mathbb{R} \quad OR \quad \mathbf{r} = \begin{pmatrix} -21 \\ 13 \\ -11 \end{pmatrix} + \alpha\begin{pmatrix} -10 \\ 7 \\ -11 \end{pmatrix},\ \alpha \in \mathbb{R}$$