RAFFLES INSTITUTION
H2 Mathematics (9758)
2024 Year 5

Term 4 RTT2: Post-Promo Revision Lesson Worksheet
Session 3: C4A to C4C: Vectors

1 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The vectors $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a} = 2p\mathbf{i} - 6p\mathbf{j} + 3p\mathbf{k} \quad \text{and} \quad \mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k},$$

where p is a constant.

Find $\dfrac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}|}$ and give a geometrical interpretation of $\dfrac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}|}$. **[2]**

Find $\mathbf{a}\times\mathbf{b}$ and give a geometrical interpretation of $|\mathbf{a}\times\mathbf{b}|$. **[2]**

Given that $\mathbf{a}$ is a unit vector, find the possible value(s) of p. **[2]**

$$\frac{|\mathbf{a}\bullet\mathbf{b}|}{|\mathbf{a}|} = \frac{\left|\begin{pmatrix}2p\\-6p\\3p\end{pmatrix}\bullet\begin{pmatrix}1\\1\\-2\end{pmatrix}\right|}{\left|p\begin{pmatrix}2\\-6\\3\end{pmatrix}\right|} = \frac{|p(2-6-6)|}{\left|p\sqrt{2^2+6^2+3^2}\right|} = \frac{10}{7}$$

What does the phrase "geometrical interpretation" mean?
Draw a picture to help determine for yourself.

Let θ be the angle AOB.

$$\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \quad \Rightarrow \quad |\mathbf{a}\bullet\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta| \quad \Rightarrow \quad \frac{|\mathbf{a}\bullet\mathbf{b}|}{|\mathbf{a}|} = |\mathbf{b}||\cos\theta|$$

$\dfrac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}|} = |\mathbf{b}||\cos(\theta)| =$ length of projection of $\mathbf{b}$ onto $\mathbf{a}$.

[REMARKS]
What is the geometrical interpretation of $|\mathbf{a}\cdot\mathbf{k}|$?
Answer : $|\mathbf{a}\cdot\mathbf{k}|$ represents the length of projection of **r** onto the *z*-axis.

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}2p\\-6p\\3p\end{pmatrix}\times\begin{pmatrix}1\\1\\-2\end{pmatrix}=p\begin{pmatrix}2\\-6\\3\end{pmatrix}\times\begin{pmatrix}1\\1\\-2\end{pmatrix}=p\begin{pmatrix}9\\7\\8\end{pmatrix}$$ which is a vector.

$\mathbf{a}\times\mathbf{b}=\left(|\mathbf{a}||\mathbf{b}|\sin\theta\right)\hat{\underset{\sim}{n}} \quad \Rightarrow \quad |\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$

Area of triangle OAB = $\frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

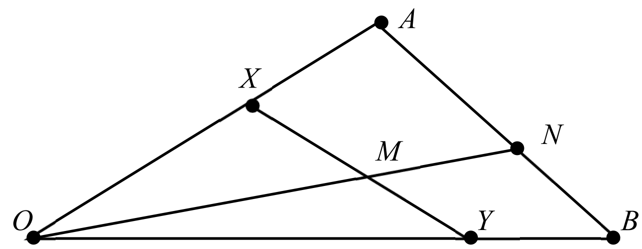
$\therefore |\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$
= area of parallelogram formed with adjacent side OA and OB

Since $\mathbf{a}$ is a unit vector, $|\mathbf{a}|=1\Rightarrow\left|\begin{pmatrix}2p\\-6p\\3p\end{pmatrix}\right|=\left|p\begin{pmatrix}2\\-6\\3\end{pmatrix}\right|=1$

$$|p|\sqrt{2^2+(-6)^2+(3)^2}=1$$
$$7|p|=1$$
$$p=\pm\frac{1}{7}$$

[REMARKS]
Be careful. It is quite common for students to miss out $p=-\frac{1}{7}$ as one of the answers.

2 DHS Prelim 9758/2020/01/Q6b modified

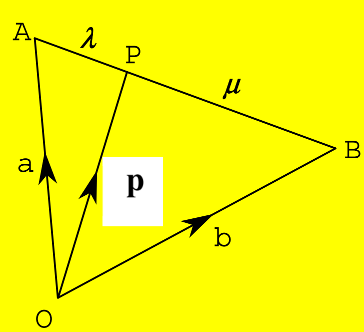


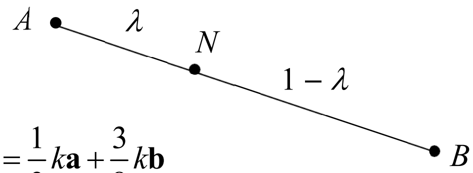
With reference to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

It is given that $\overrightarrow{OX} = \frac{2}{3}\mathbf{a}$, $\overrightarrow{OY} = \frac{3}{4}\mathbf{b}$ and the line ON bisects the line XY at the point M.

(i) By considering the ratio $XM : MY$, find the vector $\overrightarrow{OM}$ in terms of **a** and **b**. [1]

(ii) Given that $AN : NB = \lambda : 1-\lambda$ and $ON : OM = k : 1$ where λ and k are real constants, find the ratio $AN : NB$. [4]

(i)	$\overrightarrow{OX} = \frac{2}{3}\mathbf{a}$ and $\overrightarrow{OY} = \frac{3}{4}\mathbf{b}$ Since line ON bisects the line XY at the point M, $XM : MY = 1:1$ $\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OX} + \overrightarrow{OY}\right) = \frac{1}{3}\mathbf{a} + \frac{3}{8}\mathbf{b}$ Recall Ratio Theorem : 

	Let P be a point which divides AB in the ratio $\lambda : \mu$, then $\mathbf{p} = \dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$ (MF27).
(ii)	Since $AN : NB = \lambda : 1-\lambda$ for some $\lambda \in \mathbb{R}$, $\overrightarrow{ON} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$  Moreover, since $ON : OM = k : 1$, $\overrightarrow{ON} = k\overrightarrow{OM} = \frac{1}{3}k\mathbf{a} + \frac{3}{8}k\mathbf{b}$ Hence, $\frac{1}{3}k\mathbf{a} + \frac{3}{8}k\mathbf{b} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$. Vectors $\mathbf{a}$ *and* $\mathbf{b}$ are not parallel and non-zero: $\frac{3}{8}k = \lambda \Rightarrow k = \frac{8}{3}\lambda$ $\frac{1}{3}(\frac{8}{3}\lambda) = 1-\lambda \Rightarrow \lambda = \frac{9}{17}$ Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero and **non-parallel** vectors: If $\alpha\mathbf{a} + \beta\mathbf{b} = s\mathbf{a} + t\mathbf{b}$ for some $\alpha, \beta, s, t \in \mathbb{R}$, then $\alpha = s$, $\beta = t$. $\therefore AN : NB = 9 : 8$

3 **(a)** The non-zero vectors **a**, **b** and **c** are such that $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$. Given that $\mathbf{b}\neq-\mathbf{c}$, find a linear relationship between **a**, **b** and **c**. [3]

(b) The variable vector $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ satisfies the equation $\mathbf{v}\times(\mathbf{i}-3\mathbf{k})=\mathbf{j}$. Find the set of vectors **v** and describe this set geometrically. [3]

(a)	$\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$ $\mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}=\mathbf{0}$ $\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0}$ $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{0}$ Important property of cross product : $\mathbf{u}\times\mathbf{v}=-\mathbf{v}\times\mathbf{u}$ $\mathbf{a}\times\mathbf{b}=\mathbf{0}\Rightarrow\mathbf{a}=\mathbf{0}\ or\ \mathbf{b}=\mathbf{0}\ or\ \mathbf{a}\parallel\mathbf{b}$ Given that $\mathbf{a}\neq\mathbf{0}$ and $\mathbf{b}\neq-\mathbf{c}$ which implies $\mathbf{b}+\mathbf{c}\neq\mathbf{0}$, **a** is parallel to $(\mathbf{b}+\mathbf{c})$ $\therefore\mathbf{a}=\lambda(\mathbf{b}+\mathbf{c}),\ \lambda\neq0$.
(b)	$\begin{pmatrix}a\\b\\c\end{pmatrix}\times\begin{pmatrix}1\\0\\-3\end{pmatrix}=\begin{pmatrix}-3b\\3a+c\\-b\end{pmatrix}$ $\begin{pmatrix}-3b\\3a+c\\-b\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}\Rightarrow b=0\ and\ 3a+c=1$ Since cross product satisfies the equation, the values of a, b and c are restricted by RHS of the equation. Now a can still vary, but $b=0$, and value of c depends on a. $\begin{matrix}a=a\\b=0\\c=1-3a\end{matrix}\Rightarrow\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a\\0\\1-3a\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}+a\begin{pmatrix}1\\0\\-3\end{pmatrix}$ $\mathbf{v}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-3\end{pmatrix},\ \lambda\in\mathbb{R}$. This equation is in the form of 'equation of a line' $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. Hence describe it in terms of a line. **v** describe the set of position vectors of all points on the line which passes through the point (0, 0, 1) and is parallel to $\mathbf{i}-3\mathbf{k}$.

4 ACJC Promo 9758/2020/Q8

The lines l and m are defined by the equations

$$l : \mathbf{r} = \mathbf{i} - \mathbf{k} + \lambda(2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}),$$
$$m : \frac{x-1}{4} = \frac{a-y}{a} = \frac{z+3}{4}.$$

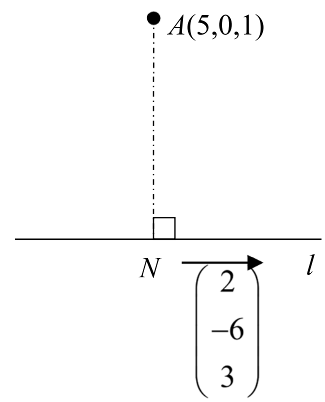
(i) Given that the lines intersect, show that $a = 6$. [2]

(ii) Find the position vector of N, the foot of perpendicular from the point $A(5, 0, 1)$ to the line l. [3]

(iii) Find the position vector of the two points on l that are 5 units from A. [3]

Solution :

(i)	$l : \mathbf{r} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-6\\3\end{pmatrix}$ $\quad m : \mathbf{r} = \begin{pmatrix}1\\a\\-3\end{pmatrix} + \mu\begin{pmatrix}4\\-a\\4\end{pmatrix}$ $m : \frac{x-1}{4} = \frac{a-y}{a} = \frac{z+3}{4}$ Let $\frac{x-1}{4} = \frac{a-y}{a} = \frac{z+3}{4} = \mu$ $\Rightarrow x = 1+4\mu, \quad y = a - a\mu, \quad z = -3+4\mu$ $\Rightarrow \mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+4\mu\\a-a\mu\\-3+4\mu\end{pmatrix} = \begin{pmatrix}1\\a\\-3\end{pmatrix} + \mu\begin{pmatrix}4\\-a\\4\end{pmatrix}$ For intersection, $\begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-6\\3\end{pmatrix} = \begin{pmatrix}1\\a\\-3\end{pmatrix} + \mu\begin{pmatrix}4\\-a\\4\end{pmatrix}$ $\Rightarrow \left.\begin{matrix}1+2\lambda = 1+4\mu\\ -6\lambda = a - a\mu\\ -1+3\lambda = -3+4\mu\end{matrix}\right\} \Rightarrow \begin{matrix}2\lambda - 4\mu = 0 \cdots (1)\\ -6\lambda = a - a\mu \cdots (2)\\ 3\lambda - 4\mu = -2 \cdots (3)\end{matrix}$ Solving the first and third equation gives $\lambda = -2, \mu = -1$ Because the lines intersect, therefore $\lambda = -2, \mu = -1$ satisfies the second equation $-6(-2) = a - a(-1) \Rightarrow 2a = 12 \Rightarrow a = 6$ (shown)

(ii)	$l:\mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-6\\3\end{pmatrix}$ $\quad \overrightarrow{OA}=\begin{pmatrix}5\\0\\1\end{pmatrix}$  Since N is on l, $\overrightarrow{ON}=\begin{pmatrix}1+2\lambda\\-6\lambda\\-1+3\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\Rightarrow \overrightarrow{AN}=\begin{pmatrix}-4+2\lambda\\-6\lambda\\-2+3\lambda\end{pmatrix}$ $\overrightarrow{AN}$ is perpendicular to l, therefore $\begin{pmatrix}-4+2\lambda\\-6\lambda\\-2+3\lambda\end{pmatrix}.\begin{pmatrix}2\\-6\\3\end{pmatrix}=0$ $\Rightarrow(-8-6)+\lambda(4+36+9)=0$ $\Rightarrow \lambda=\frac{2}{7}$ $\overrightarrow{ON}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\frac{2}{7}\begin{pmatrix}2\\-6\\3\end{pmatrix}=\frac{1}{7}\left(\begin{pmatrix}7\\0\\-7\end{pmatrix}+\begin{pmatrix}4\\-12\\6\end{pmatrix}\right)=\frac{1}{7}\begin{pmatrix}11\\-12\\-1\end{pmatrix}.$ Remarks : If the question ask for coordinates of N (instead of position vector of N), you have to conclude your answer as $\left(\frac{11}{7},-\frac{12}{7},-\frac{1}{7}\right)$.
(iii)	Let B be a point on l such that $AB=5$. Then $\overrightarrow{OB}=\begin{pmatrix}1+2\lambda\\-6\lambda\\-1+3\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\Rightarrow \overrightarrow{AB}=\begin{pmatrix}1+2\lambda\\-6\lambda\\-1+3\lambda\end{pmatrix}-\begin{pmatrix}5\\0\\1\end{pmatrix}=\begin{pmatrix}-4+2\lambda\\-6\lambda\\-2+3\lambda\end{pmatrix}$

	$\therefore AB = \left\| \begin{pmatrix} -4+2\lambda \\ -6\lambda \\ -2+3\lambda \end{pmatrix} \right\| = 5$ $\Rightarrow \sqrt{(2\lambda-4)^2+(-6\lambda)^2+(3\lambda-2)^2} = 5$ $\Rightarrow (4+36+9)\lambda^2+(-16-12)\lambda+16+4=25$ $\Rightarrow 49\lambda^2-28\lambda-5=0$ $\Rightarrow (7\lambda-5)(7\lambda+1)=0$ $\Rightarrow \lambda=\frac{5}{7}$ or $\lambda=-\frac{1}{7}$ $\therefore \overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{5}{7}\begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$ or $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{7}\begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$ $= \frac{1}{7}\begin{pmatrix} 17 \\ -30 \\ 8 \end{pmatrix}$ $\quad = \frac{1}{7}\begin{pmatrix} 5 \\ 6 \\ -10 \end{pmatrix}$ Note that point N is the midpoint of these two possible positions of B.

5 9740/2015/02/Q2

The line L has equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$.

(i) Find the acute angle between L and the x-axis. **[2]**

The point P has position vector $2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

(ii) Find the points on L which are a distance of $\sqrt{(33)}$ from P. Hence or otherwise find the point on L which is closest to P. **[5]**

(iii) Find a cartesian equation of the plane that includes the line L and the point P. **[3]**

Solution :

(i)	$L: \quad \mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$ 1) x-axis is parallel to **i** 2) Angle between **a** and **b** $= \cos^{-1} \dfrac{\mathbf{a} \bullet \mathbf{b}}{\lvert\mathbf{a}\rvert \lvert\mathbf{b}\rvert}$ Acute angle between L and the x-axis is $\cos^{-1} \dfrac{\left\lvert \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} \right\rvert \left\lvert \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rvert} = \cos^{-1} \dfrac{2}{7} = 73.4 \text{ (nearest } 0.1^\circ)$ Note the difference between 2 vectors and 2 lines.
(ii)	Let A be a point on L such that $AP = \sqrt{(33)}$. Since A lies on L, its position vector satisfies the equation of L. $\overrightarrow{OA} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. $\Rightarrow \overrightarrow{AP} = \begin{pmatrix} 1-2\lambda \\ 7-3\lambda \\ -2+6\lambda \end{pmatrix}$ $AP = \sqrt{(33)} \quad \Rightarrow (1-2\lambda)^2 + (7-3\lambda)^2 + (-2+6\lambda)^2 = 33$ $\Rightarrow 7\lambda^2 - 10\lambda + 3 = 0$ $\Rightarrow (7\lambda - 3)(\lambda - 1) = 0 \Rightarrow \lambda = \dfrac{3}{7}$ or 1

	$\therefore \overrightarrow{OA} = \frac{1}{7}\begin{pmatrix}13\\-5\\-46\end{pmatrix}$ or $\begin{pmatrix}3\\1\\-10\end{pmatrix}$ Point on L closest to P is the midpoint of the above two points. $\therefore$ Required position vector is $\frac{1}{2}\left[\frac{1}{7}\begin{pmatrix}13\\-5\\-46\end{pmatrix}+\begin{pmatrix}3\\1\\-10\end{pmatrix}\right]=\frac{1}{7}\begin{pmatrix}17\\1\\-58\end{pmatrix}$. That is, the point closest to P is $\left(\frac{17}{7},\frac{1}{7},\frac{-58}{7}\right)$.
(iii)	Let B be the point on L such that $\overrightarrow{OB}=\begin{pmatrix}1\\-2\\-4\end{pmatrix}$. Then $\overrightarrow{BP}=\begin{pmatrix}1\\7\\-2\end{pmatrix}$. $\begin{pmatrix}1\\7\\-2\end{pmatrix}\times\begin{pmatrix}2\\3\\-6\end{pmatrix}=\begin{pmatrix}-36\\2\\-11\end{pmatrix}$ $\mathbf{r}.\begin{pmatrix}-36\\2\\-11\end{pmatrix}=\begin{pmatrix}1\\-2\\-4\end{pmatrix}.\begin{pmatrix}-36\\2\\-11\end{pmatrix}=4$ Cartesian equation of plane containing L and P is $-36x+2y-11z=4$.

Vectors Equation of a plane : $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$, where **b** and **c** are parallel to the plane.

To find a vector equation of a plane, we need
- the position vector of a point on plane,
- 2 vectors parallel to the plane.

To find the scalar product or Cartesian form of a plane, we need
- the position vector of a point on plane,
- a normal to the plane (take cross product of 2 vectors parallel to the plane).

Extra Question

(iv) Let the plane in (iii) with equation $-36x+2y-11z=4$ be p_2. What is the distance from the origin to p_2?

Distance from origin to plane $\mathbf{r}\bullet\mathbf{n}=d$ is $\dfrac{|d|}{|\mathbf{n}|}$.

Alt : Consider a point $E(0,2,0)$ on the plane. Then

Required length = length of projection of $\overrightarrow{OE}$ on normal of p_2, $\begin{pmatrix}-36\\2\\-11\end{pmatrix}$.

Required distance $=\dfrac{\overrightarrow{OE}\bullet\begin{pmatrix}-36\\2\\-11\end{pmatrix}}{\left|\begin{pmatrix}-36\\2\\-11\end{pmatrix}\right|}=\dfrac{4}{\sqrt{1421}}$

(v) Find the acute angle between p_2 and the plane $\mathbf{r}\bullet\begin{pmatrix}1\\0\\1\end{pmatrix}=1$?

Let the required angle be θ.

Angles between 2 planes = angles between their normals

$$\cos\theta=\left|\frac{\begin{pmatrix}1\\0\\1\end{pmatrix}\bullet\begin{pmatrix}36\\-2\\11\end{pmatrix}}{\sqrt{2}\sqrt{1421}}\right|=\frac{47}{\sqrt{2}\sqrt{1421}}\Rightarrow\theta=28.2^\circ\,(1\text{dp})$$

6 JPJC Promo 9758/2020/Q9 modified

The plane p_1 has equation $\mathbf{r} = \lambda\begin{pmatrix}1\\2\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\1\end{pmatrix}$, where λ and μ are real parameters..

(i) Find an equation of p_1 in the form $\mathbf{r.n} = d$. [3]

The plane p_2 has equation $2x - y + z = 12$.

(ii) State the relationship between p_1 and p_2. [1]

The line l has equation $\mathbf{r} = t\begin{pmatrix}3\\2\\2\end{pmatrix}$, where t is a real parameter.

(iii) Find the acute angle between l and p_1. [2]

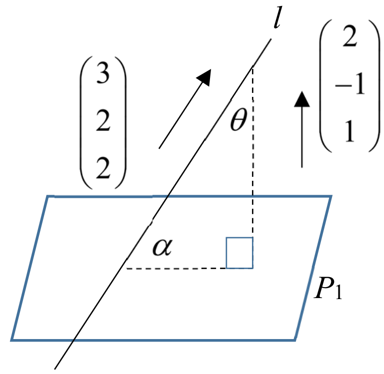
(iv) Find the foot of perpendicular from the origin to p_2. Hence, or otherwise, find the exact distance between p_1 and p_2. [4]

Solution :

(i)	$p_1 : \mathbf{r} = \lambda\begin{pmatrix}1\\2\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\1\end{pmatrix}$, where λ and μ are real parameters. Normal of p_1 is $\begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$ Since p_1 contains the origin, $p_1 : \mathbf{r}.\begin{pmatrix}2\\-1\\1\end{pmatrix} = 0$
(ii)	Since the normals of p_1 and p_2 are the same, p_1 and p_2 are parallel planes.

(iii)

$$p_1 : \mathbf{r}.\begin{pmatrix}2\\-1\\1\end{pmatrix} = 0$$



$$\cos\theta = \frac{\left|\begin{pmatrix}3\\2\\2\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix}\right|}{\sqrt{17}\sqrt{6}} = \frac{6}{\sqrt{102}}$$

$\theta = 53.552°$

Acute angle between l and $p_1 = \alpha = 90 - 53.552 = 36.4°$ (1 dp)

Alternatively,
Let the angle required be α.

$$\sin\alpha = \frac{\left|\begin{pmatrix}3\\2\\2\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix}\right|}{\sqrt{17}\sqrt{6}} = \frac{6}{\sqrt{102}}$$

$$\Rightarrow \alpha = 36.4°$$

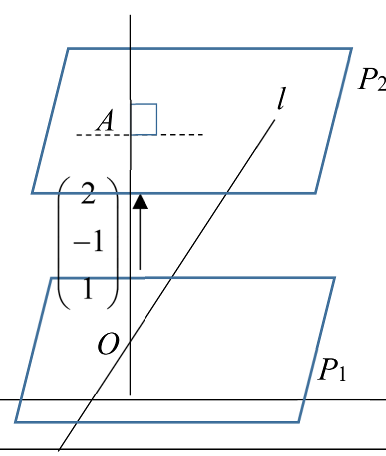
(iv)

$$p_2 : 2x - y + z = 12 \Leftrightarrow \mathbf{r}.\begin{pmatrix}2\\-1\\1\end{pmatrix} = 12$$

Note that p_1 and p_2 are parallel since they share the same normal vector $\begin{pmatrix}2\\-1\\1\end{pmatrix}$

Let A be the foot of perpendicular from O to p_2.

$$\overrightarrow{OA} = \mu\begin{pmatrix}2\\-1\\1\end{pmatrix} \text{ for some } \mu \in \mathbb{R}.$$



Since A lies on p_2,

$$\begin{pmatrix}2\mu\\-\mu\\\mu\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix} = 12$$

$4\mu + \mu + \mu = 12$

	$6\mu = 12$ $\mu = 2$ $\overrightarrow{OA} = 2\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$ Since O is on p_1 and p_1 is parallel to p_2. distance between p_1 and $p_2 = \left\|\overrightarrow{OA}\right\| = \sqrt{4^2 + (-2)^2 + 2^2} = \sqrt{24} = 2\sqrt{6}$

More Practice Questions

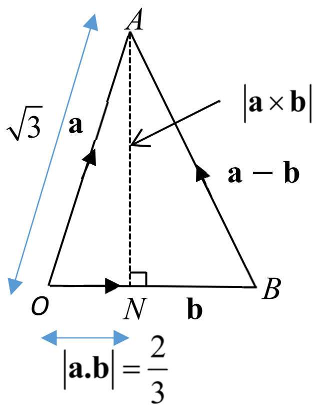
7 MI PU3 Mid-Year CT 9758/2018/01/Q3

Relative to the origin O, two points A and B have position vectors a and b respectively. It is given that $\mathbf{b}$ is a unit vector, $|\mathbf{a}| = \sqrt{3}$, and $|4\mathbf{a} - 3\mathbf{b}| = \sqrt{41}$. θ is defined as the acute angle between $\mathbf{a}$ and $\mathbf{b}$.

(i) By considering the scalar product $(4\mathbf{a} - 3\mathbf{b})\cdot(4\mathbf{a} - 3\mathbf{b})$, find θ. [4]

(ii) Give the geometrical meaning of $|(\mathbf{a} - \mathbf{b}) \times \mathbf{b}|$ and find its exact value. [2]

(i)	$|\mathbf{a}| = \sqrt{3}$, $|\mathbf{b}| = 1$ and $|4\mathbf{a} - 3\mathbf{b}| = \sqrt{41}$. $|4\mathbf{a} - 3\mathbf{b}| = \sqrt{41}$ $|4\mathbf{a} - 3\mathbf{b}|^2 = 41$ **p.p** $= |\mathbf{p}||\mathbf{p}|\cos 0$ for any vector $\mathbf{p}$ $(4\mathbf{a} - 3\mathbf{b})\bullet(4\mathbf{a} - 3\mathbf{b}) = 41$ $16\mathbf{a}\bullet\mathbf{a} - 12\mathbf{a}\bullet\mathbf{b} - 12\mathbf{a}\bullet\mathbf{b} + 9\mathbf{b}\bullet\mathbf{b} = 41$ $16|\mathbf{a}|^2 - 24\mathbf{a}\bullet\mathbf{b} + 9|\mathbf{b}|^2 = 41$ $48 - 24\mathbf{a}\bullet\mathbf{b} + 9 = 41$ $24\mathbf{a}\bullet\mathbf{b} = 16$ $\mathbf{a}\bullet\mathbf{b} = \frac{2}{3}$ $\cos\theta = \frac{2/3}{|\mathbf{a}||\mathbf{b}|} = \frac{2/3}{(\sqrt{3})(1)}$ Definition of **a.b** $= |\mathbf{a}||\mathbf{b}|\cos\theta$ $\cos\theta = \frac{2}{3\sqrt{3}}$ $\theta = \cos^{-1}\left(\frac{2}{3\sqrt{3}}\right) \approx 67.4^\circ$ (to 1 d.p.)
(ii)	$|(\mathbf{a} - \mathbf{b}) \times \mathbf{b}| = |\mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{b}| = |\mathbf{a} \times \mathbf{b}|$ since $\mathbf{b} \times \mathbf{b} = \mathbf{0}$



Geometrical meaning of $|(\mathbf{a}-\mathbf{b})\times\mathbf{b}| = |\mathbf{a}\times\mathbf{b}|$:

(1) 2 times the area of ΔOAB, or
(2) Area of the parallelogram with adjacent sides *OA* and *OB,* or
(3) Perpendicular (or shortest) distance from the point *A* to the line through *O* and *B,* since $\mathbf{b}$ is a unit vector.

Method 1: Area

$= 2\times$ Area of ΔOAB

$$= 2\times\frac{1}{2}(1)\sqrt{\left(\sqrt{3}\right)^2-\left(\frac{2}{3}\right)^2}$$

$$= \frac{\sqrt{23}}{3}$$

Method 2: Perpendicular distance

$|\mathbf{a}\times\mathbf{b}| =$ Distance of *A* to line *OB*

$$= \sqrt{OA^2-(\mathbf{a}.\mathbf{b})^2}$$

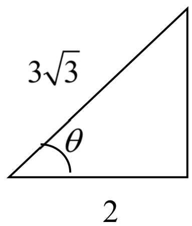
$$= \sqrt{\left(\sqrt{3}\right)^2-\left(\frac{2}{3}\right)^2}$$

$$= \frac{\sqrt{23}}{3}$$

Method 3: Cross product formula

$$\left|(\mathbf{a}-\mathbf{b})\times\mathbf{b}\right| = \left|\mathbf{a}\times\mathbf{b}\right| \quad \text{since } \mathbf{b}\times\mathbf{b}=\mathbf{0}$$
$$= |\mathbf{a}||\mathbf{b}|\sin\theta$$

Since $\cos\theta = \dfrac{2}{3\sqrt{3}}$ from part **(i)**,



$$\sqrt{\left(3\sqrt{3}\right)^2 - 2^2} = \sqrt{23}$$

$$\sin\theta = \frac{\sqrt{23}}{3\sqrt{3}}$$

$$\therefore |\mathbf{a}||\mathbf{b}|\sin\theta = \left(\sqrt{3}\right)(1)\left(\frac{\sqrt{23}}{3\sqrt{3}}\right) = \frac{\sqrt{23}}{3}$$

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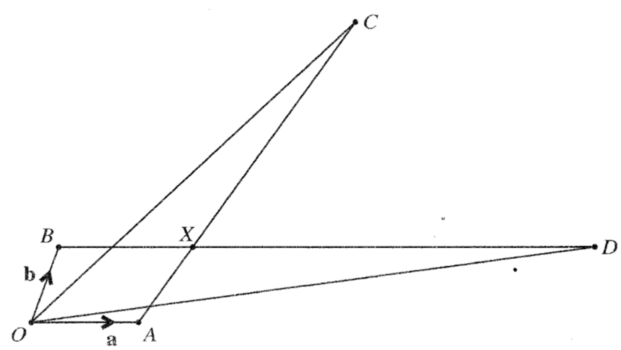

With reference to the origin O, the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = 2\mathbf{a} + 4\mathbf{b}$ and $\overrightarrow{OD} = \mathbf{b} + 5\mathbf{a}$. The lines BD and AC cross at X (see diagram).

(i) Express $\overrightarrow{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

The point Y lies on CD and is such that the points O, X and Y are collinear.

(ii) Express $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and find the ratio $OX : OY$. [6]

(i)	Since qn asks to express $\overrightarrow{OX}$ in terms of **a** and **b**, we try to find a relationship between $\overrightarrow{OX}$, $\overrightarrow{OA}$ and $\overrightarrow{OB}$. $\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX} = \underset{\sim}{b} + \lambda\overrightarrow{BD} = \underset{\sim}{b} + \lambda\left(\overrightarrow{OD} - \overrightarrow{OB}\right) = \underset{\sim}{b} + \lambda\left(\underset{\sim}{b} + 5\underset{\sim}{a} - \underset{\sim}{b}\right) = \underset{\sim}{b} + 5\lambda\underset{\sim}{a}$ $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \underset{\sim}{a} + \mu\overrightarrow{AC} = \underset{\sim}{a} + \mu\left(\overrightarrow{OC} - \overrightarrow{OA}\right) = \underset{\sim}{a} + \mu\left(2\underset{\sim}{a} + 4\underset{\sim}{b} - \underset{\sim}{a}\right) = (1+\mu)\underset{\sim}{a} + 4\mu\underset{\sim}{b}$ We form two expressions for $\overrightarrow{OX}$ and apply the following result: Let **a** and **b** be non-zero and **non-parallel** vectors: If $\alpha\mathbf{a} + \beta\mathbf{b} = s\mathbf{a} + t\mathbf{b}$ for some $\alpha, \beta, s, t \in \mathbb{R}$, then $\alpha = s$, $\beta = t$. Since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero and non-parallel, $\underset{\sim}{b} + 5\lambda\underset{\sim}{a} = (1+\mu)\underset{\sim}{a} + 4\mu\underset{\sim}{b}$ $4\mu = 1 \Rightarrow \mu = \frac{1}{4}$ $5\lambda = 1 + \mu = \frac{5}{4} \Rightarrow \lambda = \frac{1}{4}$ $\overrightarrow{OX} = \underset{\sim}{b} + \frac{5}{4}\underset{\sim}{a}$
(ii)	*Y* is collinear with *C*, *D* and we let the ratio $\overrightarrow{OY} = \alpha\overrightarrow{OD} + (1-\alpha)\overrightarrow{OC} = \alpha\left(\underset{\sim}{b} + 5\underset{\sim}{a}\right) + (1-\alpha)\left(2\underset{\sim}{a} + 4\underset{\sim}{b}\right)$ Similarly, *Y* is collinear with *O*, *X* $\overrightarrow{OY} = \beta\overrightarrow{OX} = \beta\left(\underset{\sim}{b} + \frac{5}{4}\underset{\sim}{a}\right)$ $\alpha\left(\underset{\sim}{b} + 5\underset{\sim}{a}\right) + (1-\alpha)\left(2\underset{\sim}{a} + 4\underset{\sim}{b}\right) = \beta\left(\underset{\sim}{b} + \frac{5}{4}\underset{\sim}{a}\right)$ $\beta = \alpha + 4(1-\alpha) \Rightarrow \beta + 3\alpha = 4$ ...(1) $\frac{5}{4}\beta = 5\alpha + 2(1-\alpha) \Rightarrow \frac{5}{4}\beta - 3\alpha = 2$ ...(2) $(1)+(2)$: $\frac{9}{4}\beta = 6$ $\therefore \beta = \frac{24}{9} = \frac{8}{3}$ $\overrightarrow{OY} = \beta\overrightarrow{OX} = \frac{24}{9}\left(\underset{\sim}{b} + \frac{5}{4}\underset{\sim}{a}\right) = \frac{8}{3}\underset{\sim}{b} + \frac{10}{3}\underset{\sim}{a}$ $OX : OY = 3 : 8$

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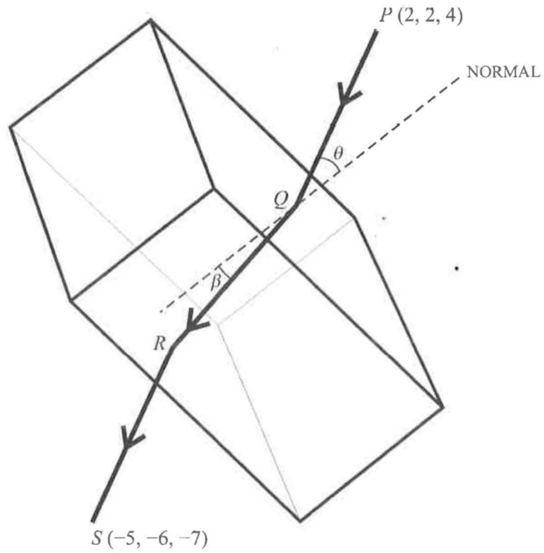

A ray of light passes from air into a material made into a rectangular prism. The ray of light is sent in direction $\begin{pmatrix}-2\\-3\\-6\end{pmatrix}$ from a light source at the point P with coordinates (2,2,4). The prism is placed so that the ray of light passes through the prism, entering at the point Q and emerging at the point R and is picked up by a sensor at point S with coordinates $(-5,-6,-7)$. The acute angle between PQ and the normal to the top of the prism at Q is θ and the acute angle between QR and the same normal is β (see diagram).

It is given that the top of the prism is a part of the plane $x+y+z=1$, and that the base of the prism is a part of the plane $x+y+z=-9$. It is also given that the ray of light along PQ is parallel to the ray of light along RS so that P, Q, R and S lie in the same plane.

(i) Find the exact coordinates of Q and R. [5]

(ii) Find the values of $\cos\theta$ and $\cos\beta$. [3]

(iii) Find the thickness of the prism measured in the direction of the normal at Q. [3]

Snell's law states that $\sin\theta = k\sin\beta$, where k is a constant called the refractive index.

(iv) Find k for the material of this prism. [1]

(v) What can be said about the value of k for a material for which $\beta > \theta$? [1]

(i)	$\overrightarrow{PQ}=\lambda\begin{pmatrix}-2\\-3\\-6\end{pmatrix}, \lambda>0$ and Q lies on $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1$ $\overrightarrow{OQ}=\lambda\begin{pmatrix}-2\\-3\\-6\end{pmatrix}+\begin{pmatrix}2\\2\\4\end{pmatrix}$ $\overrightarrow{OQ}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1\Rightarrow -2\lambda+2-3\lambda+2-6\lambda+4=1\Rightarrow \lambda=\frac{7}{11}$ Q is $\left(2-\frac{14}{11},2-\frac{21}{11},4-\frac{42}{11}\right)=\left(\frac{8}{11},\frac{1}{11},\frac{2}{11}\right)$ $\overrightarrow{RS}=\mu\begin{pmatrix}-2\\-3\\-6\end{pmatrix}, \mu>0$ and R lies on $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9$ $\overrightarrow{OR}=\begin{pmatrix}-5\\-6\\-7\end{pmatrix}-\mu\begin{pmatrix}-2\\-3\\-6\end{pmatrix}$ $\overrightarrow{OR}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9\Rightarrow -5+2\mu-6+3\mu-7+6\mu=-9\Rightarrow \mu=\frac{9}{11}$ R is $\left(-5+\frac{18}{11},-6+\frac{27}{11},-7+\frac{54}{11}\right)=\left(-\frac{37}{11},-\frac{39}{11},-\frac{23}{11}\right)$ Q can be seen as the point of intersection between line PQ and plane that contains the top of the prism. Similarly, R can be seen as the point of intersection between line RS and plane that contains the base of the prism. Answers need to be exact.

(ii)	$\cos\theta = \dfrac{\left\|\begin{pmatrix}-2\\-3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|}{\sqrt{4+9+36}\sqrt{3}} = \dfrac{11}{7\sqrt{3}}$ $\overrightarrow{RQ} = \dfrac{1}{11}\begin{pmatrix}8\\1\\2\end{pmatrix} - \dfrac{1}{11}\begin{pmatrix}-37\\-39\\-23\end{pmatrix} = \dfrac{1}{11}\begin{pmatrix}45\\40\\25\end{pmatrix}$ $\cos\beta = \dfrac{\left\|\begin{pmatrix}45\\40\\25\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|}{\sqrt{45^2+40^2+25^2}\sqrt{3}} = \dfrac{110}{\sqrt{12750}} = \dfrac{22}{\sqrt{510}}$
(iii)	Thickness of the prism is $\dfrac{\left\|\overrightarrow{RQ}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|}{\sqrt{3}} = \dfrac{\frac{1}{11}(110)}{\sqrt{3}} = \dfrac{10}{\sqrt{3}}$ Thickness of prism is length of projection of $\overrightarrow{RQ}$ onto normal of prism.
(iv)	$k = \dfrac{\sin\theta}{\sin\beta} = \sqrt{\dfrac{1-\cos^2\theta}{1-\cos^2\beta}} = \sqrt{\dfrac{1-\left(\frac{11}{7\sqrt{3}}\right)^2}{1-\left(\frac{22}{\sqrt{510}}\right)^2}} = \dfrac{\sqrt{170}}{7} = 1.86\ (3\text{ s.f.})$
(v)	Since $0^\circ < \theta < \beta < 90^\circ$, we have $0 < \sin\theta < \sin\beta < 1$. Thus, $0 < \dfrac{\sin\theta}{\sin\beta} < 1$, that is, $0 < k < 1$.