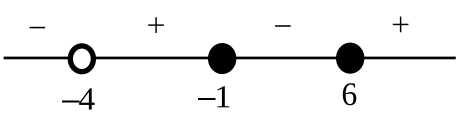
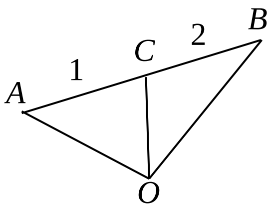
Solution to 9758/01:

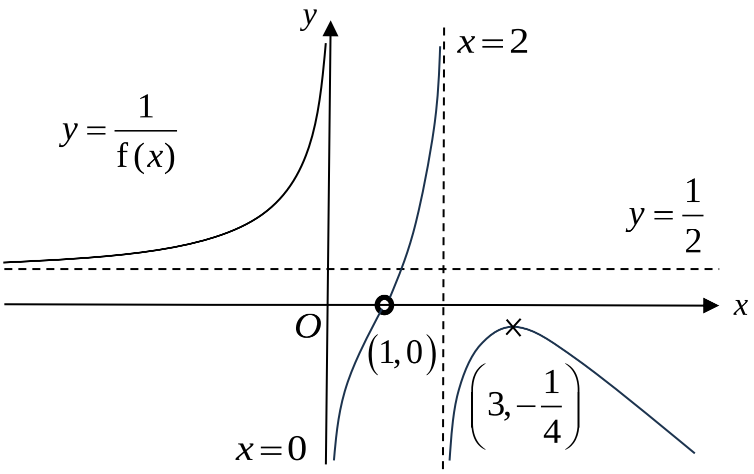
Qn	Solution
1(i) **[2]**	$S_n = n^2 + kn$ $S_{n-1} = (n-1)^2 + k(n-1)$ $\quad = n^2 - 2n + 1 + kn - k$ $u_n = S_n - S_{n-1}$ $\quad = n^2 + kn - (n^2 - 2n + 1 + kn - k)$ $\quad = n^2 + kn - n^2 + 2n - 1 - kn + k$ $\quad = 2n - 1 + k$
1(ii) **[2]**	$u_{n-1} = 2(n-1) - 1 + k$ $\quad = 2n - 2 - 1 + k$ $\quad = 2n - 3 + k$ $u_n - u_{n-1} = 2n - 1 + k - (2n - 3 + k)$ $\quad = 2n - 1 + k - 2n + 3 - k$ $\quad = 2$ Since $u_n - u_{n-1} = 2$ which is a constant independent of *n*, the sequence is an arithmetic progression.
2 **(i)** **[4]**	$\dfrac{x^2 - 3x + 2}{x+4} \geq 2$ $\dfrac{x^2 - 3x + 2}{x+4} - 2 \geq 0$ $\dfrac{x^2 - 3x + 2 - 2x - 8}{x+4} \geq 0$ $\dfrac{x^2 - 5x - 6}{x+4} \geq 0$ $\dfrac{(x+1)(x-6)}{x+4} \geq 0$  $-4 < x \leq -1$ or $x \geq 6$

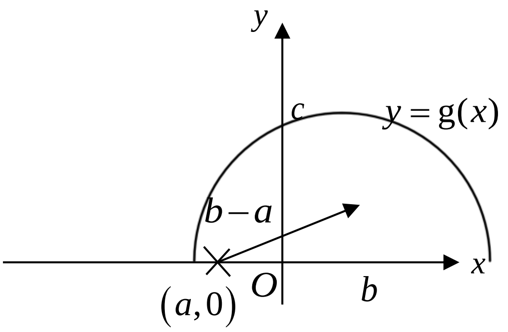
Qn	Solution
2 (ii) [4]	$\dfrac{x^2+3x+2}{4-x} \geq 2 \quad \Rightarrow \quad \dfrac{(-x)^2-3(-x)+2}{-x+4} \geq 2$ Replace x with $-x$: $-4<-x\leq -1$ or $-x\geq 6$ $4>x\geq 1$ or $x\leq -6$ $1\leq x<4$ or $x\leq -6$
3(i) [4]	$y=ax^3+bx^2+cx+d$ At $(0,7)$: $d=7$ At $(-2,-11)$: $-11=a(-2)^3+b(-2)^2+c(-2)+7$ $-8a+4b-2c=-18$ ..... (1) At $(1,1)$: $1=a(1)^3+b(1)^2+c(1)+7$ $a+b+c=-6$ ..... (2) At $(3,19)$: $19=a(3)^3+b(3)^2+c(3)+7$ $27a+9b+3c=12$ ..... (3) Using GC, $a=2$, $b=-3$ and $c=-5$
3(ii) [1]	Equation of C: $y=2x^3-3x^2-5x+7$ $y=2x^3-3x^2-5x+7$ $\xrightarrow{\text{replace } x \text{ with } -x}$ $y=2(-x)^3-3(-x)^2-5(-x)+7$ $\xrightarrow{\text{replace } y \text{ with } y-1}$ $y-1=2(-x)^3-3(-x)^2-5(-x)+7$ $\Rightarrow \ y=2(-x)^3-3(-x)^2-5(-x)+8$ $\therefore$ Equation of resulting curve: $y=-2x^3-3x^2+5x+8$

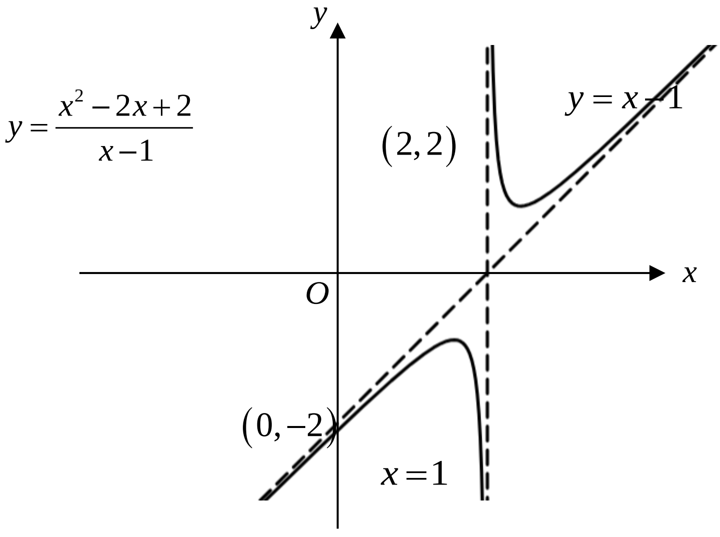
Qn	Solution	
4(i) **[2]**	**<u>Method 1</u>** $AC:AB=1:3$ $\Rightarrow AC:CB=1:2$ A 1 C 2 B $\mathbf{c}=\dfrac{(1)\mathbf{b}+(2)\mathbf{a}}{1+2}$ $=\dfrac{2\mathbf{a}+\mathbf{b}}{3}$	**<u>Method 2</u>** $AC:AB=1:3$ $\Rightarrow 3AC=AB$ $3\left(\overrightarrow{OC}-\overrightarrow{OA}\right)=\overrightarrow{OB}-\overrightarrow{OA}$ $3\left(\mathbf{c}-\mathbf{a}\right)=\mathbf{b}-\mathbf{a}$ $\mathbf{c}=\dfrac{\mathbf{b}}{3}-\dfrac{\mathbf{a}}{3}+\mathbf{a}$ $\mathbf{c}=\dfrac{2\mathbf{a}+\mathbf{b}}{3}$
4(ii) **[2]**	**<u>Method 1</u>** Area of triangle OAC $=\dfrac{1}{3}$ Area of triangle OAB  $=\dfrac{1}{3}\times\dfrac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$ $=\dfrac{1}{6}\left\|\mathbf{a}\times\mathbf{b}\right\|$ i.e. $k=\dfrac{1}{6}$ **<u>Method 2</u>** Area of triangle $OAC=\dfrac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OC}\right\|$ $=\dfrac{1}{2}\left\|\mathbf{a}\times\left(\dfrac{2\mathbf{a}+\mathbf{b}}{3}\right)\right\|$ $=\dfrac{1}{6}\left\|\mathbf{a}\times\left(2\mathbf{a}+\mathbf{b}\right)\right\|$ $=\dfrac{1}{6}\left\|2\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\right\|$ $=\dfrac{1}{6}\left\|\mathbf{a}\times\mathbf{b}\right\|$ i.e. $k=\dfrac{1}{6}$	
4(iii) **[3]**	$\left(2\mathbf{a}-\mathbf{b}\right)\cdot\mathbf{c}=\left(2\mathbf{a}-\mathbf{b}\right)\cdot\left(\dfrac{2\mathbf{a}+\mathbf{b}}{3}\right)$ $=\dfrac{1}{3}\left(2\mathbf{a}-\mathbf{b}\right)\cdot\left(2\mathbf{a}+\mathbf{b}\right)$ $=\dfrac{1}{3}\left(4\mathbf{a}\cdot\mathbf{a}+\underbrace{2\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{a}}_{0}-\mathbf{b}\cdot\mathbf{b}\right)$ $=\dfrac{1}{3}\left(4\left\|\mathbf{a}\right\|^2-\left\|\mathbf{b}\right\|^2\right)$ $=\dfrac{1}{3}\left(4(1)^2-(1)^2\right)$ since $\mathbf{a},\mathbf{b}$ are unit vectors $=1$	

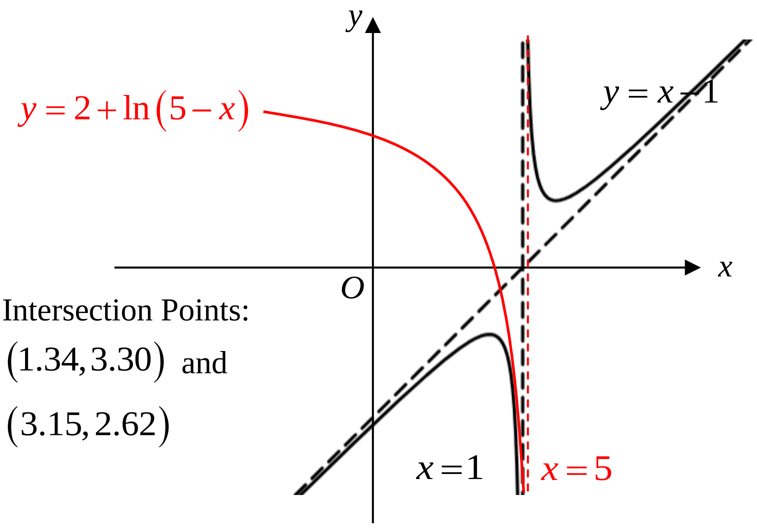
<table>
<tr><th>Qn</th><th colspan="2">Solution</th></tr>
<tr><td>5(a)
[2]</td><td><u>Sequence 1</u>
$\frac{x^2}{9}+y^2=1$
↓ Replace x with $3x$
$\frac{(3x)^2}{9}+y^2=1$
$\Rightarrow x^2+y^2=1$
↓ Replace x with $x-2$
$(x-2)^2+y^2=1$

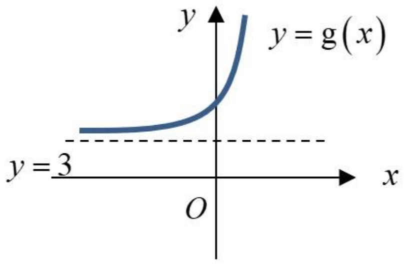
1. Scaling parallel to x-axis by scale factor $\frac{1}{3}$.
2. Translation in the positive x-direction by 2 units.</td><td><u>Sequence 2</u>
$\frac{x^2}{9}+y^2=1$
↓ Replace x with $x-6$
$\frac{(x-6)^2}{9}+y^2=1$
↓ Replace x with $3x$
$\frac{(3x-6)^2}{9}+y^2=1$
$\Rightarrow \frac{3^2(x-2)^2}{9}+y^2=1$
$\Rightarrow (x-2)^2+y^2=1$

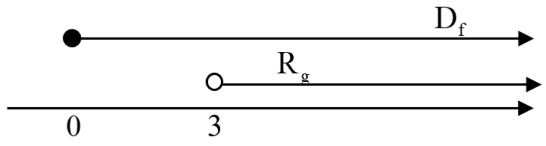
1. Translation in the positive x-direction by 6 units.
2. Scaling parallel to x-axis by scale factor $\frac{1}{3}$.</td></tr>
<tr><td>5(b)
[3]</td><td colspan="2">
</td></tr>
</table>

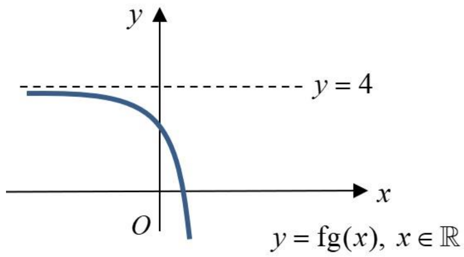
Qn	Solution
5(c) **[3]**	 Centre: (a, 0) Radius: $b - a$

Qn	Solution
6(i) **[2]**	$y=\dfrac{x^2-bx+b}{x-a}$. Asymptote: $x=1$. Turning point at $(0,-2)$. Vertical asymptote is $x=a \quad \Rightarrow \quad a=1$ At $(0,-2)$: $-2=\dfrac{b}{-1} \quad \Rightarrow \quad b=2$
6(ii) **[4]**	$y=\dfrac{x^2-2x+2}{x-1}=x-1+\dfrac{1}{x-1}$. Asymptotes: $x=1$ and $y=x-1$. 

Qn	Solution
6(iii) **[3]**	$\dfrac{x^2-2x+2}{x-1} \geq 2+\ln(5-x)$. $y=2+\ln(5-x)$ is the additional curve to be sketched. Asymptote: $5-x=0 \quad \Rightarrow \quad x=5$  Intersection Points: $(1.34, 3.30)$ and $(3.15, 2.62)$ From graph, for $\dfrac{x^2-2x+2}{x-1} \geq 2+\ln(5-x)$, $1 < x \leq 1.34$ or $3.15 \leq x < 5$ (3 s.f.) **Note:** The graph of $y=2+\ln(5-x)$ is to be sketched on the graph in part **(ii)** but is sketched separately here for clarity of illustration.

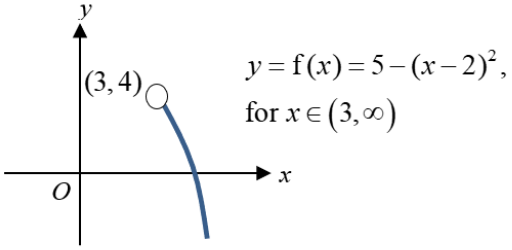
<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>7(i)
[2]</td><td>


$R_g = (3, \infty)$.
$D_f = [0, \infty)$.


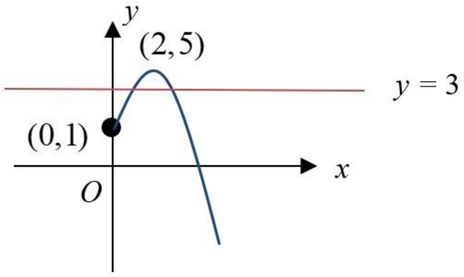
Since $R_g = (3, \infty) \subseteq [0, \infty) = D_f$, fg exists.
</td></tr>
<tr><td>7(ii)
[2]</td><td>
$\mathrm{fg}(x) = \mathrm{f}[\mathrm{g}(x)] = \mathrm{f}(e^x + 3)$
$= 5 - (e^x + 3 - 2)^2$
$= 5 - (e^x + 1)^2$
$D_{fg} = D_g = (-\infty, \infty)$ or $\mathbb{R}$
</td></tr>
<tr><td>7(iii)
[2]</td><td>
<u>Method 1: GC/Graph y = fg(x)</u>


$R_{fg} = (-\infty, 4)$

Note:
$\mathrm{fg}(x) = -(e^x + 1)^2 + 5$
$= -(e^{x^2} + 2e^x + 1) + 5$
$= -e^{x^2} - 2e^x + 4$
As $x \to \infty$, $e^x \to 0$,
$\mathrm{fg}(x) \to 4$
$\therefore y = 4$ is
horizontal asymptote

<u>Method 2: Mapping</u>
$D_{fg} = (-\infty, \infty) \xrightarrow{g} \underset{R_g}{(3, \infty)} \xrightarrow{f} (-\infty, 4) = R_{fg}$


$R_{fg} = (-\infty, 4)$
</td></tr>
</table>

7(iv) **[1]**	**<u>Method 1: Horizontal Line Test</u>**  Since there exists a horizontal line $y=3$ that intersects the graph of $y=\mathrm{f}\left(x\right)$ more than once, f is **not one-one** and f^{-1} does not exist. **<u>Method 2: Counterexamples</u>** $\mathrm{f}(0)=5-(0-2)^2=1$ $\mathrm{f}(4)=5-(4-2)^2=1$ Since $\mathrm{f}(0)=\mathrm{f}(4)=1$, f is **not one-one** and f^{-1} does not exist.
7(v) **[1]**	For f^{-1} to exists under domain $\left[k,\infty\right)$, f must be a one-one function. Hence, the smallest value of $k=2$.
7(vi) **[3]**	Let $y=\mathrm{f}(x)$ $y=5-(x-2)^2,\quad \text{for } x\in\left[2,\infty\right)$ $y-5=-\left(x-2\right)^2$ $-y+5=\left(x-2\right)^2$ $x-2=\pm\sqrt{-y+5}$ $x=2\pm\sqrt{5-y}$ Since $x\geq 2$, $x=2+\sqrt{5-y}$ $\left[\text{reject } 2-\sqrt{5-y} \text{ as } x\geq 2\right]$ $\therefore \mathrm{f}^{-1}\left(x\right)=2+\sqrt{5-x}$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\left(-\infty,5\right]$

Qn	Solution
8(i) **[4]**	Let T_n be the *n*th term of an AP. T_8, T_4, T_2 of AP are first 3 terms (consecutive terms) of a GP. $T_8 = a+7d$, $T_4 = a+3d$, $T_2 = a+d$ $r = \frac{T_4}{T_8} = \frac{T_2}{T_4}$ $\Rightarrow \frac{a+3d}{a+7d} = \frac{a+d}{a+3d}$ $(a+3d)^2 = (a+d)(a+7d)$ $a^2 + 6ad + 9d^2 = a^2 + 8ad + 7d^2$ $2ad - 2d^2 = 0$ $2d\left(a-d\right) = 0$ $d = 0$ (rejected) or $a - d = 0$ Since the terms of the arithmetic series are increasing, $d \neq 0$ $\therefore a - d = 0$ $a = d$ (shown)
8(ii) **[2]**	$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ $S_{20} = \frac{20}{2}\left[2a + (20-1)d\right]$ $630 = \frac{20}{2}\left[2a + (20-1)a\right]$ ($\because a = d$) $630 = 10(2a + 19a)$ $630 = 210a$ $a = 3$
8(iii) **[3]**	First term of GP = T_8 of AP $= a + 7d = 3 + 7\left(3\right) = 24$ Common ratio of GP $r = \frac{a+d}{a+3d} = \frac{3+3}{3+3\left(3\right)} = \frac{1}{2}$ $S_{10} = \frac{24\left(1 - \frac{1}{2}^{10}\right)}{1 - \frac{1}{2}}$ $= \frac{3069}{64}$ (or $47\frac{61}{64}$)

Qn	Solution
8(iv) **[2]**	From part (iii), $r = \frac{1}{2}$ Since $\lvert r \rvert = \frac{1}{2} < 1$, the geometric series converges. $S_\infty = \frac{24}{1-\frac{1}{2}} = 48$

Qn	Solution
9(i) **[2]**	$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix} 0 \\ 0 \\ 12 \end{pmatrix} - \begin{pmatrix} 0 \\ 15 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ -15 \\ 6 \end{pmatrix}$ $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 20 \\ 0 \\ 9 \end{pmatrix} - \begin{pmatrix} 0 \\ 15 \\ 6 \end{pmatrix} = \begin{pmatrix} 20 \\ -15 \\ 3 \end{pmatrix}$
9(ii) **[2]**	A normal vector to plane $PQR = \overrightarrow{PR} \times \overrightarrow{PQ}$ $= \begin{pmatrix} 0 \\ -15 \\ 6 \end{pmatrix} \times \begin{pmatrix} 20 \\ -15 \\ 3 \end{pmatrix}$ $= \begin{pmatrix} 45 \\ 120 \\ 300 \end{pmatrix}$ $= 15 \begin{pmatrix} 3 \\ 8 \\ 20 \end{pmatrix}$ Equation of plane PQR: $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 8 \\ 20 \end{pmatrix} \quad \Rightarrow \quad \mathbf{r} \cdot \begin{pmatrix} 3 \\ 8 \\ 20 \end{pmatrix} = 240$ (shown)
9(iii) **[2]**	Normal to plane OAB is parallel to z-axis $\Rightarrow$ Normal to plane OAB // $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Qn	Solution
	Let acute angle between plane *PQR* and the plane *OAB* be θ. $\cos\theta = \dfrac{\left\lvert \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}3\\8\\20\end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rvert \left\lvert \begin{pmatrix}3\\8\\20\end{pmatrix} \right\rvert} = \dfrac{20}{(1)\left(\sqrt{473}\right)}$ $\theta = \cos^{-1}\left(\dfrac{20}{\sqrt{473}}\right) = 23.1^{\circ}$ $\left[\text{or } 0.404 \text{ radians (3 s.f.)}\right]$
9(iv) **[4]**	Line L: $\mathbf{r} = \begin{pmatrix}-2\\8\\12\end{pmatrix} + \lambda\begin{pmatrix}20\\-13\\-4\end{pmatrix}$, where $\lambda \in \mathbb{R}$ Line PR: $\mathbf{r} = \begin{pmatrix}0\\0\\12\end{pmatrix} + \mu\begin{pmatrix}0\\-15\\6\end{pmatrix}$, where $\mu \in \mathbb{R}$ $\begin{pmatrix}20\\-13\\-4\end{pmatrix} \neq k\begin{pmatrix}0\\-15\\6\end{pmatrix}$ for some constant $k \in \mathbb{R}$. The 2 lines are not parallel. Assuming both lines intersect: $\begin{pmatrix}-2\\8\\12\end{pmatrix} + \lambda\begin{pmatrix}20\\-13\\-4\end{pmatrix} = \begin{pmatrix}0\\0\\12\end{pmatrix} + \mu\begin{pmatrix}0\\-15\\6\end{pmatrix}$ $-2 + 20\lambda = 0 \quad \Rightarrow \quad \lambda = \dfrac{1}{10}$ $8 - 13\lambda = -15\mu$ ..... (1) $12 - 4\lambda = 12 + 6\mu$ ..... (2)

Qn	Solution
	Method 1 Substituting $\lambda = \frac{1}{10}$ into (1) and (2): From (1), $\mu = -\frac{67}{150}$ From (2), $\mu = -\frac{1}{15}$ The value of μ is not unique. **Method 2** $0\mu + 20\lambda = 2$ $-15\mu + 13\lambda = 8$ $6\mu + 4\lambda = 0$ Using GC, there are no solutions. --- Hence, the 2 lines do not intersect. $\therefore$ The cable is not parallel to edge PR and does not meet the edge too.
9(v) **[3]**	Plane PQR: $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 8 \\ 20 \end{pmatrix} = 240$ Line MN: $\mathbf{r} = \begin{pmatrix} \frac{20}{3} \\ 5 \\ 8 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 8 \\ 20 \end{pmatrix}$, where $\alpha \in \mathbb{R}$ By substitution: $\begin{pmatrix} \frac{20}{3} + 3\alpha \\ 5 + 8\alpha \\ 8 + 20\alpha \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 8 \\ 20 \end{pmatrix} = 240$ $3\left(\frac{20}{3} + 3\alpha\right) + 8(5 + 8\alpha) + 20(8 + 20\alpha) = 240$ $\alpha = \frac{20}{473}$ $\overrightarrow{ON} = \begin{pmatrix} \frac{20}{3} \\ 5 \\ 8 \end{pmatrix} + \frac{20}{473} \begin{pmatrix} 3 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} \frac{9640}{1419} \\ \frac{2525}{473} \\ \frac{4184}{473} \end{pmatrix}$. i.e. $s = \frac{9640}{1419}$ and $t = \frac{4184}{473}$