National Junior College
2016 – 2017 H2 Further Mathematics
Topic: Further Special Discrete Probability Distributions **Assignment Solutions**

Qn	Suggested Solution	MS
1(a)	• It is **equally likely** to snow on any one of the days in the winter season. • Snowing occurs **independently** across all days in the winter season.	B1 B1
1(b)(i)	Let X denote the number of days from 1 December for it to first snow. Then $X \sim \text{Geo}(0.2)$. Hence $\text{P}(X = 20) = (0.2)(1-0.2)^{19} = 0.0028823$	B1
1(b)(ii)	$\text{P}(X < 5) = 1-(1-0.2)^4 = 0.5904$	B1
1(b)(iii)	$\text{P}(X > 20 \mid X > 5) = \text{P}(X > 15) = (1-0.2)^{15} = 0.035184$	M1, A1
1(b)(iv)	Let n be the least number of days from 1 December for the probability that the first snow falls on or before the n^{th} day is at least 0.95. $1-(1-0.2)^n \geq 0.95$ $0.8^n \leq 0.05$ $\ln\left(0.8^n\right) \leq \ln 0.05$ $n\ln 0.8 \leq \ln 0.05$ $n \geq \dfrac{\ln 0.05}{\ln 0.8}$ $n \geq 13.4$ Therefore, **least** value of n = 14.	M1 M1 A1
2(a)	• Particles are emitted from the material at a **constant mean rate**. • Particles are emitted from the material **independently** of one another **across the hour.**	B1 B1
2(b)(i)	Let X denote the number of particles that are emitted in a randomly chosen 10 hour period. Then $X \sim \text{Po}(2.8)$. Hence $\text{P}(X > 2) = 1-\text{P}(X \leq 2) = 0.53055$	M1 A1
2(b)(ii)	Let Y be the number of periods with more than 2 particles emitted each, out of three randomly chosen 10 hour periods. Then $Y \sim \text{B}(3, 0.53055)$. Hence $\text{P}(Y = 2) = 0.39643$	M1 A1
2(b)(iii)	Let W denote the number of particles that are emitted in a randomly chosen m minute period. Then $W \sim \text{Po}\left(\dfrac{m}{150}\times 0.7\right) \Rightarrow W \sim \text{Po}\left(\dfrac{7m}{1500}\right)$. For the probability that no particles are emitted in a m minute period to be at least 0.99, $\text{P}\left(W=0\right) \geq 0.99$ $\dfrac{e^{-\frac{7m}{1500}}\left(\frac{7m}{1500}\right)^0}{0!} \geq 0.99$ $e^{-\frac{7m}{1500}} \geq 0.99$ $-\dfrac{7m}{1500} \geq \ln 0.99$ $m \leq \dfrac{\ln 0.99}{\left(-\frac{7}{1500}\right)} = 2.1536$ Therefore, the **longest** time period is 2.15 minutes.	M1 M1 M1 A1

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2(b)(iv)	Let U and V denote the number of particles emitted in the first n hours and the last $20 - n$ hours of a randomly chosen 20 hour period. Then	
	$U \sim \text{Po}\left(\frac{7n}{25}\right),\ V \sim \text{Po}\left(\frac{140-7n}{25}\right)$ and $U+V \sim \text{Po}\left(\frac{28}{5}\right)$.	M1
	$\text{P}\left(U=4 \mid U+V=6\right)=0.324135$	
	$\frac{\text{P}\left(U=4 \cap U+V=6\right)}{\text{P}\left(U+V=6\right)}=0.324135$	M1
	$\frac{\text{P}\left(U=4 \cap V=2\right)}{\text{P}\left(U+V=6\right)}=0.324135$	
	$\frac{\text{P}\left(U=4\right)\times\text{P}\left(V=2\right)}{\text{P}\left(U+V=6\right)}=0.324135$	M1
	$\frac{\frac{e^{-\frac{7n}{25}}}{4!}\left(\frac{7n}{25}\right)^4\times\frac{e^{-\frac{140-7n}{25}}}{2!}\left(\frac{140-7n}{25}\right)^2}{\frac{e^{-\frac{28}{5}}}{6!}\left(\frac{28}{5}\right)^6}=0.324135$	
	$\frac{e^{-\frac{7n}{25}}\times e^{-\frac{140-7n}{25}}}{e^{-\frac{28}{5}}}\times\frac{6!}{4!2!}\times\frac{\left(\frac{7n}{25}\right)^4\left(\frac{140-7n}{25}\right)^2}{\left(\frac{28}{5}\right)^6}=0.324135$	M1
	$\frac{6!}{4!2!}\times\frac{\left(\frac{7n}{25}\right)^4\left(\frac{140-7n}{25}\right)^2}{\left(\frac{28}{5}\right)^4\times\left(\frac{28}{5}\right)^2}=0.324135$	
	$15\left(\frac{n}{20}\right)^4\left(1-\frac{n}{20}\right)^2=0.324135$	
	$\left(\frac{n}{20}\right)^4\left(1-\frac{n}{20}\right)^2=0.021609$	
	$\left(\frac{n}{20}\right)^2\left(1-\frac{n}{20}\right)=0.147$	M1
	$n^2\left(20-n\right)=1176$	
	$n^3-20n^2+1176=0$	
	$n=12.644,\ 14,\text{ or }-6.6436\text{ (rej.)}$	
	Therefore, there is exactly one integer solution for n, $n = 14$.	A1

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3	Since f is the probability distribution function for a geometric distribution,	
	$$\mathrm{f}(x)=p(1-p)^{x-1},\ \text{for } x=1,2,3,\ldots,$$	M1
	where p is a positive constant such that $0<p<1$. Note that $\mathrm{f}(1)=p$. Hence, $$\mathrm{g}(x)=\left(\frac{2-\mathrm{f}(1)}{1-\mathrm{f}(1)}\right)\mathrm{f}(2x)$$	
	$$=\left(\frac{2-p}{1-p}\right)\cdot p(1-p)^{2x-1}$$	M1
	Since $0<p<1$, $1<2-p<2$ and $0<1-p<1$, therefore $\mathrm{g}(x)>0$ for all $x\in\mathbb{Z}^{+}$.	B1
	$$\sum_{x=1}^{\infty}\mathrm{g}(x)=\sum_{x=1}^{\infty}\left(\frac{2-p}{1-p}\right)\cdot p(1-p)^{2x-1}$$ $$=\left(\frac{p(2-p)}{1-p}\right)\sum_{x=1}^{\infty}(1-p)^{2x-1}$$	
	$$=\left(\frac{p(2-p)}{1-p}\right)\frac{(1-p)}{1-(1-p)^2}$$	M1
	$$=\frac{p(2-p)}{1-p}\cdot\frac{1-p}{2p-p^2}$$	
	$$=\frac{p(2-p)}{1-p}\cdot\frac{1-p}{p(2-p)}=1$$	A1
	Therefore, g is a probability distribution function.	
	$$\mathrm{E}(X)=\sum_{x=1}^{\infty}x\mathrm{g}(x)$$	
	$$=\sum_{x=1}^{\infty}x\left(\frac{2-p}{1-p}\right)\cdot p(1-p)^{2x-1}$$	M1
	$$=\left(\frac{p(2-p)}{1-p}\right)\sum_{x=1}^{\infty}x(1-p)^{2x-2+1}$$ $$=p(2-p)\sum_{x=1}^{\infty}x\left[(1-p)^2\right]^{x-1}$$	
	$$=p(2-p)\left[1-(1-p)^2\right]^{-2}$$	M1
	$$=\frac{p(2-p)}{(2p-p^2)^2}$$	
	$$=\frac{p(2-p)}{p^2(2-p)^2}=\frac{1}{p(2-p)}$$	A1

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	$\mathrm{E}\left(X\left(X-1\right)\right)=\sum_{x=1}^{\infty}x(x-1)\mathrm{g}(x)$	
	$=\sum_{x=1}^{\infty}x(x-1)\left(\frac{2-p}{1-p}\right)\cdot p(1-p)^{2x-1}$	M1
	$=\left(\frac{p\left(2-p\right)}{1-p}\right)\sum_{x=1}^{\infty}x(x-1)(1-p)^{2x-2+1}$	
	$=p\left(2-p\right)\sum_{x=2}^{\infty}x(x-1)\left[(1-p)^{2}\right]^{x-1}$	
	$=p\left(2-p\right)\cdot 2\left[1-\left(1-p\right)^{2}\right]^{-3}$	M1
	$=\frac{2p\left(2-p\right)}{\left(2p-p^{2}\right)^{3}}$	
	$=\frac{2p\left(2-p\right)}{p^{3}\left(2-p\right)^{3}}=\frac{2}{p^{2}\left(2-p\right)^{2}}$	A1
	Therefore, $\mathrm{E}\left(X^{2}\right)=\mathrm{E}\left(X\left(X-1\right)\right)+\mathrm{E}\left(X\right)$	
	$=\frac{2}{p^{2}\left(2-p\right)^{2}}+\frac{1}{p\left(2-p\right)}$	M1
	and	
	and $\mathrm{Var}\left(X^{2}\right)=\mathrm{E}\left(X^{2}\right)-\left[\mathrm{E}\left(X\right)\right]^{2}$	
	$=\frac{2}{p^{2}\left(2-p\right)^{2}}+\frac{1}{p\left(2-p\right)}-\left[\frac{1}{p\left(2-p\right)}\right]^{2}$	M1
	$=\frac{1}{p^{2}\left(2-p\right)^{2}}+\frac{1}{p\left(2-p\right)}$	
	$=\frac{1}{p^{2}\left(2-p\right)^{2}}+\frac{p\left(2-p\right)}{p^{2}\left(2-p\right)^{2}}$	
	$=\frac{1+2p-p^{2}}{p^{2}\left(2-p\right)^{2}}$	A1