Chapter 11

Techniques of Integration

11.1 The Indefinite Integral

11.1.1 Introduction and Definitions

The process of **integration** is the reverse of differentiation. For example, when we differentiate x^2, $x^2 + 1$ or $x^2 - 3$ and so on, we obtain $2x$. When we integrate $2x$, the answer would be $x^2 + C$ where C is an **arbitrary constant.**

Consider two functions f(x) and F(x) which are related as follows:-

$$\frac{d}{dx}(F(x)) = f(x).$$

f(x) is called the **derivative** of F(x) with respect to x and F(x) is called an **anti-derivative** of f(x) with respect to x.

If C is a constant, then we have

$$\frac{d}{dx}[F(x)+C] = \frac{d}{dx}[F(x)] + \frac{d}{dx}[C] = f(x) + 0 = f(x).$$

Hence by our definition, the family of functions F(x) + C where C is an arbitrary constant are all anti-derivatives of f(x) if F(x) is an anti-derivative of f(x).

We denote all anti-derivatives of f(x), F(x) + C, by the notation '$\int f(x)\,dx$' which reads as '**the indefinite integral of f(x) with respect to x.**' That is, $\int f(x)\,dx = F(x) + C$.

Remarks:

1. The function f(x) within the integral sign '$\int$ ' is called the **integrand.**
2. The process of finding $\int f(x)\,dx$ is called **integration.**
3. The arbitrary constant C arising from integration is called the **constant of integration.**

11.1.2 Properties of the Indefinite Integral

1. $\int 0\,dx = C$ where C is an arbitrary constant.
2. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$.
3. $\int kf(x)\,dx = k\int f(x)\,dx$ for any real constant k.

11.1.3 Integration by Basic Formulae

Table 11.1 below are some examples of basic integrals. You have already encountered these integrals in 'O' Level.

Table 11.1 – Basic Integrals

1. $\int k\ \mathrm{d}x = kx + C$	2. $\int x^n \mathrm{d}x = \dfrac{x^{n+1}}{(n+1)} + C,\ n \neq -1$
3. $\int \mathrm{e}^x\ \mathrm{d}x = \mathrm{e}^x + C$	4. $\int \frac{1}{x}\ \mathrm{d}x = \ln\lvert x\rvert + C$ (Why do we use $\ln\lvert x\rvert$ instead of $\ln x$?)

Note: C is an arbitrary constant

11.1.4 Integration by Standard Forms

The four integrals in Table 11.2 below are some of the **standard forms** and many integrals can be classified under one of them. The basic integrals in Table 11.1 are simplest cases of the standard forms. The key to identifying standard forms is identifying the '$\mathrm{f}(x)$' and the corresponding '$\mathrm{f}'(x)$'.

Table 11.2 –Integrals of Standard Form

1. $\int \mathrm{f}'(x)[\mathrm{f}(x)]^n\ \mathrm{d}x = \dfrac{[\mathrm{f}(x)]^{n+1}}{n+1} + C$, where C is a constant , $n \neq -1$
2. $\int \dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}\ \mathrm{d}x = \ln\lvert \mathrm{f}(x)\rvert + C$, where C is a constant
3. $\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}\ \mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + C$, where C is a constant

Question: Why did we 'drop' $\mathrm{f}'(x)$ after integration?

$\frac{d}{dx}\left(\frac{[f(x)]^{n+1}}{n+1} \right) = \frac{1}{n+1} \times n+1 \times f(x)^n \times f'(x)$

Example 11.1

Find the following integrals:

(a) $\int (3x-5)^{10}\ \mathrm{d}x$ (b) $\int \frac{1}{\sqrt{1-2x}}\mathrm{d}x$ (c) $\int \frac{1}{1-2x}\mathrm{d}x$ (d) $\int \mathrm{e}^{2x+1}\ \mathrm{d}x$.

Solution:

(a) $\int (3x-5)^{10}\mathrm{d}x$ $= \frac{1}{3}\int 3(3x-5)^{10}\ dx$ $= \frac{1}{3}\ \frac{(3x-5)^{11}}{11}$	$\mathrm{f}(x) = 3x - 5$ $\mathrm{f}'(x)$ is 3 and you balance with $\frac{1}{3}$. You will then 'drop' $\mathrm{f}'(x)$ and only divide by new power.

(b) $\int \frac{1}{\sqrt{1-2x}}\,dx = \int (1-2x)^{-\frac{1}{2}}dx$ $= -\frac{1}{2}\int -2(1-2x)^{-\frac{1}{2}}dx$ $= -\frac{1}{2}\frac{(1-2x)^{\frac{1}{2}}}{\frac{1}{2}} + C = -(1-2x)^{\frac{1}{2}} + C$	Change to indices form to realise that the power is $-\frac{1}{2}$. $f(x) = 1-2x$ $f'(x)$ is -2 and you balance with $-\frac{1}{2}$. You will then 'drop' $f'(x)$ and only divide by new power.
(c) $\int \frac{1}{1-2x}\,dx = \int (1-2x)^{-1}dx$ $= -\frac{1}{2}\int -2(1-2x)^{-1}dx$ $= -\frac{1}{2}\ln\lvert 1-2x\rvert + C$	$f(x) = 1-2x$ $f'(x)$ is -2 and you balance with $-\frac{1}{2}$. You will then 'drop' $f'(x)$ and use standard form 2 in Table 11.2
(d) $\int e^{2x+1}dx = \frac{1}{2}\int 2e^{2x+1}dx = \frac{1}{2}e^{2x+1} + C$	$f(x) = 2x+1$ $f'(x)$ is 2 and you balance with $\frac{1}{2}$. You will then 'drop' $f'(x)$ and use standard form 3 in Table 11.2.

Note:

In 'O' level, you learnt the result $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C$, $n \neq -1$. Can you see that this is a special case of $\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$, $n \neq -1$, where $f(x) = ax+b$?

Example 11.2

Use appropriate standard forms in Table 11.2 to find the following integrals:

(a) $\int \frac{9x}{\sqrt{3+x^2}}\,dx$ (b) $\int \frac{(\ln x)^2}{x}\,dx$ (c) $\int \sin^6 3x \cos 3x\,dx$ (d) $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

(e) $\int \frac{x}{1-x^2}\,dx$ (f) $\int \frac{e^{2x}}{1-e^{2x}}\,dx$ (g) $\int \frac{1}{\left(\sqrt{1-4x^2}\right)\sin^{-1}2x}\,dx$

Solution:

(a) $\int \frac{9x}{\sqrt{3+x^2}}\,dx = \int 9x(3+x^2)^{-\frac{1}{2}}dx$ $= \frac{9}{2}\int 2x(3+x^2)^{-\frac{1}{2}}dx$ $= \frac{9}{2}\frac{(3+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$ $= 9(3+x^2)^{\frac{1}{2}} + c$	Rewrite the integral in the form $\int 9x(3+x^2)^{-\frac{1}{2}}dx$ and check for $f'(x)$. Here $f(x) = 3+x^2$ and $f'(x) = 2x$

(b) $\int \frac{(\ln x)^2}{x}\,dx = \int \frac{1}{x}(\ln x)^2\,dx = \frac{(\ln x)^3}{3}+C$	Here $f(x) = \ln x$ and $f'(x) = \frac{1}{x}$
(c) $\int \sin^6 3x \cos 3x\,dx$ $=\frac{1}{3}\int(3\cos 3x)(\sin 3x)^6\,dx$ $=\frac{(\sin 3x)^7}{21}+C$	Here $f(x) = \sin 3x$ and $f'(x) = 3\cos 3x$
(d) $\int \frac{e^{\tan x}}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}e^{\tan x}\,dx$ $= \int \sec^2 x\, e^{\tan x}\,dx = e^{\tan x}+C$	Here $f(x) = \tan x$ and $f'(x) = \sec^2 x$
(e) $\int \frac{x}{1-x^2}dx$ $=-\frac{1}{2}\int(-2x)(1-x^2)^{-1}\,dx=-\frac{1}{2}\ln\lvert 1-x^2\rvert+C$	Here $f(x) = 1-x^2$ and $f'(x) = -2x$
(f) $\int \frac{e^{2x}}{1-e^{2x}}\,dx = -\frac{1}{2}\int(-2e^{2x})(1-e^{2x})^{-1}dx$ $=-\frac{1}{2}\ln\lvert 1-e^{2x}\rvert+C$	Here $f(x)=1-e^{2x}$ and $f'(x)=-2e^{2x}$
(g) $\int \frac{1}{\left(\sqrt{1-4x^2}\right)\sin^{-1}2x}dx=\frac{1}{2}\int\frac{2}{\sqrt{1-4x^2}}(\sin^{-1}2x)^{-1}dx$ $=\frac{1}{2}\ln\lvert\sin^{-1}2x\rvert+C$	Here $f(x) = \sin^{-1}2x$ and $f'(x)=\frac{2}{\sqrt{1-(2x)^2}}$

Note: In examples 11.1 and 11.2, the "balancing" can only be done with **constants**.

Self-Review 11.1

Find the following integrals:

(a) $\int\sqrt{4x-3}\,dx$ $\left[\frac{1}{6}(4x-3)^{\frac{3}{2}}+C\right]$

(b) $\int\left(4x+\frac{1}{\sqrt{x}}\right)^2dx$ $\left[\frac{16}{3}x^3+\frac{16}{3}x^{3/2}+\ln\lvert x\rvert+C\right]$

(c) $\int\frac{x}{x^2+1}dx$ $\left[\frac{1}{2}\ln\left(x^2+1\right)+C\right]$

(d) $\int\frac{2x}{\sqrt{1-4x^2}}dx$ $\left[-\frac{1}{2}\sqrt{1-4x^2}+C\right]$

(e) $\int\sec^2 x\tan^3 x\,dx$ $\left[\frac{1}{4}\tan^4 x+C\right]$

(f) $\int xe^{3x^2}dx$ $\left[\frac{1}{6}e^{3x^2}+C\right]$

(g) $\int\frac{1}{(9x^2+1)\tan^{-1}3x}dx$ $\left[\frac{1}{3}\ln\lvert\tan^{-1}3x\rvert+C\right]$

11.1.5 Integration of Trigonometric Functions

Table 11.3 below are some examples of standard trigonometric integrals.

Table 11.3 – Trigonometric Integrals

1. $\int \sin x \, dx = -\cos x + C$	2. $\int \cos x \, dx = \sin x + C$
3. $\int \sec^2 x \, dx = \tan x + C$	4. $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$
5. $\int \sec x \tan x \, dx = \sec x + C$	6. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$
7. $\int \tan x \, dx = \ln\lvert\sec x\rvert + C$ **(MF 26)** $(= -\ln\lvert\cos x\rvert + C)$	8. $\int \cot x \, dx = \ln\lvert\sin x\rvert + C$ **(MF 26)**
9. $\int \sec x \, dx = \ln\lvert\sec x + \tan x\rvert + C$ **(MF 26)**	10. $\int \operatorname{cosec} x \, dx = -\ln\lvert\operatorname{cosec} x + \cot x\rvert + C$ **(MF 26)**

Use of Trigonometric Identities:

The following trigonometric identities are very useful in integrating trigonometric functions:

- $\sin 2x = 2\sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$ for $\int \cos^2 x \, dx$
  $= 1 - 2\sin^2 x$ $\int \sin^2 x \, dx$
- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$ $\int \tan^2 x \, dx$
- $1 + \cot^2 x = \operatorname{cosec}^2 x$ $\int \cot^2 x \, dx$

$$\int \sec(ax+b)\,dx = \frac{\ln(\tan(ax+b)+\sec(ax+b))}{a}$$

$$= \frac{1}{a}\left(\ln\left(\sin\left(\frac{ax+b}{2}\right) + \cos\left(\frac{ax+b}{2}\right)\right) - \ln\left(\cos\left(\frac{ax+b}{2}\right) - \sin\left(\frac{ax+b}{2}\right)\right)\right)$$

Further Trigonometric Integrals:

1. $\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + C$
2. $\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + C$
3. $\int \tan(ax+b) \, dx = \frac{1}{a}\ln\lvert\sec(ax+b)\rvert + C\left(= -\frac{1}{a}\ln\lvert\cos(ax+b)\rvert + C\right)$
4. $\int \sec^2(ax+b) \, dx = \frac{1}{a}\tan(ax+b) + C$
5. $\int f'(x)\sin[f(x)] \, dx = -\cos[f(x)] + C$
6. $\int f'(x)\cos[f(x)] \, dx = \sin[f(x)] + C$
7. $\int f'(x)\tan[f(x)] \, dx = \ln\lvert\sec[f(x)]\rvert + C\left(= -\ln\lvert\cos[f(x)]\rvert + C\right)$

Remarks:

You should have encountered integrals 1, 2 and 4 at 'O' level. Integrals 5, 6 and 7 are more general versions of integrals 1, 2 and 3 respectively. We can verify the above integrals by differentiating the RHS to obtain the integrand on the LHS.

For example, to verify integral 6, $\frac{\mathrm{d}}{\mathrm{d}x}\{\sin[f(x)]\} = \cos[f(x)]\cdot\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]}_{f'(x)} = f'(x)\cos[f(x)]$.

Verify integrals 5 and 7 in a similar manner on your own.

Example 11.3

Find the following integrals:

(a) $\int \sin(2x-1)\,\mathrm{d}x$ (b) $\int \sec(2x+1)\,\mathrm{d}x$ (c) $\int \frac{2}{\sin^2(1-4x)}\,\mathrm{d}x$ (d) $\int x\tan(x^2)\,\mathrm{d}x$

Solution:

(a) $\int \sin(2x-1)\,\mathrm{d}x$ $= -\frac{1}{2}\cos(2x-1)+C$	$\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b)+C$
(b) $\int \sec(2x+1)\,\mathrm{d}x$ $= \frac{1}{2}\ln\lvert\sec(2x+1)+\tan(2x+1)\rvert+C$	$\int \sec(ax+b)\,\mathrm{d}x = \frac{1}{a}\ln\lvert\sec(ax+b)+\tan(ax+b)\rvert+C$
(c) $\int \frac{2}{\sin^2(1-4x)}\,\mathrm{d}x$ $= 2\int \operatorname{cosec}^2(1-4x)\,\mathrm{d}x$ $= 2\left[-\left(\frac{1}{-4}\right)\cot(1-4x)\right]+C$ $= \frac{1}{2}\cot(1-4x)+C$	$\int \operatorname{cosec}^2(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cot(ax+b)+C$
(d) $\int x\tan(x^2)\,\mathrm{d}x = \frac{1}{2}\int 2x\tan(x^2)\,dx$ $= \frac{1}{2}\ln\lvert\sec(x^2)\rvert + c$	$\int f'(x)\tan[f(x)]\,\mathrm{d}x = \ln\lvert\sec[f(x)]\rvert+C$

Example 11.4

Find: (a) $\int \sin^2 x\,\mathrm{d}x$ (b) $\int 2\cos^2\frac{x}{3}\,\mathrm{d}x$ (c) $\int \tan^2 2x\,\mathrm{d}x$.

Solution:

(a) $\int \sin^2 x\,\mathrm{d}x$ $= \int \frac{1}{2}(1-\cos 2x)\,dx$ $= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$	You need to use $\cos 2x = 1 - 2\sin^2 x$ (double angle formula) and change $\sin^2 x$ to a form found in Table 11.3.

(b) $\int 2\cos^2\frac{x}{3}\,dx = \int\left(\cos\frac{2x}{3}+1\right)dx = \frac{3}{2}\sin\frac{2x}{3}+x+C$	Use $\cos 2\left(\frac{x}{3}\right) = 2\cos^2\frac{x}{3}-1$ (double angle formula)
(c) $\int \tan^2 2x\,dx = \int\left(\sec^2 2x-1\right)dx = \frac{1}{2}\tan 2x - x + C$	$\tan^2 2x$ is not found in Table 11.3. You need to use $1+\tan^2 2x = \sec^2 2x$ and change $\tan^2 2x$ to a form found in Table 11.3.

Note:
The trigonometric identities $\cos 2x = 2\cos^2 x - 1 = 1-2\sin^2 x$ are particularly useful in finding integrals of the type $\int \sin^2 kx\,dx, \int \cos^2 kx\,dx$.

Example 11.5

Show that $\tan^3 x = \tan x\sec^2 x - \tan x$. Hence, find $\int \tan^3 x\,dx$.

Solution:

$\tan^3 x = \tan^2 x\tan x = (\sec^2 x - 1)\tan x = \sec^2 x\tan x - \tan x$	$1+\tan^2 x = \sec^2 x$
$\int \tan^3 x\,dx = \int\left[\sec^2 x\tan x - \tan x\right]dx$	
$= \int\left[\underbrace{(\sec x\tan x)}_{f'(x)}\underbrace{(\sec x)^1}_{f(x)}\right]dx - \int \tan x\,dx$	$f(x) = \sec x$ $f'(x) = \sec x\tan x$
$= \frac{1}{2}\underbrace{\sec^2 x}_{[f(x)]^2} - \ln\lvert\sec x\rvert + C$	
Alternatively,	
$\int \tan^3 x\,dx = \int \underbrace{\sec^2 x}_{f'(x)}\underbrace{(\tan x)^1}_{f(x)}\,dx - \int \tan x\,dx$	$f(x) = \tan x$ $f'(x) = \sec^2 x$
$= \frac{1}{2}\underbrace{\tan^2 x}_{[f(x)]^2} - \ln\lvert\sec x\rvert + C$	
Q: The two results look different but they are actually equivalent. Can you explain why?	
$= \frac{1}{2}(\sec^2 x - 1) - \ln\lvert\sec x\rvert + C$	

Self-Review 11.2

Find the following integrals:

(a) $\int \cos^2 3x\,dx$ $\quad\left[\frac{1}{2}\left(x+\frac{1}{6}\sin 6x\right)+C\right]$

(b) $\int \sin^2 x\cos^2 x\,dx$ $\quad\left[\frac{1}{8}\left(x-\frac{1}{4}\sin 4x\right)+C\right]$

$= \int (\sin x\cos x)^2\,dx$

$= \int \left(\frac{1}{2}\sin 2x\right)^2 dx$

$= \int \frac{1}{4}\sin^2 2x\,dx$

$= \int \frac{1}{4}(1-\cos 4x)\,dx$

11.1.6 Products of Trigonometric Functions

In this section, we shall consider trigonometric integrals of the form $\int \sin mx \cos nx \, dx$, $\int \cos mx \cos nx \, dx$ or $\int \sin mx \sin nx \, dx$ where $m > n$. The **Factor Formulae** may be used to transform the product of trigonometric functions to the sum or difference of trigonometric functions of multiple angles.

Table 11.4 – The Factor Formulae

The factor formulae below are given in MF 26 ($P \neq Q$ and $P > Q$)	Not given in MF 26 ($A > B$) — Sum of A & B; "Difference" of A & B
$\sin P + \sin Q = 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$	$2\sin A\cos B = \sin(A+B) + \sin(A-B)$
$\sin P - \sin Q = 2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$	$2\cos A\sin B = \sin(A+B) - \sin(A-B)$
$\cos P + \cos Q = 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$	$2\cos A\cos B = \cos(A+B) + \cos(A-B)$
$\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$	$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$

Example 11.6

Find the following integrals:

(a) $\int 2\cos 3x \sin x \, dx$ (b) $\int \sin 2x \sin 3x \, dx$

Solution:

(a) $2\cos 3x \sin x = \sin(3x+x) - \sin(3x-x)$. Hence $\int 2\cos 3x\sin x \, dx$ $= \int (\sin(3x+x) - \sin(3x-x)) \, dx$ $= \int (\sin 4x - \sin 2x) \, dx$ $= -\frac{1}{4}\cos 4x + \frac{1}{2}\cos 2x + C$	You need not memorise the result $2\cos A\sin B = \sin(A+B) - \sin(A-B)$ but instead use the factor formula in MF26: $\sin P - \sin Q$ $= 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$ and write $\frac{P+Q}{2} = 3x$, $\frac{P-Q}{2} = x$ Adding gives $P = 4x$ Subtracting gives $Q = 2x$. $2\cos 3x\sin x = \sin 4x - \sin 2x$
(b) $\sin 2x \sin 3x = \sin 3x \sin 2x$ $= -\frac{1}{2}[\cos(3x+2x) - \cos(3x-2x)]$ $= -\frac{1}{2}[\cos 5x - \cos x]$ $\int \sin 2x \sin 3x \, dx = -\frac{1}{2}[\int \cos 5x \, dx - \int \cos x \, dx]$ $= -\frac{1}{2}[\frac{1}{5}\sin 5x - \sin x] + C$	$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$ Rearranging, $\sin A\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

$\int \frac{1}{a^2+(x+b)^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right)+c$

Self-Review 11.3

Find $\int \sin 2x \sin 4x \, dx$. $\left[-\frac{1}{12}\sin 6x+\frac{1}{4}\sin 2x+C\right]$

11.1.7 Standard Integrals

Table 11.5 below are standard integrals. The unknown 'a' denotes a positive real constant (ie $a>0$).

$\int \frac{1}{\sqrt{a^2-(x+b)^2}}dx = \sin^{-1}\frac{x+b}{a}+c$

Table 11.5 – Standard Integrals

The 4 integrals below are given in **MF 26**	A more general form, not given in MF26.
1. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+C \quad (\lvert x\rvert<a)$ *The condition $\lvert x\rvert<a$ is necessary so that $a^2-x^2>0$	$\int \frac{f'(x)}{\sqrt{a^2-(f(x))^2}}dx = \sin^{-1}\left(\frac{f(x)}{a}\right)+C$
2. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$	$\int \frac{f'(x)}{a^2+(f(x))^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right)+C$
3. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert+C$	$\int \frac{f'(x)}{a^2-(f(x))^2}dx = \frac{1}{2a}\ln\left\lvert\frac{a+f(x)}{a-f(x)}\right\rvert+C$
4. $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert+C$	$\int \frac{f'(x)}{(f(x))^2-a^2}dx = \frac{1}{2a}\ln\left\lvert\frac{f(x)-a}{f(x)+a}\right\rvert+C$

Note:

When applying the formulae in MF26, care must be taken to ensure the integrand is written in exactly the same way as those given in Table 11.5, that is $\frac{1}{\sqrt{a^2-x^2}}, \frac{1}{a^2+x^2}$ etc. and the $\boxed{\textbf{coefficient of } x^2 \textbf{ is } \pm 1}$. You may also use the more general forms on the 2nd column of Table 11.5.

$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \times \frac{1}{a} = \frac{1}{a} \times \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} = \frac{1}{\sqrt{a^2-x^2}}$

Exercise:

1. Differentiate $\sin^{-1}\left(\frac{x}{a}\right)$ with respect to x. Hence show $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+C$.

2. Differentiate $\tan^{-1}\left(\frac{x}{a}\right)$ with respect to x. Hence show $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$.

3. Differentiate $\ln\left\lvert\frac{a+x}{a-x}\right\rvert$ with respect to x. Hence show $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert+C$.

Example 11.7

Find the following integrals:

(a) $\int \frac{1}{\sqrt{1-9x^2}}dx$ (b) $\int \frac{7}{5+16x^2}dx$ (c) $\int \frac{1}{\sqrt{8-2x-x^2}}dx$ (d) $\int \frac{1}{7-6x-x^2}dx$

(e) $\displaystyle\int \frac{1}{2x^2+4x-5}\,dx$ (Self-Reading)

Solution:

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(a)

$$\int \frac{1}{\sqrt{1-9x^2}}\,dx = \int \frac{1}{\sqrt{9\left(\frac{1}{9}-x^2\right)}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}\,dx$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{x}{1/3}\right)+C$$

$$= \frac{1}{3}\sin^{-1}(3x)+C$$

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Factor out 9 so that the coefficient of x^2 is ± 1.

Alternatively, we rewrite the integral in the general form and use the result $\displaystyle\int \frac{f'(x)}{\sqrt{a^2-(f(x))^2}}\,dx = \sin^{-1}\left(\frac{f(x)}{a}\right)+C$ as below:

$$\int \frac{1}{\sqrt{1-9x^2}}\,dx = \int \frac{1}{\sqrt{1-(3x)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{3}{\sqrt{1-(3x)^2}}\,dx$$

$$= \frac{1}{3}\sin^{-1}3x + C$$

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(b)

$$\int \frac{7}{5+16x^2}\,dx$$

$$= \frac{7}{16}\int \frac{1}{\frac{5}{16}+x^2}\,dx$$

$$= \frac{7}{16}\int \frac{1}{\left(\frac{\sqrt{5}}{4}\right)^2+x^2}$$

$$= \frac{7}{16}\left(\frac{4}{\sqrt{5}}\right)\tan^{-1}\left(\frac{x}{\sqrt{5}/4}\right)+c$$

$$= \frac{7}{4\sqrt{5}}\tan^{-1}\left(\frac{4x}{\sqrt{5}}\right)+c$$

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Factor out 16 so that the coefficient of x^2 is ± 1.

Alternatively we use the general result: $\displaystyle\int \frac{f'(x)}{a^2+(f(x))^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right)+C$

$$\int \frac{7}{5+16x^2}\,dx = 7\int \frac{1}{\left(\sqrt{5}\right)^2+(4x)^2}\,dx$$

$$= \frac{7}{4}\int \frac{4}{\left(\sqrt{5}\right)^2+(4x)^2}\,dx$$

$$= \frac{7}{4}\cdot\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{4x}{\sqrt{5}}\right)+C$$

$$= \frac{7}{4\sqrt{5}}\tan^{-1}\frac{4x}{\sqrt{5}}+C$$

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(c) $\displaystyle\int \frac{1}{\sqrt{8-2x-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{-\left(x^2+2x-8\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x+1)^2+9}}\,dx$$

$$= \int \frac{1}{\sqrt{3^2-(x+1)^2}}\,dx = \sin^{-1}\frac{(x+1)}{3}+C$$

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We have used the more general result:

$$\int \frac{1}{\sqrt{a^2-(x+b)^2}}\,dx = \sin^{-1}\left(\frac{x+b}{a}\right)+C$$

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(d) $\int \frac{1}{7-6x-x^2}\,dx$ $=\int \frac{1}{16-(x+3)^2}\,dx = \int \frac{1}{4^2-(x+3)^2}\,dx$ $=\frac{1}{2(4)}\ln\left\lvert\frac{4+(x+3)}{4-(x+3)}\right\rvert+C$ $=\frac{1}{8}\ln\left\lvert\frac{7+x}{1-x}\right\rvert+C$	We have used the more general result: $\int \frac{1}{a^2-(x+b)^2}\,dx=\frac{1}{2a}\ln\left\lvert\frac{a+(x+b)}{a-(x+b)}\right\rvert+C$
(e) $\int \frac{1}{2x^2+4x-5}\,dx$ $=\int \frac{1}{2\left(x^2+2x-\frac{5}{2}\right)}\,dx$ $=\frac{1}{2}\int \frac{1}{(x+1)^2-\frac{7}{2}}\,dx$ $=\frac{1}{2}\int \frac{1}{(x+1)^2-\left(\sqrt{\frac{7}{2}}\right)^2}\,dx$ $=\frac{1}{2}\left(\frac{1}{2\sqrt{7/2}}\right)\ln\left\lvert\frac{(x+1)-\sqrt{7/2}}{(x+1)+\sqrt{7/2}}\right\rvert+C$ $=\frac{1}{2\sqrt{14}}\ln\left\lvert\frac{\sqrt{2}x+\sqrt{2}-\sqrt{7}}{\sqrt{2}x+\sqrt{2}+\sqrt{7}}\right\rvert+C$	We have used the more general result: $\int \frac{1}{(x+b)^2-a^2}\,dx=\frac{1}{2a}\ln\left\lvert\frac{(x+b)-a}{(x+b)+a}\right\rvert+C$

Self-Review 11.4

Find the following integrals:

(a) $\int \frac{1}{\sqrt{4-x^2}}\,dx$ $\quad\left[\sin^{-1}\left(\frac{x}{2}\right)+C\right]$

(b) $\int \frac{1}{2x^2+4x+20}\,dx$ $\quad\left[\frac{1}{6}\tan^{-1}\left(\frac{x+1}{3}\right)+C\right]$

(c) $\int \frac{1}{1+4x-2x^2}\,dx$ $\quad\left[\frac{\sqrt{2}}{4\sqrt{3}}\ln\left(\frac{\sqrt{3}-\sqrt{2}+\sqrt{2}x}{\sqrt{3}+\sqrt{2}-\sqrt{2}x}\right)+C\right]$

11.1.8 Integration of Rational Function of the form $\int \frac{f(x)}{g(x)}dx$, where f(x) is a polynomial and $g(x) = ax^2 + bx + c$

We consider two cases where

1. f(x) is a constant and
2. f(x) is linear, ie $f(x) = dx + e$

Note: If the fraction $\frac{f(x)}{g(x)}$ is improper, you need to do a long division. The integrand would then be reduced to either case 1 or 2.

11.1.8.1 f(x) is a constant

In this case, we perform the following steps:

1. Complete the square for $g(x)\left(= ax^2 + bx + c\right)$.
2. Refer to MF26 for appropriate formulae in Table 11.5.

Note that we have covered this in Example 11.7b, d and e.

11.1.8.2 f(x) is linear, ie $f(x) = dx + e$

In this case, we perform the following steps:

1. Let $f(x) = Ag'(x) + B$. Solve for A and B.
2. Rewrite the integrand $\int \frac{f(x)}{g(x)}dx = \int \frac{Ag'(x)+B}{g(x)}dx = \int \frac{Ag'(x)}{g(x)}dx + \int \frac{B}{g(x)}dx$ and integrate.

Example 11.8

Find the following integrals:

(a) $\int \frac{x+1}{x^2+4x+6}dx$ (b) $\int \frac{9x+7}{9x^2+6x+4}dx$

Solution:

(a)

Let $x+1 = A(2x+4)+B$

comparing coeff of $x: 1 = 2A \Rightarrow A = \frac{1}{2}$

comparing constant: $1 = 4A + B \Rightarrow B = 1-2 = -1$

$$\int \frac{x+1}{x^2+4x+6}dx$$

$$= \int \frac{\frac{1}{2}(2x+4)-1}{x^2+4x+6}dx$$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+6}dx - \int \frac{1}{x^2+4x+6}dx$$

Note that the fraction is proper.

$g(x) = x^2+4x+6$

$g'(x) = 2x+4$

We rewrite the numerator '$x+1$' as '$x+1 = \frac{1}{2}(2x+4)-1$' so as to introduce the expression '$2x+4$' which is the derivative of the denominator.

Thereafter, we "split" the integral.

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$=\frac{1}{2}\ln\left(x^2+4x+6\right)-\int\frac{1}{(x+2)^2+2}\,dx$

$=\frac{1}{2}\ln(x^2+4x+6)-\int\frac{1}{(x+2)^2+(\sqrt{2})^2}\,dx$

$=\frac{1}{2}\ln(x^2+4x+6)-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right)+c$

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Q: Why '| |' was not used for $\ln(x^2+4x+6)$?

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(b)
Let $9x+7=A(18x+6)+B$

comparing coeff of $x: 9=18A \Rightarrow A=\frac{1}{2}$

comparing constant: $7=6A+B \Rightarrow B=7-3=4$

$\int\frac{9x+7}{9x^2+6x+4}\,dx$

$=\frac{1}{2}\int\frac{18x+6}{9x^2+6x+4}\,dx+\int\frac{4}{9x^2+6x+4}\,dx$

$=\frac{1}{2}\ln\left(9x^2+6x+4\right)+\frac{4}{9}\int\frac{1}{x^2+\frac{2}{3}x+\frac{4}{9}}\,dx$

$=\frac{1}{2}\ln\left(9x^2+6x+4\right)+\frac{4}{9}\int\frac{1}{\left(x+\frac{1}{3}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2}\,dx$

$=\frac{1}{2}\ln\left(9x^2+6x+4\right)+\frac{4}{9}\left[\frac{1}{1/\sqrt{3}}\tan^{-1}\left(\frac{x+1/3}{1/\sqrt{3}}\right)\right]+C$

$=\frac{1}{2}\ln\left(9x^2+6x+4\right)+\frac{4}{3\sqrt{3}}\tan^{-1}\left[\sqrt{3}\left(x+\frac{1}{3}\right)\right]+C$

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Note that the fraction is proper.

$g(x)=9x^2+6x+4$

$g'(x)=18x+6$

Rewrite the numerator '$9x+7$' as '$9x+7=\frac{1}{2}(18x+6)+4$' to introduce the derivative '$18x+6$' in the numerator. Then split the integral.

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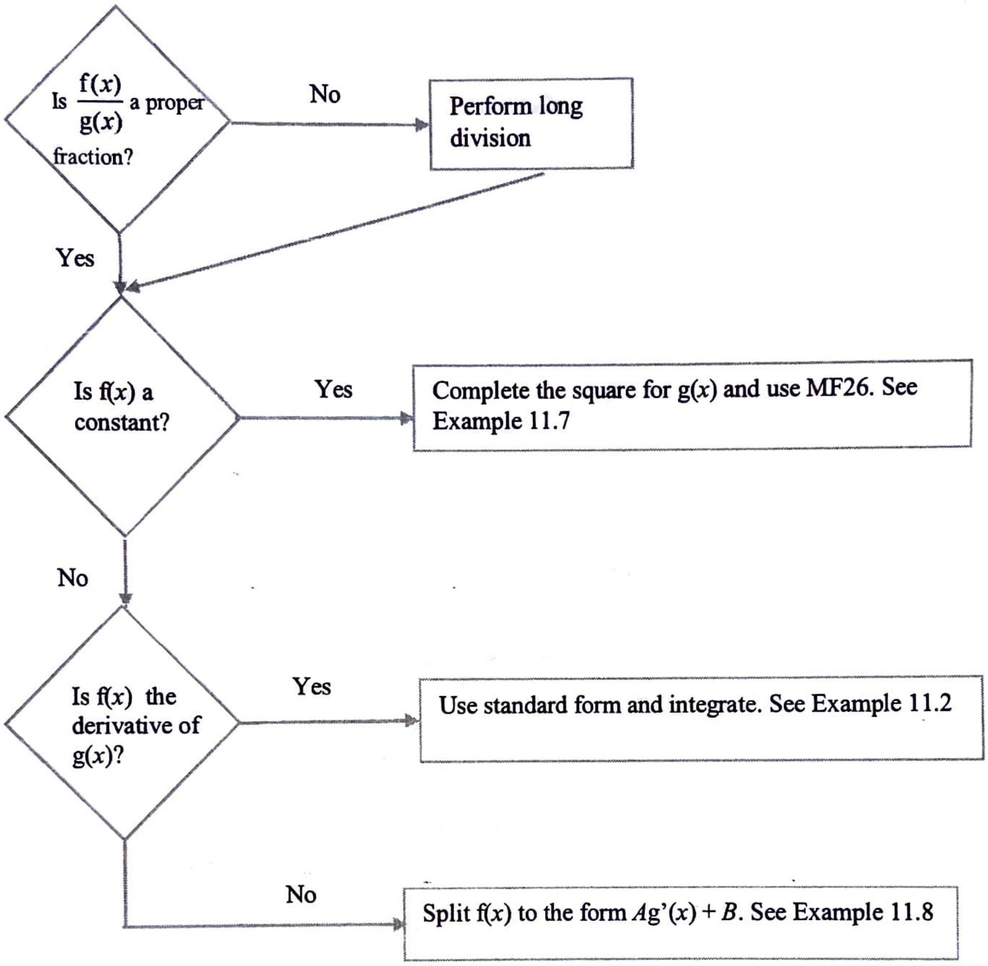
Self-Review 11.5

Show that $\int\frac{x}{x^2+x-1}\,dx=\frac{1}{2}\ln\left|x^2+x-1\right|-\frac{1}{2\sqrt{5}}\ln\left|\frac{2x+1-\sqrt{5}}{2x+1+\sqrt{5}}\right|+C$.

Process map for finding $\int \frac{f(x)}{g(x)}dx$, where g(x) is a quadratic polynomial



11.1.9 Integration of Rational Function of the form $\int \frac{f(x)}{\sqrt{g(x)}}dx$, where f($x$) is a polynomial and $g(x) = ax^2 + bx + c$

11.1.9.1 f(x) is a constant (See Example 11.7a and c)

In this case, we perform the following steps:

1. Complete the square for $g(x)\left(=ax^2+bx+c\right)$.
2. Refer to MF26 for appropriate formulae in Table 11.5.

11.1.9.2 f(x) is linear, ie $f(x) = dx + e$

In this case, we perform the following steps:

1. Let $f(x) = Ag'(x) + B$. Solve for A and B.
2. Rewrite the integrand $\int \frac{f(x)}{\sqrt{g(x)}}dx = \int \frac{Ag'(x)+B}{\sqrt{g(x)}}dx = \int \frac{Ag'(x)}{\sqrt{g(x)}}dx + \int \frac{B}{\sqrt{g(x)}}dx$

$$= A\int g'(x)\left[g(x)\right]^{-\frac{1}{2}}dx + B\int \frac{1}{\sqrt{g(x)}}dx$$

Example 11.9

Find $\int \frac{x}{\sqrt{2x-x^2}}\,dx$.

Solution:

Write $x = A(2-2x)+B$. Solving gives $A=-\frac{1}{2}, B=1$ $\int \frac{x}{\sqrt{2x-x^2}}\,dx$ $= -\frac{1}{2}\int \frac{2-2x}{\sqrt{2x-x^2}}\,dx + \int \frac{1}{\sqrt{2x-x^2}}\,dx$ $= -\frac{1}{2}\int (2-2x)(2x-x^2)^{-½}\,dx + \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$ $= -\frac{1}{2}\frac{(2x-x^2)^{½}}{½} + \sin^{-1}(x-1)+C$ $= -\sqrt{2x-x^2} + \sin^{-1}(x-1)+C$	$g(x)=2x-x^2$ $g'(x)=2-2x$ Rewrite the numerator 'x' as '$x=-\frac{1}{2}(2-2x)+1$' to introduce the expression '$2-2x$'. Complete the square for $2x-x^2$ in the denominator.

Self-Review 11.6

Find: (a) $\int \frac{x+3}{x^2+16}\,dx$ $\left[\frac{1}{2}\left(\ln(x^2+16)+\frac{3}{2}\tan^{-1}\left(\frac{x}{4}\right)+C\right)\right]$

(b) $\int \frac{2x+1}{\sqrt{3+2x-x^2}}\,dx$ $\left[3\sin^{-1}\left(\frac{x-1}{2}\right)-2\sqrt{(3+2x-x^2)}+C\right]$

11.1.10 Integrating Rational Fraction by Partial Fractions

Another method to integrate rational function is via partial fractions decomposition. You need to ensure that the rational function is a proper rational fraction.

The following partial fractions decomposition are given in MF 26:

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{ax+b}+\frac{B}{cx+d}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{ax+b}+\frac{Bx+C}{x^2+c^2}$$

Example 11.10

Find $\displaystyle\int \frac{1}{x^2+x-2}\,\mathrm{d}x$.

Solution:

$\dfrac{1}{x^2+x-2}=\dfrac{1}{(x-1)(x+2)}=\dfrac{A}{x-1}+\dfrac{B}{x+2}$ By multiplying the entire equation by $(x-1)(x+2)$, $1=A(x+2)+B(x-1)$ Sub $x=-2,\ B=-\dfrac{1}{3}$ Sub $x=1,\ A=\dfrac{1}{3}$ $\displaystyle\int \frac{1}{x^2+x-2}\,\mathrm{d}x=\int \frac{1/3}{x-1}+\frac{-1/3}{x+2}\,\mathrm{d}x=\frac{1}{3}\ln\lvert x-1\rvert-\frac{1}{3}\ln\lvert x+2\rvert+C$	1. You can use the 'cover-up' method to obtain the values of A and B 2. Remember to put | | sign 3. Can you solve this via completing the square?

Example 11.11

Find $\displaystyle\int \frac{2x+9}{x^3+9x}\,\mathrm{d}x$.

Solution:

$\dfrac{2x+9}{x^3+9x}=\dfrac{2x+9}{x(x^2+9)}=\dfrac{Ax+B}{x^2+9}+\dfrac{C}{x}$ $2x+9=(Ax+B)x+C(x^2+9)$ Sub $x=0,\quad 9=9C\Rightarrow C=1$ Comparing coefficient of x^2 : $0=A+C\ \Rightarrow A=-1$ Comparing coefficient of x : $2=B$ $\displaystyle\int\left(\frac{-x+2}{x^2+9}+\frac{1}{x}\right)\mathrm{d}x=-\int\frac{x}{x^2+9}\,\mathrm{d}x+\int\frac{2}{x^2+9}\,\mathrm{d}x+\int\frac{1}{x}\,\mathrm{d}x$ $\displaystyle=-\frac{1}{2}\int\frac{2x}{x^2+9}\,\mathrm{d}x+\int\frac{2}{x^2+3^2}\,\mathrm{d}x+\int\frac{1}{x}\,\mathrm{d}x$ $\displaystyle=-\frac{1}{2}\ln(x^2+9)+\frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right)+\ln\lvert x\rvert+C$	$\ln\lvert x^2+9\rvert=\ln(x^2+9)$ (Why?)

Self-Review 11.7

Find $\displaystyle\int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,\mathrm{d}x$.

$[-\frac{7}{15}\ln\lvert 3x-1\rvert+\frac{2}{5}\ln(x^2+1)+\frac{3}{5}\tan^{-1}x+C\ ]$

11.1.11 Integration by Substitution

The basic idea of **integration by substitution** is to transform the original integral $\int f(x)\,dx$ by a change of variable into a new integral $\int g(u)\,du$ which is easy to find. Upon integrating, you must remember to change the answer back in terms of x. Note that whenever the question requires the use of substitution, the substitution is always given.

Example 11.12

Use the suggested substitutions (in parenthesis) to find the following integrals:

(a) $\int x^2\cos\left(x^3\right)dx \quad (u=x^3)$ (b) $\int \frac{x^3}{\sqrt{1-x^8}}\,dx \quad (u=x^4)$ (c) $\int\sqrt{25-x^2}\,dx \quad (x=5\sin u)$

Solution:

(a) $u=x^3 \Rightarrow \frac{du}{dx}=3x^2 \Rightarrow dx=\frac{du}{3x^2}$ $\int x^2\cos\left(x^3\right)dx = \int x^2\cos\left(x^3\right)\frac{du}{3x^2}$ $=\int\cos(u)\frac{1}{3}du$ $=\frac{1}{3}\int\cos(u)\,du$ $=\frac{1}{3}\sin u + C$ $=\frac{1}{3}\sin\left(x^3\right)+C$	Make "dx" the subject. Remember to change the answer back to in terms of x.
(b) $u=x^4 \Rightarrow \frac{du}{dx}=4x^3 \Rightarrow dx=\frac{du}{4x^3}$ $\int\frac{x^3}{\sqrt{1-x^8}}dx=\int\frac{x^3}{\sqrt{1-x^8}}\frac{du}{4x^3}=\frac{1}{4}\int\frac{du}{\sqrt{1-u^2}}$ $=\frac{1}{4}\sin^{-1}u+C=\frac{1}{4}\sin^{-1}(x^4)+c$	Make "dx" the subject. Remember to change the answer back to in terms of x.
(c) $x=5\sin u \Rightarrow \frac{dx}{du}=5\cos u \Rightarrow dx=5\cos u\,du$ $\int\sqrt{25-x^2}\,dx=\int\sqrt{25-25\sin^2 u}\,(5\cos u)\,du$ $=\int\sqrt{25\left(1-\sin^2 u\right)}\,(5\cos u)\,du$ $=\int\sqrt{25\cos^2 u}\,(5\cos u)\,du$ $=25\int\cos^2 u\,du$ $=25\int\frac{1}{2}(1+\cos 2u)\,du$	Use $\cos 2x = 2\cos^2 x - 1$

$= \dfrac{25}{2}\left(u + \dfrac{1}{2}\sin 2u\right) + C$	Use $\sin 2x = 2 \sin x \cos x$
$= \dfrac{25}{2}\left[u + \dfrac{1}{2}(2 \sin u \cos u)\right] + C$	$x = 5\sin u \Rightarrow u = \sin^{-1}\left(\dfrac{x}{5}\right)$
$= \dfrac{25}{2}(u + \sin u \cos u) + C$	$x = 5 \sin u$ $\Rightarrow \sin u = \dfrac{x}{5}$
$= \dfrac{25}{2}\left[\sin^{-1}\left(\dfrac{x}{5}\right) + \left(\dfrac{x}{5}\right)\left(\dfrac{\sqrt{25-x^2}}{5}\right)\right] + C$	$\cos u = \sqrt{1-\sin^2 u}$
$= \dfrac{1}{2}\left[25\sin^{-1}\left(\dfrac{x}{5}\right) + x\sqrt{25-x^2}\right] + C$	$= \sqrt{1-\left(\dfrac{x}{5}\right)^2} = \sqrt{\dfrac{25-x^2}{25}}$ $= \dfrac{\sqrt{25-x^2}}{5}$

Self-Review 11.8

(a) Find $\displaystyle\int \frac{e^x}{1+e^{2x}}\, dx$ using the substitution $u = e^x$ $\quad\left[\tan^{-1}(e^x) + C\right]$

(b) Find $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\, dx$ using the substitution $x = \sin\theta$ $\quad\left[\frac{1}{2}\left(\sin^{-1}x - x\sqrt{1-x^2}\right) + C\right]$

11.1.12 Integration by Parts

Let u and v to be two functions of x

Then $\quad \dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$ $\quad$ by the product rule of differentiation.

Integrating, $\quad \displaystyle\int \frac{d}{dx}(uv)\, dx = \int v\frac{du}{dx}\, dx + \int u\frac{dv}{dx}\, dx$

$$uv = \int v\frac{du}{dx}\, dx + \int u\frac{dv}{dx}\, dx$$

$\int u v'\, dx = uv - \int v u'\, dx$

Rearranging, we have the following useful result:

$$\boxed{\int u\frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx}$$

$\cancel{\int u(x)\, v'(x)\, dx = u(x)v(x)}$

Integrating a product using this formula is called **integration by parts**. The aim is to ensure that $v\dfrac{du}{dx}$ is easier to integrate than $u\dfrac{dv}{dx}$. Thus, care must be taken in the choice of u and $\dfrac{dv}{dx}$.

In the method of integration by parts, we will let u to be part of the expression to be differentiated while the remaining part, $\dfrac{dv}{dx}$, to be integrated.

As a general guideline, we assign 'u' to take the following functions in *descending order* of priority (The so-called 'LIATE' rule):

L:	Logarithmic functions (e.g. $\ln x$)
I:	Inverse trigonometric functions (e.g. $\sin^{-1} x$, $\tan^{-1} x$)
A:	Algebraic functions (e.g. x, x^2)
T:	Trigonometric functions (e.g. $\sin x$, $\cos x$)
E:	Exponential functions (e.g. e^{-x}, e^{2x})

After assigning 'u' to the appropriate function, '$\frac{dv}{dx}$' will be assigned to the remaining function.

Note that in some cases the constant function '1' will be assigned as $\frac{dv}{dx}$ (see Example 11.13(c) and (d)).

Note:

1. The part assigned as '$\frac{dv}{dx}$' must be easily integrable. Moreover the resulting integrand '$v\frac{du}{dx}$' must be easier to integrate than the original integrand '$u\frac{dv}{dx}$'.
2. The 'LIATE' rule provides only a general guideline to choose the appropriate 'u'. It should not be adhered to rigidly (see example 11.15).

Example 11.13

Find the following using integration by parts:

(a) $\int x^2 \ln x \, dx$ (b) $\int x \sin x \, dx$ (c) $\int \ln 2x \, dx$ (d) $\int \sin^{-1} x \, dx$

Solution:

(a) $\int \underset{\frac{dv}{dx}}{x^2}\ \underset{u}{\ln x}\, dx$	Assign u to $\ln x$ and $\frac{dv}{dx}$ to x^2.
$u = \ln x \qquad \frac{dv}{dx} = x^2$	L I A T E
$\frac{du}{dx} = \frac{1}{x} \qquad v = \int x^2 dx = \frac{1}{3}x^3$	$\boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$
$\int x^2 \ln x \, dx = (\ln x)\frac{x^3}{3} - \int \frac{x^3}{3}\left(\frac{1}{x}\right)dx$ $= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 \, dx$ $= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$	If u and $\frac{dv}{dx}$ is written in this order, the flow of the arrows corresponds to the formula above.

(b) $\int x\sin x\,dx$ $u=x \quad \frac{dv}{dx}=\sin x$ $\frac{du}{dx}=1 \quad v=-\cos x$ $\int x\sin x\,dx = -x\cos x-\int(-\cos x)(1)\,dx$ $= -x\cos x+\sin x + C$	LIATE Assign u to x and $\frac{dv}{dx}$ to $\sin x$.
(c) $\int \ln 2x\,dx = \int 1\cdot\ln 2x\,dx$ $u=\ln 2x \quad \frac{dv}{dx}=1$ $\frac{du}{dx}=\frac{2}{2x} \leftarrow v=\int 1\,dx = x$ $\int \ln 2x\,dx = (\ln 2x)x-\int x\left(\frac{1}{x}\right)dx = x\ln 2x - x + C$	Add in the constant function "1" to create an algebraic function, "A". LIATE
(d) $\int \sin^{-1}x\,dx = \int 1\cdot\sin^{-1}x\,dx$ $u=\sin^{-1}x \quad \frac{dv}{dx}=1$ $\frac{du}{dx}=\frac{1}{\sqrt{1-x^2}} \leftarrow v=\int 1\,dx = x$ $\int \sin^{-1}x\,dx = x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}\,dx$ $= x\sin^{-1}x-\frac{1}{(-2)}\int(-2x)\left(1-x^2\right)^{-1/2}dx$ $= x\sin^{-1}x+\frac{1}{2}\frac{\left(1-x^2\right)^{1/2}}{1/2}+C$ $= x\sin^{-1}x+\sqrt{1-x^2}+C$	Add in the constant function "1" to create an algebraic function, "A". LIATE The standard form $\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1}+C$ is used here

Example 11.14

Find $\int e^x\cos x\,dx$.

Solution:

$u=\cos x \quad \frac{dv}{dx}=e^x$ $\frac{du}{dx}=-\sin x \leftarrow v=\int e^x dx = e^x$	

$\int e^x \cos x \, dx = (\cos x)e^x - \int e^x(-\sin x)\, dx$ $= e^x \cos x + \int e^x \sin x \, dx$ $u = \sin x \qquad \frac{dv}{dx} = e^x$ $\frac{du}{dx} = \cos x \qquad v = \int e^x dx = e^x$ $\int e^x \cos x \, dx = e^x \cos x + \left[e^x \sin x - \int e^x \cos x \, dx \right]$ -----(1) $2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$ $\int e^x \cos x \, dx = \frac{1}{2} e^x(\cos x + \sin x)$ $\therefore \int e^x \cos x \, dx = \frac{1}{2} e^x(\cos x + \sin x) + C$	Need to apply Integration by Parts again with $u = \sin x$ and $\frac{dv}{dx} = e^x$. Arrange (1) to make $\int e^x \cos x \, dx$ the subject. Since inclusion of the arbitrary constant is mandatory, we include the constant of integration at the final step in our answer. **Q:** What if we let $u = e^x$ and $\frac{dv}{dx} = \sin x$? What do you get? Investigate it yourself.

Example 11.15

Find: (a) (i) $\int xe^{x^2} \, dx$, (ii) $\int x^3 e^{x^2} \, dx$.

(b) (i) $\int x\sqrt{x^2 - 1} \, dx$, (ii) $\int x^3\sqrt{x^2 - 1} \, dx$.

Solution:

(a) (i) $\int xe^{x^2} \, dx = \frac{1}{2}\int 2xe^{x^2} \, dx = \frac{1}{2}e^{x^2} + C$	We can just use the standard form $\int f'(x)e^{f(x)} \, dx = e^{f(x)} + C$
(a) (ii) $\int x^3 e^{x^2} \, dx = \int x^2\left(xe^{x^2}\right) dx$ $u = x^2 \qquad \frac{dv}{dx} = xe^{x^2}$ $\frac{du}{dx} = 2x \qquad v = \int xe^{x^2} dx = \frac{1}{2}e^{x^2}$ $\int x^3 e^{x^2} \, dx = \left(x^2\right)\frac{e^{x^2}}{2} - \int\left(\frac{e^{x^2}}{2}\right) 2x \, dx$ $= \frac{1}{2}x^2 e^{x^2} - \left(\frac{1}{2}\right)\int (2x)e^{x^2} \, dx$ $= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$	Note that LIATE cannot be used. We choose $\frac{dv}{dx} = xe^{x^2}$ because we have $\int xe^{x^2} dx$ in part (i).

(b)(i) $\int x\sqrt{x^2-1}\,\mathrm{d}x$ $= \frac{1}{2}\int 2x\sqrt{x^2-1}\,dx$ $= \frac{1}{2}\frac{(x^2-1)^{3/2}}{3/2} = \frac{1}{3}(x^2-1)^{\frac{3}{2}}+C$	
(b)(ii) $\int x^3\sqrt{x^2-1}\,\mathrm{d}x = \int x^2\left(x\sqrt{x^2-1}\right)\mathrm{d}x$ $u = x^2$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x} = x\sqrt{x^2-1}$ $\frac{du}{\mathrm{d}x} = 2x \quad v = \int x\sqrt{x^2-1}\,\mathrm{d}x = \frac{1}{3}(x^2-1)^{\frac{3}{2}}$ $\int x^3\sqrt{x^2-1}\,\mathrm{d}x = \frac{1}{3}x^2(x^2-1)^{\frac{3}{2}} - \frac{1}{3}\int 2x(x^2-1)^{\frac{3}{2}}\,\mathrm{d}x$ $= \frac{1}{3}x^2(x^2-1)^{\frac{3}{2}} - \frac{1}{3}\left[\frac{(x^2-1)^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}\right] + C$ $= \frac{1}{3}x^2(x^2-1)^{\frac{3}{2}} - \frac{2}{15}(x^2-1)^{\frac{5}{2}} + C$	We choose $\frac{\mathrm{d}v}{\mathrm{d}x} = x\sqrt{x^2-1}$ because we have $\int x\sqrt{x^2-1}\,\mathrm{d}x$ in part (i).

Note: Example 11.15 illustrates the fact that the 'LIATE' rule should not be followed rigidly. For instance in (a)(ii), do you know that we cannot apply integration by parts by letting $u = x^3$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{x^2}$?

Self-Review 11.9

Find: (a) $\int x\ln x\,\mathrm{d}x$ $\quad \left[\frac{1}{4}x^2(2\ln x - 1) + C\right]$

(b) $\int \cos^{-1}2x\,\mathrm{d}x$ $\quad \left[x\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2} + C\right]$

(c) $\int x^2\mathrm{e}^x\,\mathrm{d}x$ $\quad \left[\mathrm{e}^x\left(x^2-2x+2\right)+C\right]$ (Note: You will need to do by parts twice)

11.2 The Definite Integral

Let F(x) be an anti-derivative of f(x) with respect to x i.e. $\int f(x)\,dx = F(x) + C$. The **definite integral of f(x) with respect to x from $x = a$ to $x = b$**, denoted by the symbol '$\int_a^b f(x)\,dx$', is defined to be F(b) – F(a). That is, $\int_a^b f(x)\,dx = F(b) - F(a)$, where $\int f(x)\,dx = F(x) + C$.

The values 'a' and 'b' are called the lower and upper limit of the definite integral respectively. Note that unlike the indefinite integral, the definite integral has a numerical value. We will give the precise geometrical meaning of the definite integral in chapter 12.

11.2.1 Properties of the Definite Integral

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
4. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ for any real constant k.
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a \le c \le b$

Example 11.16

Evaluate (a) $\int_0^{1/3} \frac{1}{\sqrt{1-9x^2}}\,dx$, (b) $\int_1^2 x^2 \ln x\,dx$

Solution:

(a) $\int_0^{1/3} \frac{1}{\sqrt{1-9x^2}}\,dx = \int_0^{1/3} \frac{1}{\sqrt{9\left(\frac{1}{9}-x^2\right)}}dx$ $= \frac{1}{3}\int_0^{1/3} \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}dx$ $= \frac{1}{3}\left[\sin^{-1} 3x\right]_0^{1/3}$ $= \frac{1}{3}\left[\sin^{-1}(1) - \sin^{-1} 0\right]$ $= \frac{1}{3}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{6}$	Recall $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C$ Remember to factor out "9" so that the coefficient of x^2 is ± 1 When working with differentiation and integration, we need to use **radians** for angles.

(b) From Example 11.13(a) $\int x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$ $\therefore \int_1^2 x^2 \ln x \, dx = \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^2 = \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(0 - \frac{1}{9}\right)$ $= \frac{8}{3}\ln 2 - \frac{7}{9}$	

Example 11.17

Use integration by parts to find the exact value of $\int_0^1 \tan^{-1}x \, dx$.

Solution:

Let $u = \tan^{-1}x$ and $\frac{dv}{dx} = 1$ $\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}$ $\Rightarrow v = x$ $\int_0^1 \tan^{-1}x \, dx = [x\tan^{-1}x]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$ $= \tan^{-1}1 - \frac{1}{2}[\ln(1+x^2)]_0^1$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2$	For definite integral $\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$ Alternatively, we can find the indefinite integral (i.e. $\int \tan^{-1}x \, dx$) first and then find the value of the definite integral like Example 11.16(b)

11.2.2 Evaluating the Definite Integral Using Substitution

When evaluating a definite integral using substitution, extra care must be taken to ensure that the limits of integration are changed to the corresponding values of the new variable.

Example 11.18

Use the substitution $u = x + 4$ to evaluate $\int_0^5 \frac{x}{\sqrt{x+4}} dx$ exactly.

Solution:

$u = x + 4 \Rightarrow \frac{du}{dx} = 1 \Rightarrow dx = du$ $x = 0 \Rightarrow u = 4$ and $x = 5 \Rightarrow u = 9$ $\int_0^5 \frac{x}{\sqrt{x+4}} dx = \int_4^9 \frac{u-4}{\sqrt{u}} du$ $= \int_4^9 \left(u^{1/2} - 4u^{-1/2}\right) du$ $= \left[\frac{2}{3}u^{3/2} - 8u^{1/2}\right]_4^9 = \frac{14}{3}$	The word "exactly" means that we cannot use GC to evaluate the integral. However, we can still use it to check our answer. Remember to change limits of integration. Note that the new limits for the new integral **must correspond** to those of the original integral and there is **no need** to change back to in terms of x. (Why?)

Example 11.19

Evaluate $\int_0^1 \sqrt{1-x^2}\ \mathrm{d}x$ using the substitution $x=\cos\theta$.

Solution:

$x=\cos\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=-\sin\theta$ $x=0 \Rightarrow \theta=\dfrac{\pi}{2}$ and $x=1 \Rightarrow \theta=0$	Change limits of integration. **Note:** Since the substitution is one-one, we take principal angles of the trigonometric function (pg 3 of MF26).
$\int_0^1 \sqrt{1-x^2}\ \mathrm{d}x=\int_{\frac{\pi}{2}}^{0} \sqrt{\left(1-\cos^2\theta\right)}\left(-\sin\theta\right)\ \mathrm{d}\theta$ $=-\int_{\frac{\pi}{2}}^{0} \sin^2\theta\ \mathrm{d}\theta$	Note that the new limits for the new integral must **correspond** to those of the original integral.
$=\int_0^{\frac{\pi}{2}} \sin^2\theta\ \mathrm{d}\theta$	Use $\int_a^b \mathrm{f}(x)\mathrm{d}x=-\int_b^a \mathrm{f}(x)\mathrm{d}x$.
$=\int_0^{\frac{\pi}{2}} \dfrac{1-\cos 2\theta}{2}\ \mathrm{d}\theta$ $=\dfrac{1}{2}\left[\theta-\dfrac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$ $=\dfrac{1}{2}\left(\dfrac{\pi}{2}-0-(0-0)\right)=\dfrac{\pi}{4}$	Use double angle formula $\cos 2\theta=1-2\sin^2\theta$.

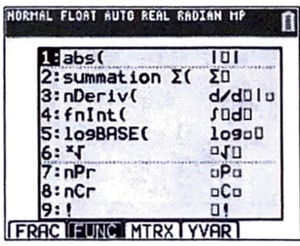
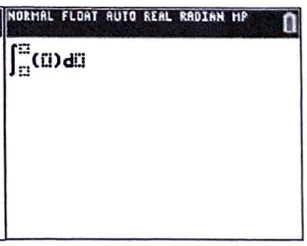
11.2.3 Evaluating the Definite Integral Using the Graphic Calculator

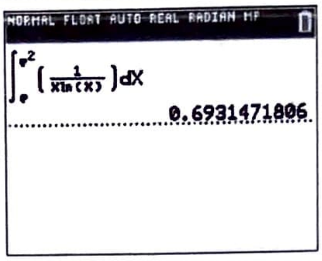
The GC can be used to evaluate a definite integral but if the question requires an exact answer, then the analytical (non-calculator) method is implied.

Example 11.20

Evaluate $\int_{\mathrm{e}}^{\mathrm{e}^2} \dfrac{1}{x\ln x}\mathrm{d}x$ to 3 significant figures.

Solution:

Step 1: Press ALPHA WINDOW and Select 4:fnInt(		

Step 2: Key in the lower and upper limits and the function to be integrated. Press ENTER and the definite integral can be evaluated.	
Hence, $\int_{e}^{e^2} \frac{1}{x \ln x} \mathrm{d}x = 0.693$ (to 3 s.f.)	**Note:** You can also press MATH 9 to call out the same command.

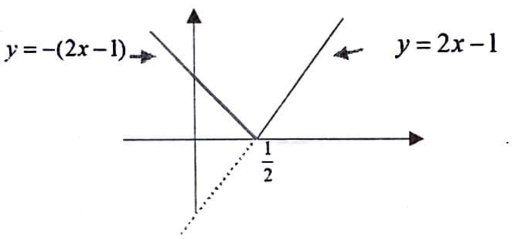
11.3 Miscellaneous Examples

Example 11.21

Without using the graphing calculator, evaluate the following integrals:

(a) $\int_{-1}^{1} x\|2x-1\| \, \mathrm{d}x$ (b) $\int_{-1}^{1} \mathrm{e}^{\|2x-1\|} \, \mathrm{d}x$ (c) $\int_{0}^{\frac{\pi}{2}} \cos(\|2x-1\|) \, \mathrm{d}x$

Solution:

 (a) $\int_{-1}^{1} x\|2x-1\| \, \mathrm{d}x = \int_{-1}^{\frac{1}{2}} x[-(2x-1)] \, dx + \int_{\frac{1}{2}}^{1} x(2x-1) \, dx$ $= -\left[\frac{2}{3}x^3 - \frac{1}{2}x^2\right]_{-1}^{\frac{1}{2}} + \left[\frac{2}{3}x^3 - \frac{1}{2}x^2\right]_{\frac{1}{2}}^{1}$ $= -\frac{11}{12}$	You should never integrate with a modulus function. To remove the modulus, draw the function without the modulus, i.e. $y=2x-1$ The portion of the graph below the x-axis will be reflected about the x-axis. The equation of this portion is $y=-(2x-1)$. The portion of the graph that is not reflected has the equation $y=(2x-1)$. You will "split" the integral with the appropriate limits. (We "split" the integral at $x=\frac{1}{2}$)
(b) $\int_{-1}^{1} \mathrm{e}^{\|2x-1\|} \, \mathrm{d}x = \int_{-1}^{\frac{1}{2}} \mathrm{e}^{-(2x-1)} \, \mathrm{d}x + \int_{\frac{1}{2}}^{1} \mathrm{e}^{2x-1} \, \mathrm{d}x$ $= -\frac{1}{2}\left[\mathrm{e}^{-2x+1}\right]_{-1}^{\frac{1}{2}} + \frac{1}{2}\left[\mathrm{e}^{2x-1}\right]_{\frac{1}{2}}^{1} = \frac{1}{2}\left(\mathrm{e}^3 + \mathrm{e} - 2\right)$	Split integral at $x=\frac{1}{2}$

(c) $\int_0^{\frac{\pi}{2}} \cos(\lvert 2x-1\rvert)\,dx$ $=\int_0^{\frac{1}{2}} \cos[-(2x-1)]\,dx+\int_{\frac{1}{2}}^{\frac{\pi}{2}} \cos(2x-1)\,dx$ $=-\frac{1}{2}\left[\sin(-2x+1)\right]_0^{\frac{1}{2}}+\frac{1}{2}\left[\sin(2x-1)\right]_{\frac{1}{2}}^{\frac{\pi}{2}}$ $=-\frac{1}{2}[-\sin 1]+\frac{1}{2}\sin(\pi-1)$ $=\frac{1}{2}[\sin 1+\sin\pi\cos 1-\cos\pi\sin 1]=\sin 1$	

Self-Review 11.10

Evaluate (a) $\int_1^3 (x-1)^3\,dx$ [4]

(b) $\int_4^9 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$ using the substitution $u=1+\sqrt{x}$ $[3+2\ln\frac{4}{3}]$

(c) $\int_{-2}^3 \lvert x^2-1\rvert\,dx$ $[\frac{28}{3}]$ [Check all your answers using the GC]

Example 11.22

(a) Find $\int_0^n \frac{1}{1+4x^2}\,dx$ in terms of n. Deduce the exact value of $\int_0^\infty \frac{1}{1+4x^2}\,dx$.

[You may assume that $\tan^{-1}x \to \frac{\pi}{2}$ as $x\to\infty$. Can you see why?]

(b) Given that n is a positive integer, find $\int_0^\pi x\cos nx\,dx$ in terms of n in the simplest form.

Solution:

(a) $\int_0^n \frac{1}{1+4x^2}\,dx=\frac{1}{4}\int_0^n \frac{1}{\frac{1}{4}+x^2}\,dx=\frac{1}{4}\int_0^n \frac{1}{\left(\frac{1}{2}\right)^2+x^2}\,dx$ $=\frac{1}{4}\times\frac{1}{1/2}\left[\tan^{-1}(2x)\right]_0^n=\frac{1}{2}\tan^{-1}(2n)$ $\int_0^\infty \frac{1}{1+4x^2}\,dx$ $=\lim_{n\to\infty}\int_0^n \frac{1}{1+4x^2}\,dx=\lim_{n\to\infty}\left[\frac{1}{2}\tan^{-1}(2n)\right]$ $=\frac{1}{2}\left(\frac{\pi}{2}\right)=\frac{\pi}{4}$	

(b) Using integration by parts, $\int_0^{\pi} x\cos nx \, dx = \left[\frac{x}{n}\sin nx\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx \, dx$ $= \frac{\pi}{n}\sin n\pi - \frac{1}{n^2}\left[-\cos nx\right]_0^{\pi}$ $= \frac{1}{n^2}(\cos n\pi - 1)$ $= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{2}{n^2} & \text{if } n \text{ is odd} \end{cases}$	$u = x$ and $\frac{dv}{dx} = \cos nx$ $\frac{du}{dx} = 1$ and $v = \frac{1}{n}\sin nx$ $\cos 2\pi = \cos 4\pi$ $= \cos 6\pi = \ldots. = 1$ ie $\cos(\text{even no.} \times \pi) = 1$ $\cos \pi = \cos 3\pi$ $= \cos 5\pi = \ldots. = -1$ ie $\cos(\text{odd no.} \times \pi) = -1$

Example 11.23 (MC N98 / I / 18 modified)

(a) Find the positive integer a if $\int_{-2}^{a} \frac{1}{4+x^2} \, dx = \int_0^{\frac{\pi}{2}} \cos^2 ax \, dx$.

(b) By considering $\frac{d}{dx}\left[\frac{1}{3}\sin\left(x^3\right)\right]$ or otherwise, show using integration by parts, that

$$\int x^5\cos\left(x^3\right)dx = \frac{1}{3}x^3\sin\left(x^3\right) - \int x^2\sin\left(x^3\right)dx.$$

Find $\int x^2\sin\left(x^3\right)dx$ and hence obtain the exact value of $\int_0^{\sqrt[3]{\pi}} x^5\cos\left(x^3\right)dx$.

Solution:

(a) LHS $= \int_{-2}^{a} \frac{1}{4+x^2} \, dx = \int_{-2}^{a} \frac{1}{2^2+x^2} \, dx = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{a}$ $= \frac{1}{2}\left[\tan^{-1}\left(\frac{a}{2}\right) - \tan^{-1}(-1)\right]$ $= \frac{1}{2}\left[\tan^{-1}\left(\frac{a}{2}\right) - \left(-\frac{\pi}{4}\right)\right]$ $= \frac{1}{2}\tan^{-1}\left(\frac{a}{2}\right) + \frac{\pi}{8}$	
RHS $= \int_0^{\pi/2} \cos^2 ax \, dx = \frac{1}{2}\int_0^{\pi/2} (\cos 2ax + 1) \, dx$ $= \frac{1}{2}\left[\frac{1}{2a}\sin 2ax + x\right]_0^{\pi/2} = \frac{1}{2}\left[\frac{1}{2a}\sin a\pi + \frac{\pi}{2}\right] = \frac{\pi}{4}$	Use double angle formula $\cos 2ax = 2\cos^2 ax - 1$ **Q:** Why is $\sin a\pi = 0$?
Hence $\frac{1}{2}\tan^{-1}\left(\frac{a}{2}\right) + \frac{\pi}{8} = \frac{\pi}{4}$ $\Rightarrow \tan^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{4} \Rightarrow \frac{a}{2} = 1 \Rightarrow a = 2$	

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{3}\sin\left(x^3\right)\right]=\dfrac{1}{3}\left(3x^2\right)\cos\left(x^3\right)=x^2\cos\left(x^3\right)$	Rewrite the result as $\int x^2\cos\left(x^3\right)\mathrm{d}x$ $=\dfrac{1}{3}\sin\left(x^3\right)+C$
Rewrite the integral $\int x^5\cos\left(x^3\right)\,\mathrm{d}x=\int x^3\cdot x^2\cos\left(x^3\right)\,\mathrm{d}x$ Let $u=x^3$ $\qquad \dfrac{\mathrm{d}v}{\mathrm{d}x}=x^2\cos\left(x^3\right)$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}=3x^2$ $\qquad v=\dfrac{1}{3}\sin\left(x^3\right)$ from the above result	Split $x^5\cos\left(x^3\right)$ into a product of x^3 and $x^2\cos\left(x^3\right)$ because we know the result of $\int x^2\cos\left(x^3\right)\mathrm{d}x$.
Using integration by parts, $\int x^5\cos\left(x^3\right)\mathrm{d}x=\dfrac{1}{3}x^3\sin\left(x^3\right)-\int 3x^2\cdot\dfrac{1}{3}\sin\left(x^3\right)\mathrm{d}x$ $=\dfrac{1}{3}x^3\sin\left(x^3\right)-\int x^2\sin\left(x^3\right)\mathrm{d}x$ $\int x^2\sin\left(x^3\right)\,\mathrm{d}x=\dfrac{1}{3}\int 3x^2\sin\left(x^3\right)\,\mathrm{d}x$ $=\dfrac{1}{3}[-\cos\left(x^3\right)+C\,]==-\dfrac{1}{3}\cos\left(x^3\right)+C$ Hence $\int_0^{\sqrt[3]{\pi}} x^5\cos\left(x^3\right)\,\mathrm{d}x=\left[\dfrac{1}{3}x^3\sin\left(x^3\right)+\dfrac{1}{3}\cos\left(x^3\right)\right]_0^{\sqrt[3]{\pi}}$ $=\dfrac{1}{3}\pi\,\sin\pi+\dfrac{1}{3}\cos\pi-\dfrac{1}{3}=-\dfrac{2}{3}$	Note that the integral $\int x^2\sin\left(x^3\right)\,\mathrm{d}x$ can be expressed in the form $\dfrac{1}{3}\int \mathrm{f}'(x)\sin\mathrm{f}(x)\,\mathrm{d}x$ $=-\dfrac{1}{3}\cos\mathrm{f}(x)+C$ where $\mathrm{f}(x)=x^3$

11.4 Summary

1. Integration is the reverse operation of differentiation. Given a function f, the objective of integration is to recover a function F such that $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{F}(x)\right]=\mathrm{f}(x)$.	
2. A definite integral is an integral which has limits of integration while an indefinite integral does not have limits of integration. A definite integral produces a definite value while an indefinite integral produces a function F with an arbitrary constant C. The function F is called the anti-derivative.	
3a $\int \mathrm{f}'(x)[\mathrm{f}(x)]^n\,\mathrm{d}x=\frac{[\mathrm{f}(x)]^{n+1}}{n+1}+C,\ n\neq-1$ 3b $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x=\ln\lvert \mathrm{f}(x)\rvert+C$ 3c $\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}\mathrm{d}x=\mathrm{e}^{\mathrm{f}(x)}+C$	These three integrals are usually called standard forms. Many integrals belong to one of these three forms. When performing integration, it is a good strategy to check if the integral can be rewritten as a standard form.
4a $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\sin^{-1}\left(\frac{x}{a}\right)+C\ \left(\lvert x\rvert<a\right)$ * 4b $\int \frac{1}{a^2+x^2}\,\mathrm{d}x=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$ * 4c $\int \frac{1}{a^2-x^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert+C$ * 4d $\int \frac{1}{x^2-a^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert+C$ *	These four standard integrals are very important. When performing integration, it is a good strategy to check if the integral belongs to one of these types. Note that the coefficient of x^2 in the integrand is ±1. * formulae given in MF26
5. The following trigonometric identities are very useful in integrating trigonometric functions: • $\sin 2x=2\sin x\cos x$ * • $\cos 2x=\cos^2 x-\sin^2 x$ $=2\cos^2 x-1$ * $=1-2\sin^2 x$ • $1+\tan^2 x=\sec^2 x$ • $1+\cot^2 x=\operatorname{cosec}^2 x$	$\cos 2x=2\cos^2 x-1\ =1-2\sin^2 x$ The above identities are particularly useful in finding integrals of the type $\int \sin^2 kx\ \mathrm{d}x, \int \cos^2 kx\ \mathrm{d}x$ * formulae given in MF26
6. In dealing with integrals of the type $\int \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\,\mathrm{d}x$ where $\mathrm{g}(x)$ is a quadratic function, it is often necessary to complete the square for $\mathrm{g}(x)$ and apply integrals '3b' and '4b' or '4c' or '4d' above.	Express $\int \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\,\mathrm{d}x$ as follows: $\int \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\,\mathrm{d}x=A\int \frac{\mathrm{g}'(x)}{\mathrm{g}(x)}\,\mathrm{d}x+B\int \frac{1}{\mathrm{g}(x)}\,\mathrm{d}x$ where A and B are real constants. Then use '3b' for the first integral and '4b' for the second integral.

7. In dealing with integrals of the type $\int \frac{f(x)}{\sqrt{g(x)}}\,dx$ where $g(x)$ is a quadratic function which cannot be factorised into real factors, it is often necessary to complete the square for $g(x)$ and apply integrals '3a' and '4a' above.	Express $\int \frac{f(x)}{\sqrt{g(x)}}\,dx$ as follows: $\int \frac{f(x)}{\sqrt{g(x)}}\,dx$ $= A\int g'(x)\left[g(x)\right]^{-\frac{1}{2}}\,dx + B\int \frac{1}{\sqrt{g(x)}}\,dx$ where A and B are real constants. Then use '3a' for the first integral and '4a' for the second integral.
8. For rational fraction, if the denominator is factorisable in linear or quadratic factors, we can find its partial fractions before integrating.	
9. The technique of substitution transforms the original integral in 'x' into a new integral in the new variable 'u'. Remember to change the original limits of the integral to the corresponding limits in the new variable.	Use the replacement '$dx \equiv \frac{dx}{du}du$'.
10. The integration by parts formula states: $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$	This technique is usually applied to integrals consisting of a product of two different functions, for example, $\int x\sin x\,dx, \int x^2\ln x\,dx$. The choice of '$u$' is guided by the 'LIATE' rule.

Appendix A

Table of Integrals

1. $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C \qquad (n \neq -1)$

2. $\int \dfrac{1}{ax+b} \, dx = \dfrac{1}{a}\ln|ax+b| + C$

3. $\int e^{ax+b} \, dx = \dfrac{1}{a}e^{ax+b} + C$

4. $\int f'(x)[f(x)]^n \, dx = \dfrac{[f(x)]^{n+1}}{n+1} + C$

5. $\int \dfrac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

6. $\int f'(x)e^{f(x)} \, dx = e^{f(x)} + C$

7. $\int \sin x \, dx = -\cos x + C$

8. $\int \cos x \, dx = \sin x + C$

9. $\int \tan x \, dx = \ln(\sec x) + C \qquad \left(|x| < \frac{1}{2}\pi\right)$ *

10. $\int \cot x \, dx = \ln(\sin x) + C \qquad (0 < x < \pi)$ *

11. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$ *

12. $\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + C$ *

13. $\int \sec^2 x \, dx = \tan x + C$

14. $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

15. $\int \dfrac{1}{\sqrt{(a^2-x^2)}} \, dx = \sin^{-1}\left(\dfrac{x}{a}\right) + C \qquad (|x| < a)$ *

16. $\int \dfrac{1}{a^2+x^2} \, dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C \qquad (x \in \mathbb{R})$ *

17. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right) + C \qquad (|x| < a)$ *

18. $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\ln\left(\dfrac{x-a}{x+a}\right) + C \qquad (|x| > a)$ *

* formulae given in MF 26

Chapter 12

Applications of Integration

In chapter 11, we introduced the definite integral. In this chapter, we conceptualize the definite integral as a limit of a sum and geometrically as the area under a curve and volume of a solid obtained when a region under a curve is rotated about the axes.

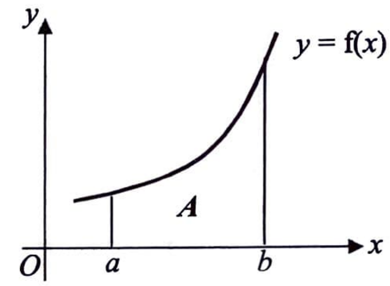
12.1 Area of Region

12.1.1 Area under a curve

Area above the x-axis

If a region A is completely above the x-axis, i.e. $f(x) \geq 0$ for $a \leq x \leq b$, then the integral $\int_a^b y \, dx > 0$.

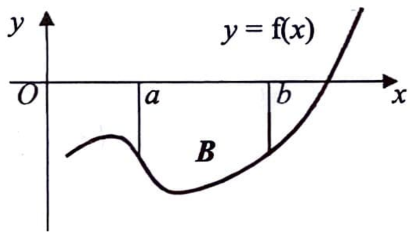
$$\boxed{\text{Area of the region } A = \int_a^b y \, dx}$$



Area below the x-axis

If a region B is completely below the x-axis, i.e. $f(x) \leq 0$ for $a \leq x \leq b$ then the integral $\int_a^b y \, dx < 0$. Since area is always positive,

$$\boxed{\text{Area of region } B = -\int_a^b y \, dx}$$

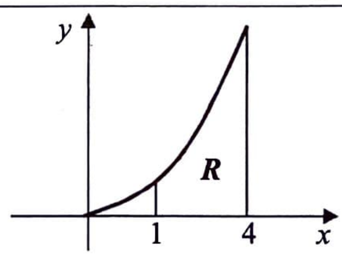


Note that the negative sign is necessary since the integral is negative.

Example 1

Find the exact area of the region bounded by the graph of $y = x^2$, the lines $x = 1$, $x = 4$ and the x-axis.

Solution:

The area of region $R = \int_a^b y \, dx$  $= \int_1^4 y \, dx = \int_1^4 x^2 \, dx$ $= \left[\frac{x^3}{3}\right]_1^4 = \frac{64}{3} - \frac{1}{3} = 21$ units	Always draw a sketch of the curve to check whether the required region is above or below the x-axis. Note that $\int_1^4 y \, dx > 0$ since R is above the x-axis.

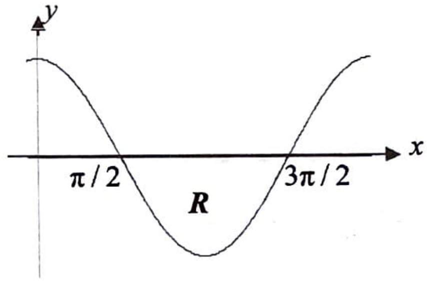
Self-Review 1

Show that the exact area under the curve $y = \dfrac{1}{1-x}$ between the lines $x = -3$ and $x = \dfrac{1}{2}$ is $\ln 8$ units².

Example 2

(i) Find the exact value of $\displaystyle\int_{\pi/2}^{3\pi/2} \cos x \, dx$.

(ii) The figure below shows a curve given by the equation $y = \cos x$ for values of x between 0 and 2π. Find the exact area of the region R enclosed by the curve and the x-axis between $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$.



Solution:

(i) $\displaystyle\int_{\pi/2}^{3\pi/2} \cos x \, dx = [\sin x]_{\pi/2}^{3\pi/2}$ $= \sin\frac{3\pi}{2} - \sin\frac{\pi}{2} = -1 - 1 = -2$ (ii) The area of region $R = -\displaystyle\int_{\pi/2}^{3\pi/2} \cos x \, dx = -(-2) = 2 \text{ unit}^2$	Note that $\displaystyle\int_{\pi/2}^{3\pi/2} \cos x \, dx < 0$ since R is below the x-axis.

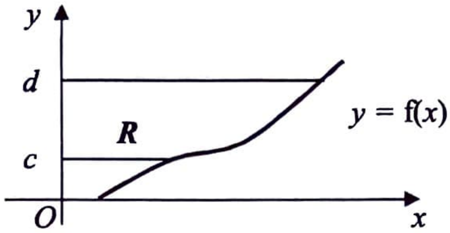
Self-Review 2

Show that the exact area under the curve $y = \dfrac{1}{1-x}$ between the lines $x = 2$ and $x = 4$ is $\ln 3$ units².

12.1.2 Area of the Region Bounded by a Curve $y = f(x)$, the Horizontal Lines $y = c$ and $y = d$ and the y-axis

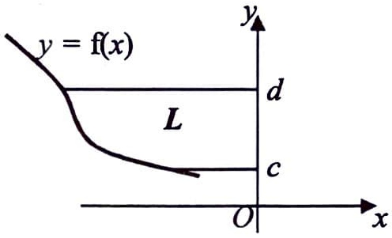
If a region $\boldsymbol{R}$ is completely to the right of the y-axis, then the integral $\int_c^d x \, dy > 0$.

$$\text{Area of the region } \boldsymbol{R} = \int_c^d x \, dy$$



If a region $\boldsymbol{L}$ is completely to the left of the y-axis, then the integral $\int_c^d x \, dy < 0$.

$$\text{Area of the region } \boldsymbol{L} = -\int_c^d x \, dy$$

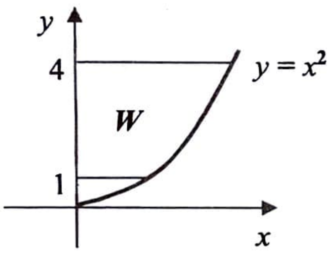


Again the negative sign is necessary since the integral is negative.

Example 3

Find the exact area of the region bounded by the graph of $y = x^2$ where $x \geq 0$, the lines $y = 1$, $y = 4$ and the y-axis.

Solution:



$y = x^2 \Rightarrow x = \pm\sqrt{y}$ (since $x \geq 0$, $\therefore x = \sqrt{y}$)

The area of the region W

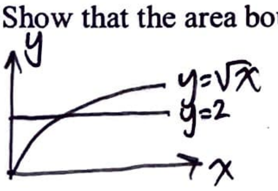
$= \int_1^4 x \, dy = \int_1^4 \sqrt{y} \, dy = \left[\frac{y^{3/2}}{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3}$ units^2

You should always sketch the graph to determine the appropriate expression to use when finding area under the graph.

What is exact area of the region bounded by the graph of $y = x^2$ where $x \leq 0$, the lines $y = 1$, $y = 4$ and the y-axis?

Self-Review 3

Show that the area bounded by the graph of $y = \sqrt{x}$, the line $y = 2$ and the y-axis is $\frac{8}{3}$ units^2.

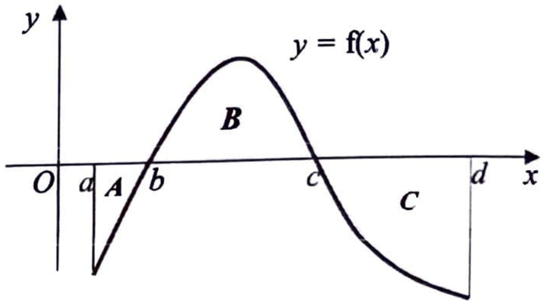


area $= \int_0^2 y^2 \, dy$

12.1.3 Area of Regions on both sides of the x-axis or y-axis

Suppose we want to calculate the total area of two or more regions which lies on either side of the x-axis.
If we compute the total area (sum of areas of regions ***A***, ***B*** and ***C***)
by simply writing $\int_a^d y\ \mathrm{d}x$, our answer will be **wrong**
as $\int_a^d y\ \mathrm{d}x$ = Area of A + Area of B + Area of C

$$= \underbrace{\int_a^b y\ \mathrm{d}x}_{\text{-ve}} + \int_b^c y\ \mathrm{d}x + \underbrace{\int_c^d y\ \mathrm{d}x}_{\text{-ve}}.$$



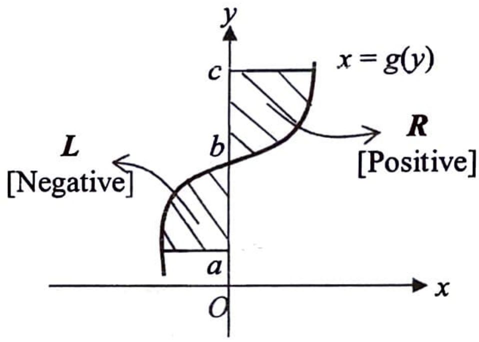
Hence, the **total area is given by**

$$-\int_a^b y\ \mathrm{d}x + \int_b^c y\ \mathrm{d}x + \left(-\int_c^d y\ \mathrm{d}x\right).$$

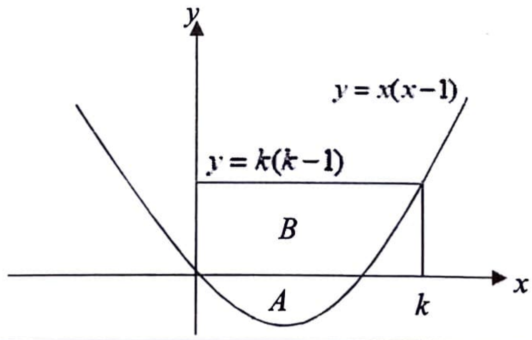
Note: It is important to sketch the graph of $y = \mathrm{f}(x)$ when finding the area under the curve.
Similarly, for the shaded region bounded by the curve $x = \mathrm{g}(y)$, the y-axis, and the lines $y = a$ and $y = c$,

total area = Area of L + Area of R

$$= -\int_a^b x\ \mathrm{d}y + \int_b^c x\ \mathrm{d}y.$$



Example 4
A curve C is given by the cartesian equation $y = x(x-1)$.
(i) Find the exact area of the region enclosed by the curve C, the x-axis and the lines $x = -1$ and $x = 3$.
(ii) The area of region B bounded by the curve, the positive x-axis, y-axis and the line $y = k\,(k-1)$ is 5 times the area of region A bounded by the curve and the x-axis. Find the value of k.



Solution:

i) Required area $= \boxed{\int_{-1}^{0}(x^2-x)\,dx - \int_0^1 (x^2-x)\,dx + \int_1^3 (x^2-x)\,dx}$

$$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-1}^{0} - \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{0}^{1} + \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{1}^{3}$$

$$= \frac{15}{6} - \left(-\frac{1}{6}\right) + \frac{28}{6} = \cancel{5}\ \frac{17}{3} \text{ units}^2$$

Note that the curve cuts the x-axis at $x = 1$.

(ii) Area of region B = $\boxed{k^2(k-1) - \int_1^k (x^2 - x)\,dx}$

Given: Area of region $B = 5 \times$ Area of region A

$$k^2(k-1) - \int_1^k (x^2 - x)\,\mathrm{d}x = 5 \times \left(-\int_0^1 (x^2 - x)\,\mathrm{d}x\right)$$

$$k^2(k-1) - \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^k = 5\left(\frac{1}{6}\right)$$

$$k^2(k-1) - \left(\frac{1}{3}k^3 - \frac{1}{2}k^2 - \frac{1}{3} + \frac{1}{2}\right) = \frac{5}{6}$$

$$\Rightarrow \frac{2}{3}k^3 - \frac{1}{2}k^2 - 1 = 0$$

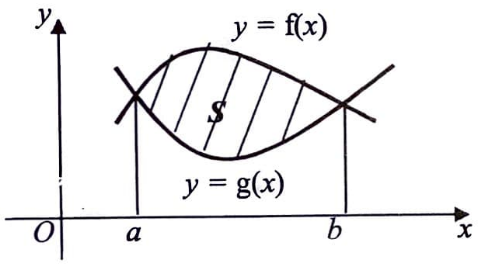
Using GC, $k = 1.46$ (3 s.f.)

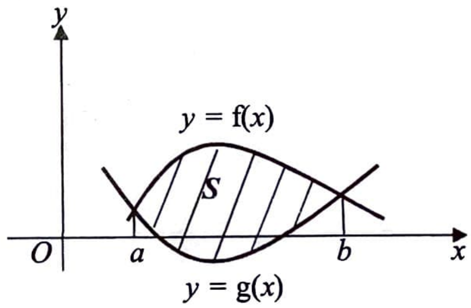
Clearly, the area of region B is the area of the rectangle minus the area under the curve from $x = 1$ to $x = k$.

Self-Review 4

Find the area enclosed by the curve $y = x^3 - x^2 - 2x$ and the x-axis. $[\frac{37}{12}$ units$^2]$

12.1.4 Area of a Region Bounded between Two Curves





For both figures above,
Area of region S = Area under the curve $y = \mathrm{f}(x)$ – Area under the curve $y = \mathrm{g}(x)$

$$= \int_a^b \mathrm{f}(x)\,\mathrm{d}x - \int_a^b \mathrm{g}(x)\,\mathrm{d}x$$

$$= \int_a^b [\mathrm{f}(x) - \mathrm{g}(x)]\,\mathrm{d}x, \quad \text{where } \mathrm{f}(x) \geq \mathrm{g}(x) \text{ for } a \leq x \leq b.$$

Note: 1. Always use "upper" curve – "lower" curve to find the area between two curves.
2. Even if some portion of the region S is below the x–axis, the area of the region S is still $\int_a^b [\mathrm{f}(x) - \mathrm{g}(x)]\,\mathrm{d}x$.

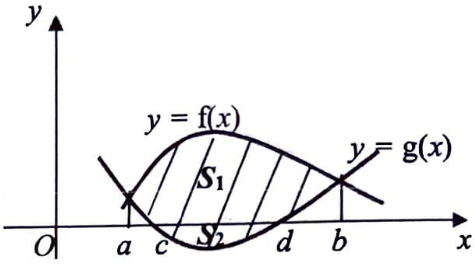
Independent reading

Let S_1 be the region above x–axis and S_2 be the region below x–axis.

Area of region S_1

$$= \int_a^b \mathrm{f}(x)\,\mathrm{d}x - \int_a^c \mathrm{g}(x)\,\mathrm{d}x - \int_d^b \mathrm{g}(x)\,\mathrm{d}x$$

Area of region $S_2 = -\int_c^d \mathrm{g}(x)\,\mathrm{d}x$



Area of region S

= Area of region S_1 + Area of region S_2

$$= \left[\int_a^b \mathrm{f}(x)\,\mathrm{d}x - \int_a^c \mathrm{g}(x)\,\mathrm{d}x - \int_d^b \mathrm{g}(x)\,\mathrm{d}x\right] + \left(-\int_c^d \mathrm{g}(x)\,\mathrm{d}x\right)$$

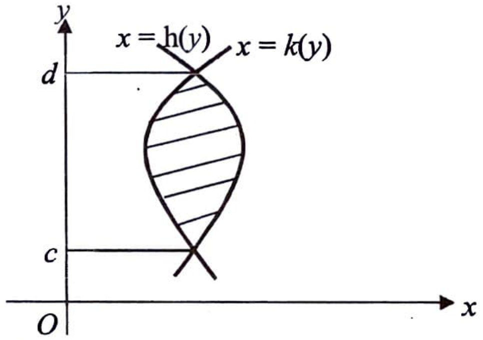
$$= \int_a^b \mathrm{f}(x)\,\mathrm{d}x - \left(\int_a^c \mathrm{g}(x)\,\mathrm{d}x + \int_c^d \mathrm{g}(x)\,\mathrm{d}x + \int_d^b \mathrm{g}(x)\,\mathrm{d}x\right)$$

$$= \int_a^b \mathrm{f}(x)\,\mathrm{d}x - \int_a^b \mathrm{g}(x)\,\mathrm{d}x$$

$$= \int_a^b [\mathrm{f}(x) - \mathrm{g}(x)]\,\mathrm{d}x$$

If we wish to obtain the area of the region as shown on the right, then it is not feasible to perform the integration with respect to x. Instead, we integrate with respect to y as follows:

Shaded area $= \int_c^d [\mathrm{h}(y) - \mathrm{k}(y)]\,\mathrm{d}y$

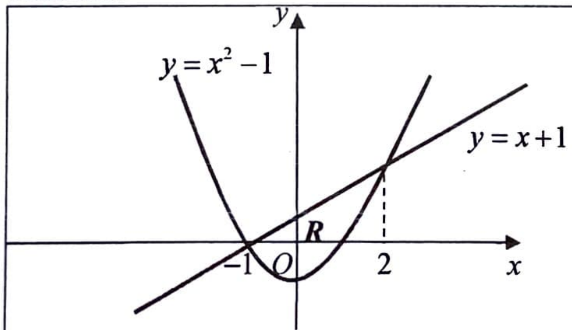


Note: Always use 'right' curve – 'left' curve to find the area between two curves

Example 5

Find the exact area of the region bounded by the curve $y = x^2 - 1$ and the straight line $y = x + 1$.

Solution:



For intersections of the two curves,

Let $x^2 - 1 = x + 1$

$\Rightarrow\ x^2 - x - 2 = 0 \ \Rightarrow\ (x+1)(x-2) = 0$

$\Rightarrow\ x = -1$ or $x = 2$

Area of the region R = $\int_{-1}^{2} [(x+1) - (x^2-1))]\,dx$

Key steps to find area as a definite integral:

1) Draw diagram and identify correct region
2) Find points of intersection if any
3) Write down the correct definite integral (is it a sum of areas, subtraction or a combination of both?)

$$= \int_{-1}^{2} x - x^2 + 2\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2}$$

$$= \left[2 - \frac{8}{3} + 4\right] - \left[\frac{1}{2} - (-\frac{1}{3}) - 2\right] = 4\frac{1}{2} \text{ units}^2$$

Alternative Solution

Consider region R as two parts. One portion, R_1, is above the x-axis and the other portion, R_2, is below the x-axis.

Area of region $R_1 = \int_{-1}^{2} (x+1)\,dx - \int_{1}^{2} (x^2 - 1)\,dx$

$\int_{-1}^{2}(x+1)\,dx$ is the area of the triangle bounded by $y = x + 1$, the x-axis and $x = 2$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^{2} - \left[\frac{x^3}{3} - x\right]_{1}^{2}$$

$$= \left[4 - (-\frac{1}{2})\right] - \left[\frac{2}{3} - (-\frac{2}{3})\right] = 4\frac{1}{2} - \frac{4}{3} \text{ units}^2$$

Area of region R_2

$$= -\int_{-1}^{1} (x^2 - 1)\,dx = -\left[\frac{x^3}{3} - x\right]_{-1}^{1} = -\left[\left(\frac{1}{3} - 1\right) - \left(-\frac{1}{3} + 1\right)\right]$$

$$= \frac{4}{3} \text{ units}^2$$

Area of the region R = Area of region R_1 + Area of region R_2

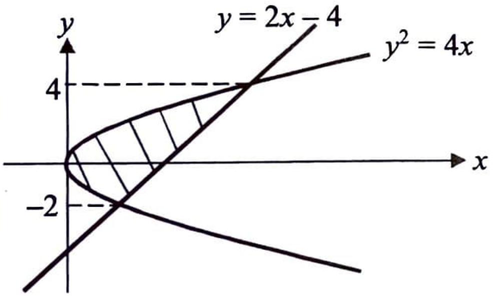
$$= \left(4\frac{1}{2} - \frac{4}{3}\right) + \frac{4}{3} = 4\frac{1}{2} \text{ units}^2$$

Which is the easier method?

Example 6

Find the exact area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

Solution:



For the points of intersection,

$(2x - 4)^2 = 4x \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x - 1)(x - 4) = 0 \Rightarrow x = 1$ or $x = 4$

When $x = 1$, $y = 2(1) - 4 = -2$

$x = 4$, $y = 2(4) - 4 = 4$

$y^2 = 4x \Rightarrow x = \frac{1}{4}y^2$ and $y = 2x - 4 \Rightarrow x = \frac{1}{2}(y + 4)$.

$\therefore$ Shaded area

$= \int_{-2}^{4} \cancel{(2x-4)} \; \frac{1}{2}(y+4) - \frac{1}{4}y^2 \; dy$

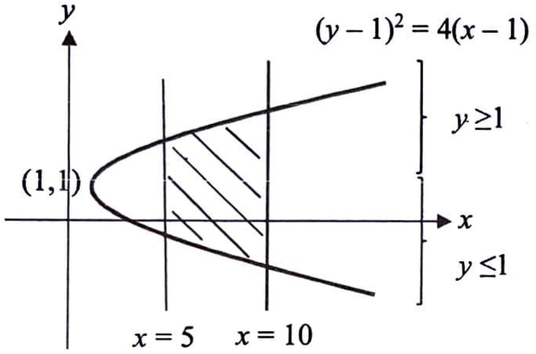
$$=\left[\frac{1}{4}y^2+2y-\frac{1}{12}y^3\right]_{-2}^{4}$$

$$=\left[\frac{1}{4}(4)^2+2(4)-\frac{1}{12}(4)^3\right]-\left[\frac{1}{4}(-2)^2+2(-2)-\frac{1}{12}(-2)^3\right]=9\text{ units}^2$$

Example 7

Find the exact area of the region bounded by the parabola $(y-1)^2=4(x-1)$, $x\geq 1$ and the lines $x=5$ and $x=10$.

Solution :



Note that

$(y-1)^2=4(x-1)\ \Rightarrow\ y-1=\pm\sqrt{4(x-1)}\ \Rightarrow\ y=1\pm 2\sqrt{x-1}$

Area required $=\int_5^{10}(y_1-y_2)\,\mathrm{d}x$

$$=\int_5^{10} 1+\sqrt{4(x-1)}-(1-2\sqrt{x-1})\,dx$$

$$=\int_5^{10}4(x-1)^{\frac{1}{2}}\,dx$$

$$=4\left[\frac{2}{3}(x-1)^{3/2}\right]_5^{10}=\frac{8}{3}\left[9^{\frac{3}{2}}-4^{\frac{3}{2}}\right]$$

$$=\frac{8}{7}(27-8)=\frac{152}{3}\text{ units}^2$$

We can consider that the required region is bounded by the two curves, $y_1=1+2\sqrt{x-1}$ ("upper" curve), $y_2=1-2\sqrt{x-1}$ ("lower" curve), and the lines $x=5$ and $x=10$.

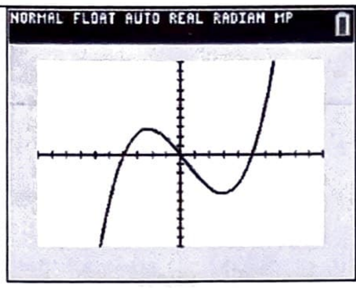
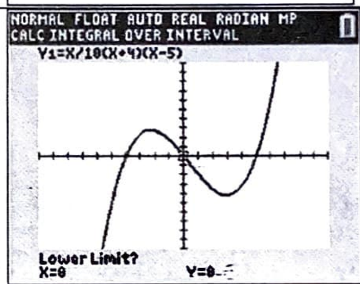
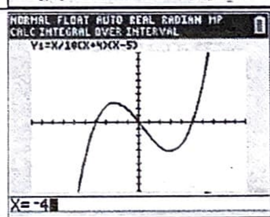
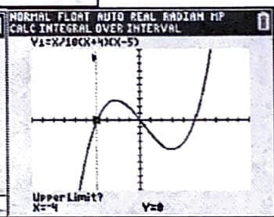
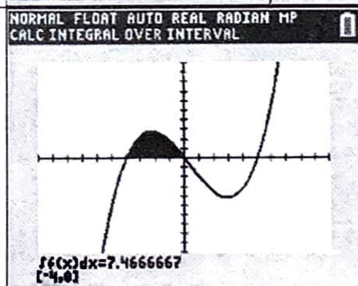
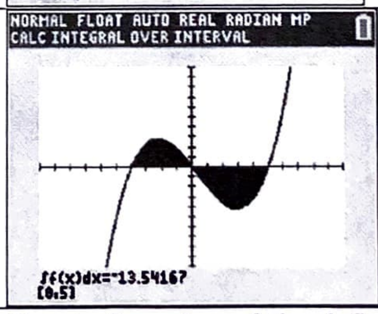
12.1.5 Using GC to Calculate Areas

We mentioned before that when finding the area bounded by a curve, it is important to sketch the graph so that we can actually 'see' the area we wish to find so as to determine the appropriate method and formulae to use. The GC can help us achieve just that by shading the required area and at the same time compute the required area. Quite often, we will have to find the points of intersection of the curve and the x-axis to determine the limits of integration.

Example 8

Sketch the graph $y=\frac{1}{10}x(x+4)(x-5)$ and shade the area bounded by the curve and the x-axis using a GC. Write down the value of the shaded area correct to 3 significant figures.

Solution:

1. Press Y= and enter **Y_1 = 0.1X(X + 4)(X – 5)** to graph out the curve.	
2. Press 2nd TRACE to call out the calculate function.	
3. Select **7:∫ *f*(*x*)dx**. You will return to the graph screen.	
4. Enter the value $x = -4$ as the lower limit and press ENTER.	 
5. Enter the value $x = 0$ for the upper limit and press ENTER to find the shaded region above the x-axis.	
6. Repeat steps 4 and 5 using lower limit 0 and upper limit 5 to find the area of the shaded region below the x-axis. Note that the value is negative since the shaded region is below the x-axis.	
7. The area bounded by the curve and the x-axis is $7.467 + 13.542 \approx \underline{21.0}$ (to 3 sig. figs.)	Negate the value of the definite integral to retrieve correct area

Note:

1. It is **wrong** to calculate the area of the shaded region by simply setting the lower limit as –4 and the upper limit as 5. i.e. $\int_{-4}^{5} \frac{1}{10}x(x+4)(x-5)\,dx = -6.075$.
2. In general, an integral of the form $\int_a^b f(x)\,dx$ does not necessarily give the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. This happens when part of the curve lies below the x-axis (see section 12.1.3).
3. Use of GC is **NOT** allowed if the question ask for the **EXACT** area.

12.1.6 Area Under a Curve Given its Parametric Equations

When the parametric equations, $x = f(t)$ and $y = g(t)$, of the curve are given, the formulae for areas in the earlier sections still hold. The evaluation is then **similar to integration by substitution** method, i.e. start off with either

$$\int_a^b y\,dx \quad \text{or} \quad \int_c^d x\,dy,$$

depending on the required area. Using the parametric equations to do substitution, it is either

$$\int_a^b y\,dx = \int_{t_1}^{t_2} g(t)\,f'(t)\,dt \quad \text{or} \quad \int_c^d x\,dy = \int_{t_1}^{t_2} f(t)\,g'(t)\,dt$$

Note: You **need** to change the limits and the integrand.

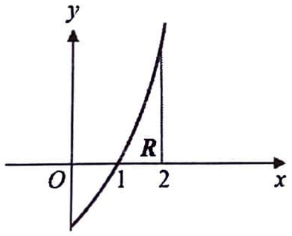
Replace dx by $f'(t)\,dt$ for $\int_a^b y\,dx$. (Since $x = f(t) \Rightarrow dx = f'(t)\,dt$).

Replace dy by $g'(t)\,dt$ for $\int_c^d x\,dy$. (Since $y = g(t) \Rightarrow dy = g'(t)\,dt$)

Example 9

The curve C is defined parametrically by $x = \sqrt{t}$, $y = t^2 - 1$ where $t \ge 0$. The region bounded by the curve C, the x-axis, $x = 1$ and $x = 2$ is denoted by R. Find the area of the region R.

Solution:



When $x = 1$, $t = \boxed{1}$

$x = 2$, $t = \boxed{4}$

$x = \sqrt{t}$ and $y = t^2 - 1$

$dx = \frac{1}{2\sqrt{t}}\,dt$

The required area $= \int_1^2 y\,dx$

Hence area $= \int_1^2 y\,\mathrm{d}x = \int_1^4 \left(t^2-1\right)\frac{1}{2\sqrt{t}}\,\mathrm{d}t$ $= 5.2$ units^2 by GC	We need to change $x=1$ to $t=1$, $x=2$ to $t=4$, and replace $\mathrm{d}x$ by $\frac{1}{2\sqrt{t}}\mathrm{d}t$.

Example 10 (N2016/2/3)
A curve D has parametric equations $x=t-\cos t$, $y=1-\cos t$, for $0 \le t \le 2\pi$.

(i) Sketch the graph of D. Give in exact form the coordinates of the points where D meets the x-axis, and also give in exact form the coordinates of the maximum point on the curve. [4]

(ii) Find, in terms of a, the area under D for $0 \le t \le a$, where a is a positive constant less than 2π. [3]

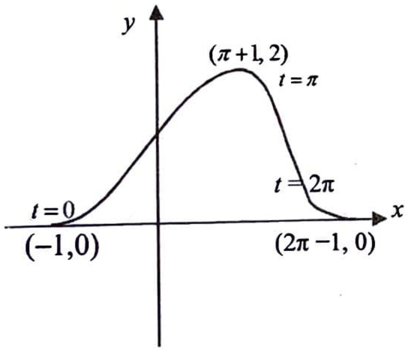
The normal to D at the point where $t=\frac{\pi}{2}$ cuts the x-axis at E and the y-axis at F.

(iii) Find the exact area of triangle OEF, where O is the origin. [4]

Solution:

(i)
When D meets the x-axis, then
$y = 1-\cos t = 0 \Rightarrow$ $\boxed{\cos t = 1,\ t \Rightarrow 0, 2\pi}$

When $t=0$, $x=$ $\boxed{x = 1-\cos t = 0-\cos 0 = -1}$
When $t=2\pi$, $x=$ $\boxed{x = 1-\cos t = 2\pi-\cos 2\pi = 2\pi-1}$
Hence D meets the x-axis at $(-1, 0)$ and $(2\pi-1, 0)$



$x = t-\cos t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1+\sin t$

$y = 1-\cos t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \sin t$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sin t}{1+\sin t}$

For maximum point, let $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin t}{1+\sin t} = 0$, then

$\sin t = 0 \Rightarrow t = 0,\ \pi,\ 2\pi$

Note that at $t=0,\ \pi,\ 2\pi$, the gradient of the curve is 0. Notice the shape of the curve at these points.

When $t=0$, $x=-1$, $y=0$.

When $t=2\pi$, $x=2\pi-1$, $y=0$

When $t=\pi$,

$x = t-\cos t = \pi - \cos\pi = \pi - (-1) = \pi + 1$

$y = 1-\cos t = 1-\cos t = 1 - \cos\pi = 1-(-1) = 2$

The coordinates of the stationary points are $(-1,0)$, $(\pi+1, 2)$ and $(2\pi-1,\ 0)$.

From the graph, we see that $(\pi+1, 2)$ is the maximum point.

(ii) When $t=0$, $x=-1$ and when $t=a$, $x=a-\cos a$.

Area under D for $0\le t\le a$

$$=\int_{-1}^{a-\cos a} y\,\mathrm{d}x$$

$$=\int_0^a (1-\cos t)(1+\sin t)\,\mathrm{d}t$$

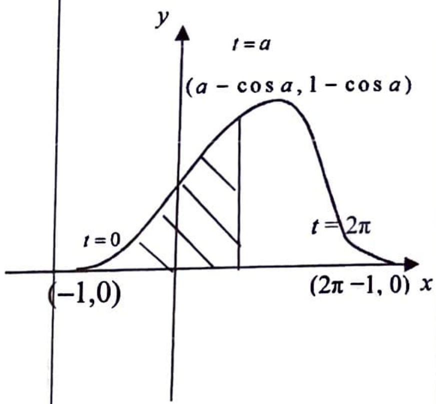
$$=\int_0^a 1-\cos t+\sin t-\sin t\cos t\,\mathrm{d}t$$

$$=\int_0^a 1-\cos t+\sin t-\tfrac{1}{2}\sin 2t\,\mathrm{d}t$$

$$=\left[t-\sin t-\cos t+\tfrac{1}{4}\cos 2t\,\mathrm{d}t\right]_0^a$$

$$=(a-\sin a-\cos a+\tfrac{1}{4}\cos 2a)-(0-0-1+\tfrac{1}{4})$$

$$=a-\sin a-\cos a+\tfrac{1}{4}\cos 2a+3/4$$



(iii) When $t=\dfrac{\pi}{2}$,

$$x_1 = t-\cos t = \frac{\pi}{2}-\cos\frac{\pi}{2} = \frac{\pi}{2}-0=\frac{\pi}{2}$$

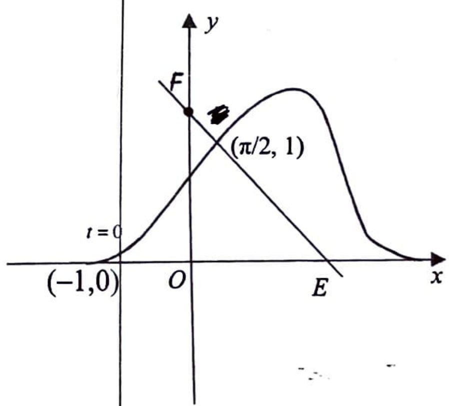
$$y_1 = 1-\cos t = 1-\cos\frac{\pi}{2}=1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\sin t}{1+\sin t}=\frac{\sin\frac{\pi}{2}}{1+\sin\frac{\pi}{2}}=\frac{1}{2},$$

Hence, gradient of normal $=-2$

The equation of normal at $t=\dfrac{\pi}{2}$ is

$$y-y_1=-2(x-x_1)\ \Rightarrow\ y-1=-2(x-\frac{\pi}{2}).$$



At point F,

$$x=0, y-1=-2(0-\frac{\pi}{2})\ \Rightarrow\ y=1-2(-\frac{\pi}{2})=1+\pi$$

At point E, $y=0$,

$$0-1=-2(x-\frac{\pi}{2})\ \Rightarrow\ \frac{1}{2}=x-\frac{\pi}{2}\ \Rightarrow\ x=\frac{1}{2}+\frac{\pi}{2}=\frac{\pi+1}{2}$$

Area of triangle OEF

$$=\frac{1}{2}\times OF\times OE=\frac{1}{2}(1+\pi)(\frac{1+\pi}{2})=\frac{(1+\pi)^2}{4}\text{ units}^2.$$

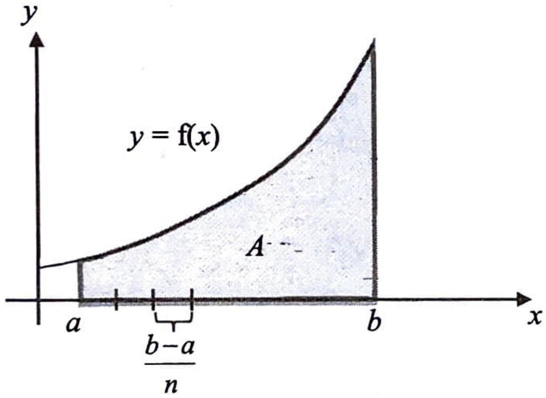
Self Review 5 (MJC Prelim 2007/1/6 modified)
A curve is defined by the parametric equations

$$x = t^3,\ y = \frac{1}{1+t^2},\ -2 \le t \le 2.$$

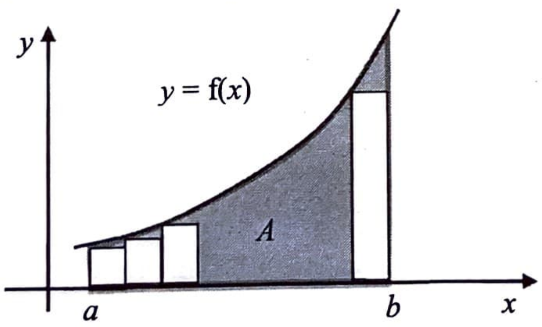
Show that the region R bounded by the axes, the curve and the line $x = a^3$, where $a > 0$ is given by $\int_0^a \frac{3t^2}{1+t^2}\,dt$. Hence find the exact area of region which is bounded by the axes, the curve and the line $x = 1$. $[3\left(1-\frac{\pi}{4}\right)]$

12.2 Area under a Curve as a Limit of a Sum of the Areas of Rectangles

Let f(x) be a function defined on a closed interval $[a, b]$.

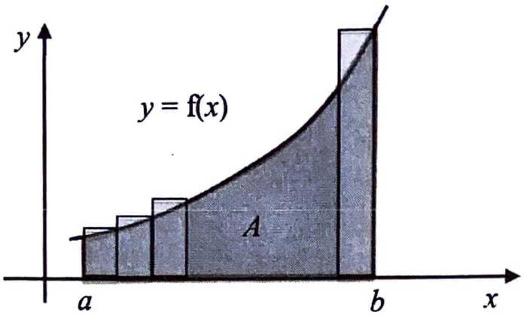


Suppose we would like to approximate the area of the shaded region A using n rectangles. To do so, we divide $[a,b]$ into n equal subintervals such that the width of each interval $= \frac{b-a}{n}$. We then construct a rectangle on each subinterval as shown where the height of each rectangle is given by the y-coordinate on the curve corresponding to the **"left"** endpoint of each subinterval. For example, the height of first rectangle is f(a).

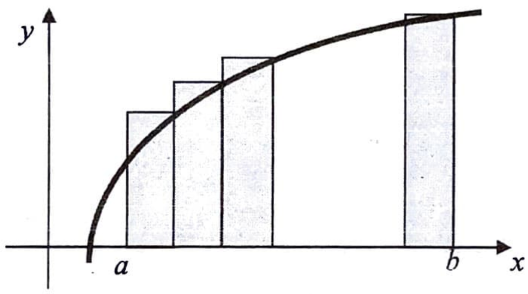


Summing up the area of all the rectangles will give us an approximation of the area under the curve.
In this case, the total area of the rectangles is an **under-estimate** of the actual area under the curve.

If the **"right"** endpoints of each subinterval are used to construct the rectangles instead, the total area of the rectangles will be an **over-estimate** of the actual area under the curve.



Note:
Whether total area of rectangles is an under-estimate or over-estimate of the actual area under the curve depends on the **shape** of the graph (e.g. increasing & concave downwards as shown in the figure) and the way the heights of the rectangles are constructed (i.e. "left" or "right" endpoints used). Therefore, you should always have a graph to visualize.

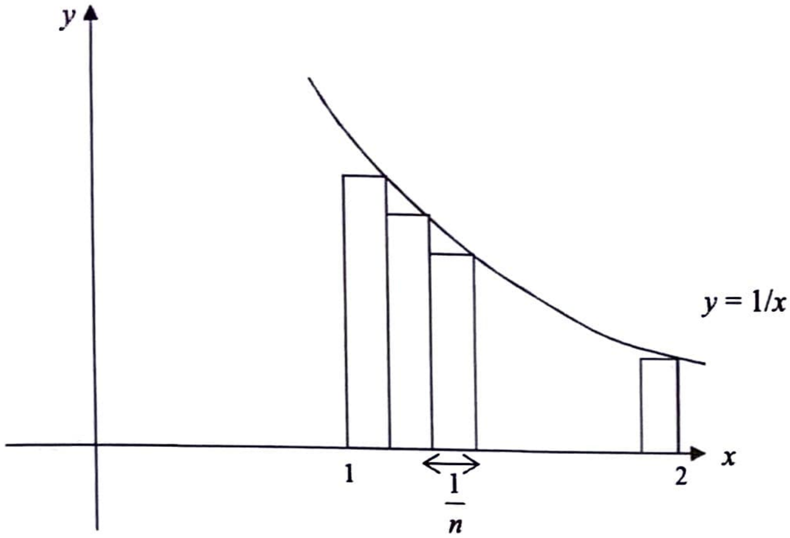


Example 11

The diagram shows the graph of $y = \frac{1}{x}$ in the interval $[1, 2]$ with n rectangles of equal width.

Show that the total area of n rectangles, $S = \sum_{r=1}^{n} \frac{1}{n+r}$.

Deduce the exact value of $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n+r}$

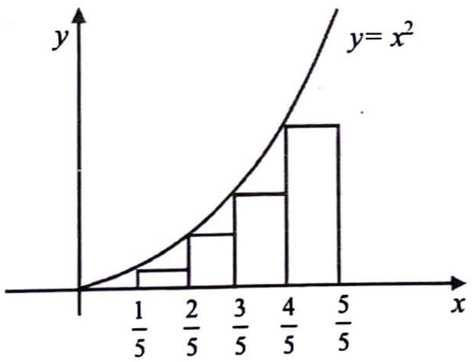


Solution:

Width of each rectangle $= \dfrac{2-1}{n} = \dfrac{1}{n}$ S = Area of the n rectangles $= \sum_{r=1}^{n}\left(\frac{1}{n}\right)\left(\frac{1}{1+\frac{r}{n}}\right) = \sum_{r=1}^{n}\left(\frac{1}{n+r}\right)$ $= \frac{1}{n} \times \frac{1}{1+\frac{1}{n}} + \frac{1}{n} \times \frac{1}{1+\frac{2}{n}} + \dots \frac{1}{n} \times \frac{1}{1+\frac{n}{n}}$ $= \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n} = \sum_{r=1}^{n}\left(\frac{1}{n+r}\right)$ $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r} = \int_1^2 \frac{1}{x}\,dx = \ln 2$	Note that $2 = 1 + \dfrac{n}{n}$ $\lim_{n\to\infty}\sum_{r=1}^{n}\dfrac{1}{n+r}$ = the area under the curve $y = \dfrac{1}{x}$ over the interval [1, 2]

Example 12

The diagram below shows part of the graph of $y = x^2$, with rectangles of equal width approximating the area under the curve between $x = 0$ and $x = 1$.



(i) Show that the total area of the four rectangles shown may be expressed as $\dfrac{1}{5^3}\left(\sum_{r=1}^{4} r^2\right)$.

(ii) Let A be the total area of the $(n-1)$ rectangles, each of width $\dfrac{1}{n}$, under the curve. Given that $\sum_{r=1}^{n} r^2 = \dfrac{n}{6}(n+1)(2n+1)$, show that $A = \dfrac{(n-1)(2n-1)}{6n^2}$.

(iii) Deduce the area bounded by the curve, the x-axis and the line $x = 1$.

Solution:

(i) Area of the four rectangles under the curve

$$=\frac{1}{5}\times\left(\frac{1}{5}\right)^2+\frac{1}{5}\times\left(\frac{2}{5}\right)^2+\frac{1}{5}\times\left(\frac{3}{5}\right)^2+\frac{1}{5}\times\left(\frac{4}{5}\right)^2$$

$$=\frac{1}{5^3}\left(1^2+2^2+3^2+4^2\right)$$

$$=\frac{1}{5^3}\left(\sum_{r=1}^{4} r^2\right) \qquad \text{(shown)}$$

The width of each of the 4 rectangles is $\frac{1}{5}$.

Their heights are $\left(\frac{1}{5}\right)^2$, $\left(\frac{2}{5}\right)^2$, $\left(\frac{3}{5}\right)^2$ and $\left(\frac{4}{5}\right)^2$ respectively.

(ii)

$$A=\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{n-1}{n}\right)^2\right]$$

$$=\frac{1}{n^3}\left[1^2+2^2+\ldots(n-1)^2\right]$$

$$=\frac{1}{n^3}\left(\sum_{r=1} r^2\right)$$

$$=\frac{1}{n^3}\left(\frac{n-1}{6}\right)\left((n-1)+1\right)\left(2(n-1)+1\right)$$

$$=\frac{(n-1)(2n-1)}{6n^2} \qquad \text{(shown)}$$

(iii)

Required area $= \lim_{n\to\infty}\frac{(n-1)(2n-1)}{(6n^2)}$

$$=\frac{1}{6}\lim_{n\to\infty}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right)$$

$$=\frac{1}{6}\times 1\times 2=\frac{1}{3}\text{ units}^2$$

Check:

$$\text{Area}=\int_0^1 x^2\,dx=\left[\frac{1}{3}x^3\right]_0^1=\frac{1}{3}$$

<u>Important idea</u>

When n (the number of rectangles) $\to\infty$,

the sum of area of the rectangles $\to$ actual area bounded by the curve and the x-axis from a to b.

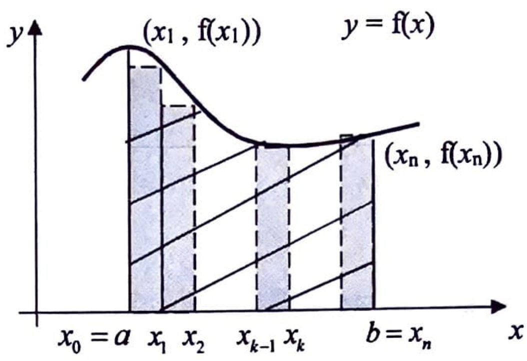
<u>Independent reading</u>

Let f(x) be a function defined on $[a, b]$. The region bounded by the curve y = f(x), the x-axis and the lines $x = a$ and $x = b$ is divided into n equal subintervals $[x_0, x_1]$, $[x_1, x_2]$,…., $[x_{n-1}, x_n]$, where $a = x_0$ and $b = x_n$, and the width of each interval $\delta x = \frac{b-a}{n}$.

We then construct a rectangle on each subinterval where the height of the k^{th} rectangle is given by the y-coordinate $\mathrm{f}(x_k)$ corresponding to x_k. Then the area of the k^{th} rectangle is given by $[\mathrm{f}(x_k)]\delta x$ and the total area of the n rectangles is $\sum_{k=1}^{n}[\mathrm{f}(x_k)]\delta x$.

Thus the area, A, bounded by the curve and the x-axis from a to b is

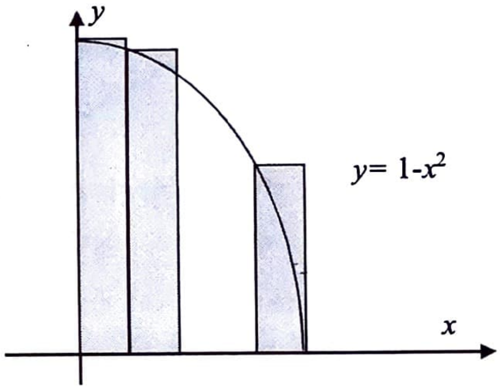
$A = \lim_{n\to\infty}\sum_{k=1}^{n}[\mathrm{f}(x_k)]\delta x = \int_a^b [\mathrm{f}(x)]\,\mathrm{d}x$.



Self-Review 6

The diagram shows the graph of $y = 1 - x^2$ in the interval [0,1] with n rectangles of equal width. Show that the total area of the n rectangles is given by

$S = 1 - \frac{1}{n^3}\sum_{r=1}^{n-1} r^2$.

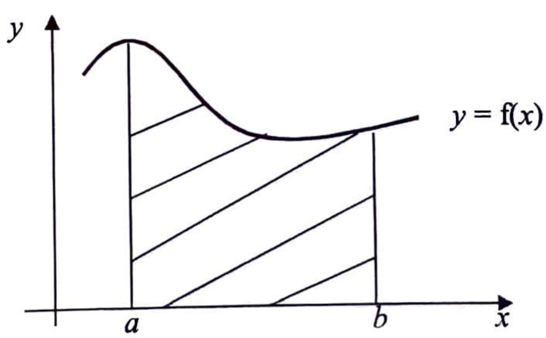


Deduce the exact value of $\lim_{n\to\infty}(1 - \frac{1}{n^3}\sum_{r=1}^{n-1} r^2)$

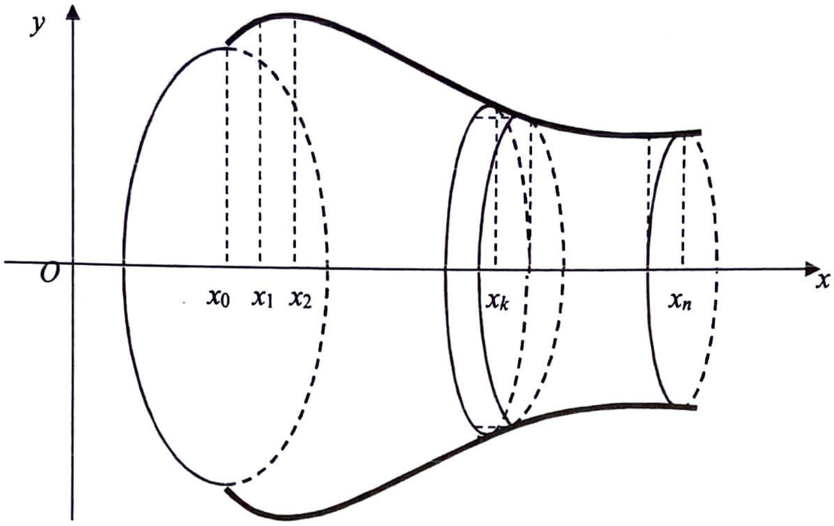
[Ans: 2/3]

12.3 Volume of a Solid of Revolution as a Limit of a Sum of the Volume of Cylindrical Discs

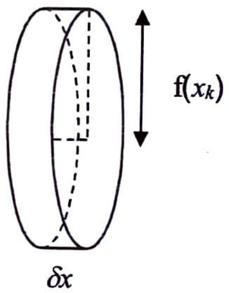
12.3.1 Volume of a Solid of Revolution Formed by Rotation About the x-axis Through 2π radians



Let $f(x)$ be a function defined on $[a, b]$. The region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is rotated about the x-axis through 360° to form a solid of revolution. We wish to find the volume, V, of the solid of revolution.



Divide the interval $[a,b]$ into n equal subintervals $[x_0, x_1]$, $[x_1, x_2]$,…., $[x_{n-1}, x_n]$, each of length $\delta x = \dfrac{b-a}{n}$, where $a = x_0$ and $b = x_n$.



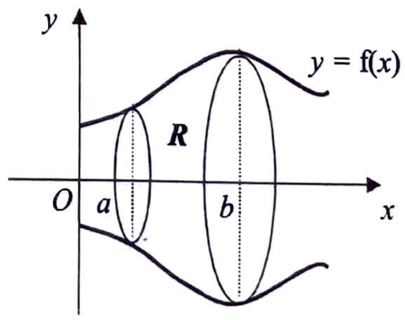
Let δV_k denotes the volume of the k^{th} circular disc, with radius $f(x_k)$ and width δx.
Then $\delta V_k = \pi[f(x_k)]^2\delta x$.
There are n such discs.
Thus, the total volume of the disks is

$V_n = \sum_{k=1}^{n} \pi[f(x_k)]^2 \delta x$ which can be regarded as an approximation to the value of the actual volume V of the solid.
We see that as the number of discs increases, i.e as $n \to \infty$, $V_n \to V$.
Therefore

$$\boxed{V = \lim_{n\to\infty} \sum_{k=1}^{n} \pi[f(x_k)]^2 \delta x = \int_a^b \pi\left[f(x)\right]^2 dx}$$

From our discussion above, the volume of the solid generated when the region R is rotated through 2π radians about the x-axis is given by

$$\boxed{V_x = \pi\int_a^b y^2\, dx = \pi\int_a^b \left[f(x)\right]^2 dx}$$



Scan the QR code to visualize the volume of solid of revolution about the x- axis

Example 13

The region R is bounded by the curve $y = x^2$, the x-axis and the lines $x = 1$ and $x = 2$.
Find the exact volume of the solid of revolution obtained by rotating R about the x-axis through four right angles.

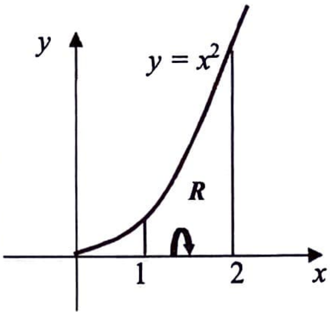
Solution:

The required volume

$= \pi \int_1^2 y^2 \, dx$

$= \pi \int_1^2 x^4 \, dx$

$= \pi \left[\frac{x^5}{5}\right]_1^2 = \pi\left[\frac{32}{5} - \frac{1}{5}\right] = \frac{31}{5}\pi$ units^3



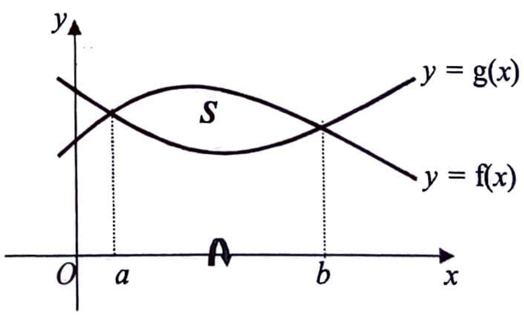
Self-Review 7

The region bounded by the graph of $y = \sqrt{x}$, the line $x = 2$ and the x-axis is rotated completely about the x-axis. Show that the volume of the solid formed is 2π units^3.

12.3.2 Volume of a Solid of Revolution Formed by Rotating Region Bounded Between Two Curves About the x-axis

Volume of the solid formed when the region ***S*** bounded between the curves with equations $y = \mathrm{g}(x)$ and $y = \mathrm{f}(x)$ is rotated through 2π radians about the x-axis is given by

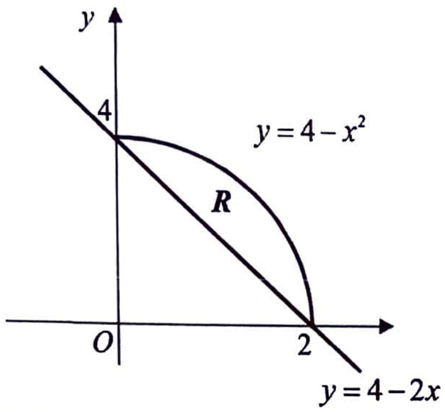
$$V_x = \pi \int_a^b [\mathrm{f}(x)]^2 \, dx - \pi \int_a^b [\mathrm{g}(x)]^2 \, dx$$
$$= \pi \int_a^b \left\{[\mathrm{f}(x)]^2 - [\mathrm{g}(x)]^2\right\} dx$$



Example 14

The region R is bounded by the curve $y = 4 - x^2$ and the line $y = 4 - 2x$. Show this region clearly on a sketch and find the exact volume of the solid formed when this region is rotated through four right angles about the x-axis.

Solution:



To find the points of intersections of the two curves, we set

$4-2x=4-x^2 \Rightarrow x(x-2)=0 \Rightarrow x=0$ or 2.

When $x=0$, $y=4-2(0)=4$

When $x=2$, $y=4-2(2)=0$

Volume, $V_x = \boxed{\pi\int_0^2 (4-x^2)^2 dx - \pi\int_0^2 (4-2x)^2 dx}$

$$=\pi\int_0^2 (16-8x^2+x^4)-(16-16x+4x^2)\,dx$$

$$=\pi\int_0^2 16x-12x^2+x^4\,dx$$

$$=\pi\left[8x^2-4x^3+\frac{1}{5}x^5\right]_0^2$$

$$=\pi\left(32-32+\frac{32}{5}\right)=\frac{32}{5}\pi \text{ units}^3$$

Alternatively,

$$\pi\int_0^2 (4-2x)^2\,dx$$

$$=\pi\left[\frac{(4-2x)^3}{3(-2)}\right]_0^2$$

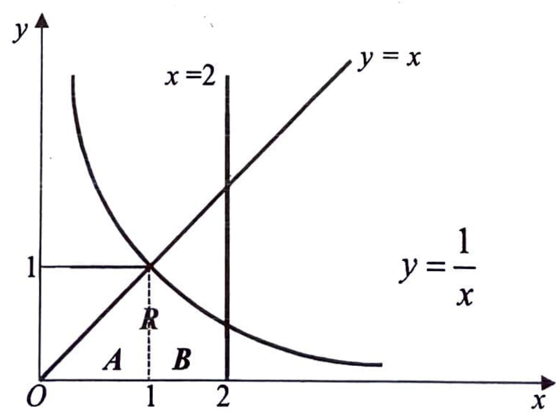
$$=\frac{32}{3}\pi$$

Could we use volume of cone to replace $\pi\int_0^2 (4-2x)^2\,dx$? What would be the radius and height of this cone?

Example 15 (MB N74/1/7) (Self-Reading)

The region R in the 1st quadrant bounded by the lines $y=x$, $x=2$, $y=0$ and the curve $y=\frac{1}{x}$ is rotated completely about the x-axis. Compute the exact volume of the solid formed.

Solution:



We divide the region R into regions A and B as shown above.

V_A = volume of cone with height 1 unit and radius 1 unit $=\frac{1}{3}\pi(1)^2(1)=\frac{1}{3}\pi$ units^3

$$V_B = \pi\int_1^2 \frac{1}{x^2}\,dx = -\pi\left[\frac{1}{x}\right]_1^2 = -\pi\left[\frac{1}{2}-1\right] = \frac{1}{2}\pi \text{ units}^3$$

Required volume $= \frac{1}{3}\pi + \frac{1}{2}\pi = \frac{5}{6}\pi$ units^3.

Self-Review 8

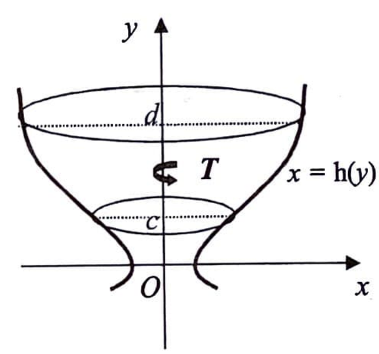
The region bounded by the curve $y = x^2 + 1$ and the straight line $y = x + 3$ is denoted by R. Show that the volume of the solid formed when R is rotated through 2π radians about the x-axis is $\frac{117}{5}\pi$ units^3.

12.3.3 Volume of a Solid of Revolution Formed by Rotation about the *y*-axis through 2π radians

The region T is bounded by the curve $x = \mathrm{h}(y)$, the lines $y = c$, $y = d$ and the y-axis.
Volume of the solid formed when the region T is rotated through 2π radians about the y-axis is

$$\boxed{V_y = \pi\int_c^d x^2\,dy = \pi\int_c^d [\mathrm{h}(y)]^2\,dy}$$

Scan the QR code to visualize the volume of solid of revolution about the y- axis



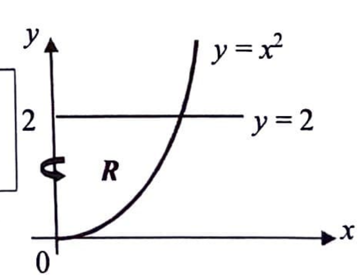
Example 16

The region R is bounded by the curve $y = x^2$ where $x \geq 0$, the y-axis and the line $y = 2$. Find in terms of π, the volume of the solid generated when R is rotated through 4 right angles about the y-axis.

Solution:

$$V_y = \pi\int_0^2 x^2\,dy = \pi\int_0^2 y\,dy = \boxed{\frac{\pi}{2}\left[y^2\right]_0^2 = \frac{\pi}{2}(2)^2 = 2\pi}$$

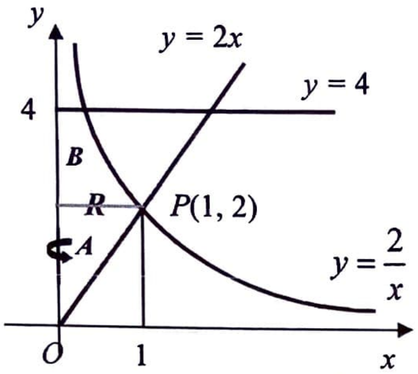
units 3



Example 17

A region R is bounded by the curve $y = \frac{2}{x}$ and the lines $y = 2x$, $x = 0$ and $y = 4$. Show this region clearly on a sketch. Find, in terms of π, the volume of the solid formed when this region is rotated through 4 right angles about the y-axis.

Solution:



The points of intersection of $y=\dfrac{2}{x}$ and $y = 2x$:

Set $\dfrac{2}{x} = 2x \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

So P has coordinates (1, 2) as shown.
By dividing the region R into regions A and B as shown,
required volume

$$\frac{1}{3}\pi(1)^2(2)+\pi\int_2^4\left(\frac{2}{y}\right)^2 dy = \frac{2\pi}{3} + 4\pi\left[-\frac{1}{y}\right]$$

$\frac{5}{3}\pi$ units

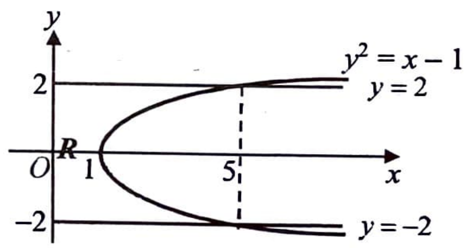
Rotating region A about y- axis results in a cone

Example 18

The diagram shows the graph of $y^2 = x - 1$. The region bounded by the curve, the y-axis and the lines $y = -2$ and $y = 2$ is denoted by R. Find the volume of the solid generated when R is rotated through

(i) 2π radians about the y-axis,

(ii) π radians about the x-axis.



Solution:

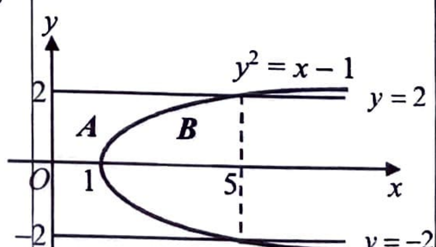
(i) $V_y = \pi\int_{-2}^{2} x^2 \mathrm{d}y = 2\pi\int_0^2 \left(y^2+1\right)^2 \mathrm{d}y = 2\pi\int_0^2 (y^4+2y^2+1)\,\mathrm{d}y$

$= 2\pi\left[\dfrac{y^5}{5}+\dfrac{2y^3}{3}+y\right]_0^2 = \dfrac{412}{15}\pi$ units^3

Do you know where the symmetrical property of the graph is used?

(ii) Let B be the region in the first quadrant bounded by the curve, the line $x = 5$ and the x-axis.
Required volume

= $\boxed{\text{Volume of cylinder } (A+B) \text{ of radius 2 units and height 5 units}}$ − $\boxed{\text{Volume generated by } B \text{ (rotated about the } x\text{-axis).}}$

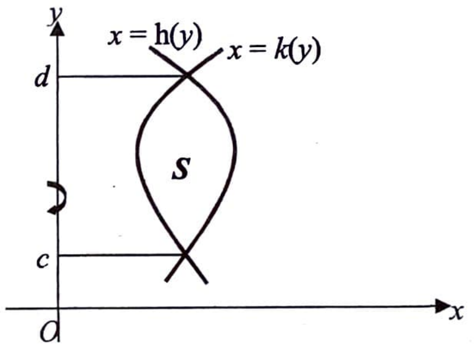


$=\pi(2)^2(5) - \pi\int_1^5 y^2\,dx$ $= 20\pi - \pi\int_1^5 (x-1)\,dx$ $= 20\pi - \pi\left[\frac{1}{2}x^2 - x\right]_1^5 = 20\pi - 8\pi$ $= 12\pi$ units^2	Note the angles of rotation in this example. Can you explain the difference?

12.3.4 Volume of a Solid of Revolution Formed by Rotating Region Bounded Between Two Curves About the *y*-axis

Volume of the solid formed when the region S bounded between the curves with equations $x = \mathrm{g}(y)$ and $x = \mathrm{f}(y)$ is rotated through 2π radians about the y-axis is given by

$$V_y = \pi\int_c^d [\mathrm{h}(y)]^2\,dy - \pi\int_c^d [\mathrm{k}(y)]^2\,dy$$
$$= \pi\int_c^d \left\{[\mathrm{h}(y)]^2 - [\mathrm{k}(y)]^2\right\}\,dy$$



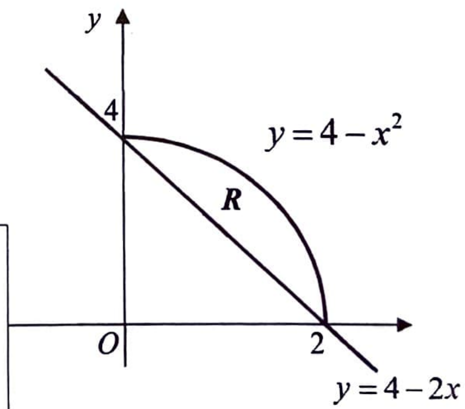
Example 19 (Example 14 revisited)
The region R is bounded by the curve $y = 4 - x^2$ and the line $y = 4 - 2x$. Find the exact volume of the solid formed when R is rotated through four right angles about the y-axis.

Solution:

Points of intersections of the two curves:
$(0, 4)$ and $(2, 0)$.

$y = 4 - x^2 \quad \Rightarrow \quad x^2 = 4 - y$

$y = 4 - 2x \quad \Rightarrow \quad x = \dfrac{4-y}{2}$



Volume $V_y = \pi\int_0^4 (4-y)\,dy - \frac{1}{3}\pi(2^2)(4)$

$= \pi\left[4y - \frac{y^2}{2}\right]_0^4 - \frac{16}{3}\pi$

$= 8\pi - \frac{16}{3}\pi = \frac{8}{3}\pi$ units^3

12.3.5 Volume of a Solid of Revolution Formed by Rotation through 2π radians about a line parallel to the x- or y-axis

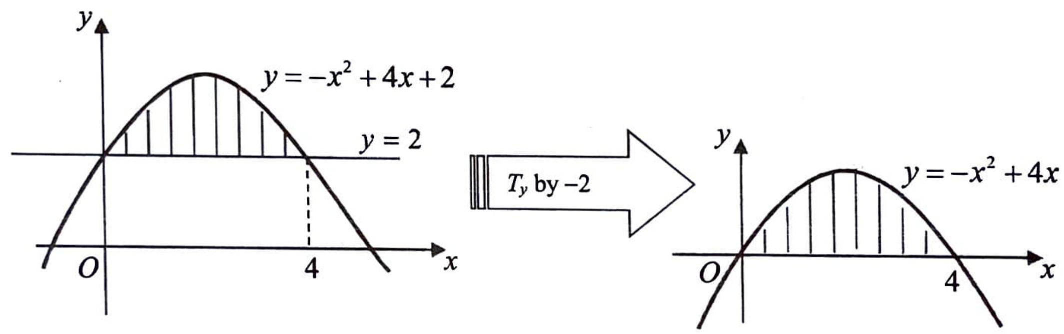
In handling problems of this nature, it is a common practice to translate the graph parallel to the x- or y-axis to obtain a new graph so that the line which the region is rotated about is correspondingly translated to coincide with the x or y-axis. The desired volume is then computed in the usual way.

Example 20

The graph of $y=-x^2+4x+2$ is rotated about the line $y = 2$ from $x = 0$ to $x = 4$. By performing an appropriate transformation or otherwise, find the exact volume of the solid thus formed.

Solution:

Perform a translation in the direction of y-axis by -2 units. Then the line $y = 2$ will be mapped to the x-axis; the graph of $y=-x^2+4x+2$ will be mapped to the graph of $y=-x^2+4x$. Thus the volume required = volume of the solid formed by rotating the graph of $y=-x^2+4x$ in the x-axis.



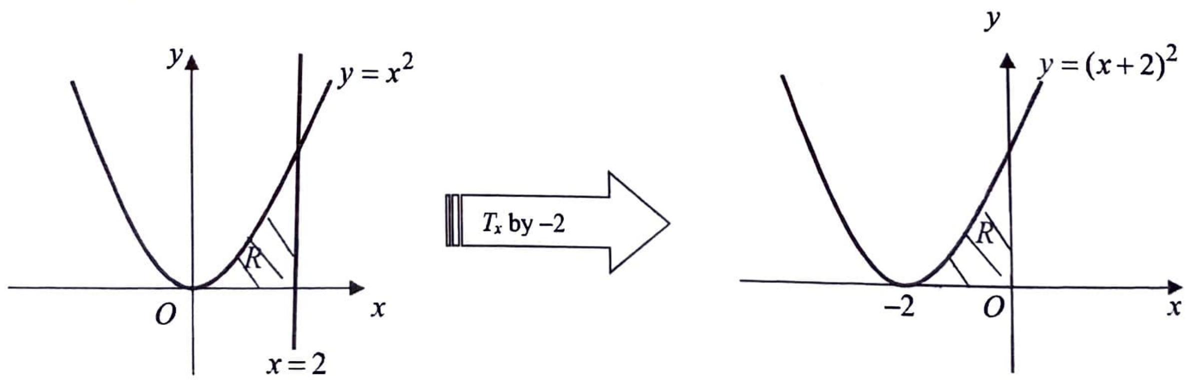
Volume of solid formed $\pi\int_0^4(-x^2+4x)^2\,dx = \pi\int_0^4(x^4-8x^3+16x^2)\,dx$

$= \pi\left[\frac{1}{5}x^5-2x^4+\frac{16}{3}x^3\right]_0^4 = \frac{512}{15}\pi$ units^3

Example 21

The region R is bounded by the curve $y = x^2$, the x-axis and the line $x = 2$. Find the volume of the solid generated when R is rotated completely about the line $x = 2$.

Solution:



$f(x) \to f(x+2)$

$y = x^2 \xrightarrow{T_x} y = (x+2)^2$

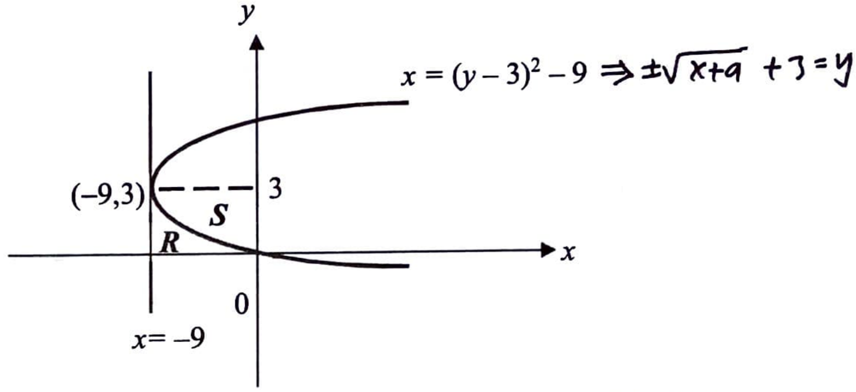
$\pm\sqrt{y} - 2 = x$

$x = \sqrt{y} - 2$ as $x > -2$

$V_y = \pi\int_0^4 x^2\,dy = \pi\int_0^4 (\sqrt{y} - 2)^2\,dy = 8.38$ units^3 (to 3sf)

Example 22

The diagram shows the region R bounded by the curve with equation $x = (y-3)^2 - 9$, the x-axis and the line $x = -9$.



Find

(i) the area of the region R,

(ii) the volume of the solid of revolution formed when the region R is rotated through 360° about the y-axis, leaving your answer in terms of π, leaving your answers exact.

Solution:

Area of $R = \int_{-9}^{0} 3 - \sqrt{x+9}\,dx = \left[3x - \frac{2}{3}(x+9)^{3/2}\right]_{-9}^{0} = 9$ units^2

$x = (y-3)^2 - 9 \Rightarrow 3 \pm (x+9)^{\frac{1}{2}}$

since $y \le 3$ ∴ $y = 3 - (x+9)^{\frac{1}{2}}$

Alternatively,

Area of R = Area of rectangle – Area of S

$$= 9(3) - \left(-\int_0^3 \left[(y-3)^2 - 9\right] dy\right)$$

$$= 27 + \left[\frac{1}{3}(y-3)^3 - 9y\right]_0^3$$

$= 27 - 18 = 9$ units^2 as above

(ii) Volume generated

$= \pi(9)^2(3) - \pi\int_0^3 x^2\,dy$

$= 243\pi - \pi\int_0^3 \left[(y-3)^2 - 9\right]^2 dy$

Vol generated

= volume of cylinder with $r = 9$, $h = 3$

– volume S (rotated about the y-axis)

$$= 243\pi - \pi\int_0^3 \left(y^2 - 6y\right)^2 dy$$

$$= 243\pi - \pi\int_0^3 \left(y^4 - 12y^3 + 36y^2\right) dy$$

$$= 243\pi - \pi\left[\frac{1}{5}y^5 - 3y^4 + 12y^3\right]_0^3 = \frac{567}{5}\pi \text{ units}^3$$

12.3.6 Motion problems – displacement function, velocity function & acceleration function

Recall that, for a particle moving in a straight line, if $s(t)$ is the displacement function, $v(t)$ is the velocity function, then $v(t) = \frac{ds}{dt}$.

Thus, the displacement function of the particle is given by: $s(t) = \int v(t)\, dt$.

Also the displacement in a time interval $a \le t \le b$ is given by: $s(t) = \int_a^b v(t)\, dt = s(b) - s(a)$.

If $a(t)$ if the acceleration function of the particle, then $a(t) = \frac{dv}{dt}$. Thus $v(t) = \int a(t)\, dt$.

Example 23 (Independent learning)

A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2$ ms^{-1}. Given that at time $t = 0$, P is 3 metres away from a given point O on the line.

(i) What is the initial velocity and acceleration of P?
(ii) Find the displacement function of P.
(iii) Find the times when P is at instantaneous rest.
(iv) How far does P travel in the first 4 seconds of motion?
(v) Find the displacement of P at the end of 4 seconds.

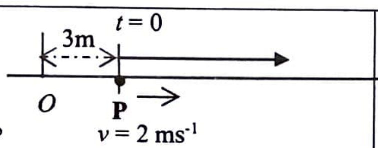
Solution:

(i) $a(t) = \frac{dv}{dt} = 2t - 3$

When $t = 0$, we have

$v(0) = 0^2 - 3(0) + 2 = 2$ ms^{-1},

$a(0) = 2(0) - 3 = -3$ ms^{-2}.



This means that P starts to move away from point O with velocity 2 ms^{-1}, and the velocity is initially decreasing at the rate of 3ms^{-2}.

(ii) The displacement function is:

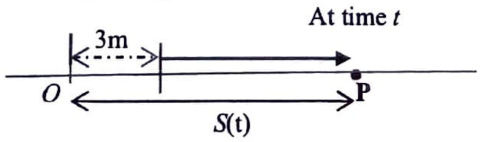
$$s(t) = \int v(t)\, dt = \int t^2 - 3t + 2\, dt = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + c$$

At $t = 0$, $s(0) = 3$, we have

$$3 = \frac{1}{3}(0)^3 - \frac{3}{2}(0)^2 + 2(0) + c$$

$\Rightarrow \quad c = 3$

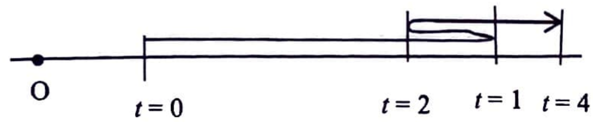
Therefore, $s(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + 3$



Can $s(t)$ be negative?

(iii) When P is at instantaneous rest, then $v(t)=0$

$$t^2-3t+2=0 \Rightarrow (t-1)(t-2)=0 \Rightarrow t=1,2$$



The velocity function, $v(t)$, changes sign at $t=1$ s and $t=2$ s.
From the diagram, we can see that at $t=0$ s, the particle P starts to move away from point O. At $t=1$ s, it reverses its direction to move towards O and then at $t=2$ s, it reverses its direction again and moves away from O.

(iv) We have $s(0)=3$

$$s(1)=\frac{1}{3}(1)^3-\frac{3}{2}(1)^2+2(1)+3=\frac{23}{6}$$
$$s(2)=\frac{1}{3}(2)^3-\frac{3}{2}(2)^2+2(2)+3=\frac{11}{3}$$
$$s(4)=\frac{1}{3}(4)^3-\frac{3}{2}(4)^2+2(4)+3=\frac{25}{3}$$

Distance travelled from $t=0$ to $t=1$ is :

$$d_1=\left|\int_0^1 v(t)\,dt\right|=\left|s(1)-s(0)\right|=\left|\frac{23}{6}-3\right|=\left|\frac{5}{6}\right|=\frac{5}{6}$$

Distance travelled from $t=1$ to $t=2$ is :

$$d_2=\left|\int_1^2 v(t)\,dt\right|=\left|s(2)-s(1)\right|=\left|\frac{11}{3}-\frac{23}{6}\right|=\left|-\frac{1}{6}\right|=\frac{1}{6}$$

Distance travelled from $t=2$ to $t=4$ is :

$$d_3=\left|\int_2^4 v(t)\,dt\right|=\left|s(4)-s(2)\right|=\left|\frac{25}{3}-\frac{11}{3}\right|=\left|\frac{14}{3}\right|=\frac{14}{3}$$

Total distance travelled in the first 4 s $=\frac{5}{6}+\frac{1}{6}+\frac{14}{3}=5\frac{2}{3}$ (m)

Note that distance is not a vector

Alternatively,
Total distance travelled in the first 4 s

$$=\int_0^4 |v(t)|\,dt=\int_0^4 \left|t^2-3t+2\right|\,dt=5.67 \text{ (m) (By GC)}$$

(v) Displacement after 4 s
= final position – original position

$$= s(4)-s(0)=\frac{25}{3}-3=5\frac{1}{3}\text{ (m)}$$

Alternatively, displacement after 4 s

$$=\int_0^4 v(t)\,dt=s(4)-s(0)=\frac{25}{3}-3=5\frac{1}{3}\text{ (m)}$$

Note that displacement is a vector

That is, after 4 s, P has a displacement of $5\frac{1}{3}$ m in the direction away from the fixed point O.

12.3.7 Miscellaneous Examples

Example 24 [ACJC Prelim 2017/2/2]

A curve C has parametric equations $x=\cos t$, $y=\frac{1}{2}\sin 2t$, where $\frac{\pi}{2}\le t\le\frac{3\pi}{2}$.

(i) Find the equation of the normal to C at the point P with parameter p. [2]

The normal to C at the point when $t=\frac{2\pi}{3}$ cuts the curve again. Find the coordinates of the point of intersection. [2]

(ii) Sketch C, clearly labelling the coordinates of the points where the curve crosses the x- and y-axes. [1]

(iii) Find the Cartesian equation of C. [2]

The region bounded by C is rotated through π radians about the x-axis. Find the exact volume of the solid formed. [3]

Solution:

(i) $x=\cos t \Rightarrow \frac{dx}{dt}=-\sin t$ $y=\frac{1}{2}\sin 2t \Rightarrow \frac{dy}{dt}=\frac{1}{2}(2\cos 2t)=\cos 2t$ $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\cos 2t}{-\sin t} \Rightarrow \left.\frac{dy}{dx}\right\|_{t=p}=\frac{\cos 2p}{-\sin p}$ $\Rightarrow$ gradient of normal $=\frac{\sin p}{\cos 2p}$ Equation of normal at the point $\left(\cos p, \frac{1}{2}\sin 2p\right)$: $y-\frac{1}{2}\sin 2p=\frac{\sin p}{\cos 2p}(x-\cos p)$ $\Rightarrow y=\frac{1}{2}\sin 2p+\frac{\sin p}{\cos 2p}x-\frac{\sin p\cos p}{\cos 2p}$ $\Rightarrow y=\frac{\sin p}{\cos 2p}x+\frac{\sin 2p}{2}-\frac{2\sin p\cos p}{2\cos 2p}$ $\Rightarrow y=\frac{\sin p}{\cos 2p}x+\frac{1}{2}(\sin 2p-\tan 2p)$ Substitute $p=\frac{2\pi}{3}$ into the above to find the equation of normal at this point, $y=\frac{\sin\frac{2\pi}{3}}{\cos 2(\frac{2\pi}{3})}x+\frac{1}{2}\left(\sin 2(\frac{2\pi}{3})-\tan 2(\frac{2\pi}{3})\right)$	

$$y=\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}x+\frac{1}{2}\left(-\frac{\sqrt{3}}{2}-\sqrt{3}\right)$$

$$\Rightarrow \quad y=-\sqrt{3}x-\frac{1}{4}\left(3\sqrt{3}\right) \quad \cdots\cdots(1)$$

To find the other point of intersection of normal and C:

Substitute $x=\cos t$ and $y=\frac{1}{2}\sin 2t$ into (1):

$$\frac{1}{2}\sin 2t=-\sqrt{3}\left(\cos t\right)-\frac{1}{4}\left(3\sqrt{3}\right)$$

$$\frac{1}{2}\sin 2t+\sqrt{3}\left(\cos t\right)+\frac{1}{4}\left(3\sqrt{3}\right)=0$$

From GC,

$t=2.094395$ (corresponds to $t=\frac{2\pi}{3}$) or $t=3.495928$

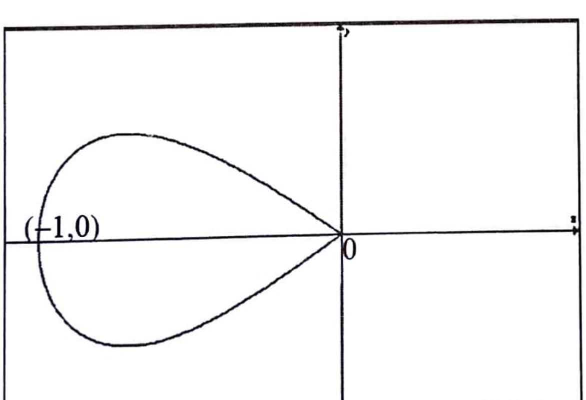
Hence, $t = 3.495928$ is the parameter of the point where the normal cut the curve again.

Thus $x=\cos t=\cos 3.495928=-0.938$,

$y=\frac{1}{2}\sin 2(3.495928)=0.325$

Therefore, the coordinates of the point is $\left(-0.938, \quad 0.325\right)$

(ii)



At x-intercept,

$y=0$

$\sin 2t=0$

$2t=0,\pi,2\pi,3\pi$

Since $\frac{\pi}{2}\le t\le\frac{3\pi}{2}$,

$t=\frac{\pi}{2},\pi,\frac{3\pi}{2}$

$x=\cos\frac{\pi}{2}=0$

$x=\cos\pi=-1$

$x=\cos\frac{3\pi}{2}=0$

(iii) To find the Cartesian equation of the curve C, we just need to eliminate the parameter t from the equations

$x=\cos t, \quad y=\frac{1}{2}\sin 2t$

$x=\cos t \quad \Rightarrow \quad x^2=\cos^2 t$

$y=\frac{1}{2}\sin 2t \quad \Rightarrow \quad y=\sin t\cos t$

$\Rightarrow \quad y^2=\sin^2 t\cos^2 t=\left(1-\cos^2 t\right)\cos^2 t=\left(1-x^2\right)x^2$

$\therefore$ Cartesian equation: $y^2=\left(1-x^2\right)x^2$

Volume required $=\int_{-1}^{0} \pi y^2 \, dx$ $=\pi\int_{-1}^{0}(1-x^2)x^2 \, dx=\pi\int_{-1}^{0} x^2-x^4 \, dx=\pi\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_{-1}^{0}$ $=\frac{2}{15}\pi$ units^3	

Example 25 (N2009/1/4)
It is given that

$$f(x)=\begin{cases}7-x^2 & \text{for } 0<x\le 2,\\ 2x-1 & \text{for } 2<x\le 4,\end{cases}$$

and that $f(x)=f(x+4)$ for all real values of x.

(i) Evaluate $f(27)+f(45)$.

(ii) Sketch the graph of $y=f(x)$ for $-7\le x\le 10$.

(iii) Find $\int_{-4}^{3} f(x)\,dx$.

Solution:

(i) $f(27)$

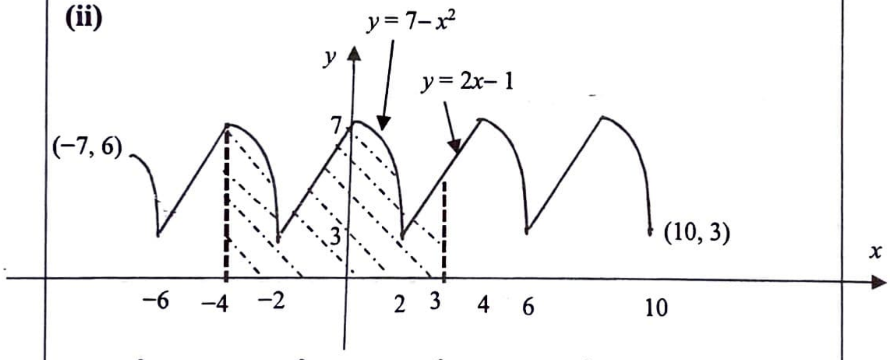
$=f(23+4)=f(23)=f(19+4)=f(19)=\cdots=f(3)=5$

(In fact, $27=4\times 6+3 \Rightarrow f(27)=f(3)=5$)

Similarly, $f(45)=f(11\times 4+1)=f(1)=6$

$\therefore f(27)+f(45)=5+6=11$

(ii)



(iii) $\int_{-4}^{3} f(x)\,dx=\int_{-4}^{-2} f(x)\,dx+\int_{-2}^{0} f(x)\,dx+\int_{0}^{2} f(x)\,dx+\int_{2}^{3} f(x)\,dx$

From the graph, we see that

$\int_{-4}^{-2} f(x)\,dx=\int_{0}^{2} f(x)\,dx=\int_{0}^{2}(7-x^2)\,dx=\frac{34}{3}$,

$\int_{-2}^{0} f(x)\,dx=\int_{2}^{4} f(x)\,dx=\int_{2}^{4}(2x-1)\,dx=10$,

$\int_{2}^{3} f(x)\,dx=\int_{2}^{3}(2x-1)\,dx=4$

Hence, $\int_{-4}^{3} f(x)\,dx=\frac{34}{3}+10+\frac{34}{3}+4=36\frac{2}{3}$