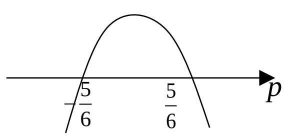
2022 PHSS Prelim AMATH Paper 2 Solutions

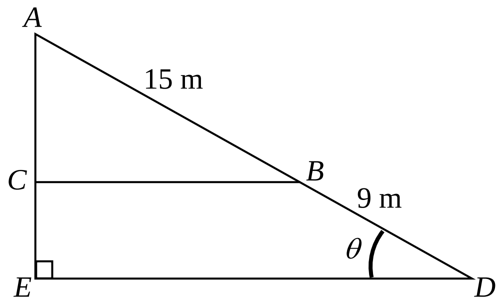
Question	Solutions
1	Solve the equation $3\left(e^{2x}-\frac{5}{e^{2x}}\right)=-4$, giving your answer(s) in exact form.
	$3\left(e^{2x}-\frac{5}{e^{2x}}\right)=-4$ $3\left(e^{2x}\right)^2+4e^{2x}-15=0$ Let $y=e^{2x}$ $3y^2+4y-15=0$ $(3y-5)(y+3)=0$ $y=\frac{5}{3}$ *or* $y=-3$ $e^{2x}=\frac{5}{3}$ *or* $e^{2x}=-3$ (reject as $e^{2x}>0$) $2x=\ln\frac{5}{3}$ $x=\frac{1}{2}\ln\frac{5}{3}$
2	The equation of a polynomial is given by $\mathrm{f}(x)=2x^3+x^2-8x+21$. **(i)** Show that $x+3$ is a factor of $\mathrm{f}(x)$.
	$\mathrm{f}(x)=2x^3+x^2-8x+21$ Subst $x=-3$ into f(x) $\mathrm{f}(-3)=2(-3)^3+(-3)^2-8(-3)+21$ $=-54+9+24+21$ $=0$ Since $\mathrm{f}(-3)=0$, by Factor Theorem, $x+3$ is a factor of $\mathrm{f}(x)$.
2	**(ii)** Hence, show that the equation $\mathrm{f}(x)=0$ has only one real root.
	By inspection, $2x^3+x^2-8x+21=(x+3)\left(2x^2+bx+7\right)$

	By comparing Coeff. of *x:* $3b+7=-8$ $b=-5$ $\therefore 2x^3+x^2-8x+21=(x+3)(2x^2-5x+7)$ For $f(x)=0$ $(x+3)(2x^2-5x+7)=0$ $2x^2-5x+7=0 \quad or \quad x=-3$ $D=(-5)^2-4(2)(7)$ $=-31<0$ Since discriminant is less than 0, $2x^2-5x+7=0$ has no real roots. Therefore, the equation $f(x)=0$ has only one real root which is $x=-3$.
3 (i)	Differentiate $\frac{3\ln 2x}{x^3}$ with respect to *x*.
	$\frac{d}{dx}\left(\frac{3\ln 2x}{x^3}\right)=\frac{\frac{3}{x}(x^3)-3x^2(3\ln 2x)}{x^6}$ $=\frac{3(x^2)-9x^2(\ln 2x)}{x^6}$ $=\frac{3-9\ln 2x}{x^4}$
3 (ii)	Hence show that $\int_1^2 \frac{4\ln 2x}{x^4}\,dx=\frac{1}{18}(a+b\ln 2)$, where *a* and *b* are integer values to be determined.

	$\int_1^2 \frac{4\ln 2x}{x^4}\,\mathrm{d}x$ $=\frac{4}{9}\int_1^2 \frac{9\ln 2x}{x^4}\,\mathrm{d}x$ $=\frac{4}{9}\left[\int_1^2 \frac{3}{x^4}\,\mathrm{d}x-\left[\frac{3\ln 2x}{x^3}\right]_1^2\right]$ $=\frac{4}{9}\left[-\frac{1}{x^3}-\frac{3\ln 2x}{x^3}\right]_1^2$ $=\frac{4}{9}\left[-\frac{1}{8}-\frac{3\ln 4}{8}-\left(-1-\frac{3\ln 2}{1}\right)\right]$ $=\frac{4}{9}\left(\frac{7}{8}-\frac{3\ln 2}{4}+3\ln 2\right)$ $=\frac{4}{9}\left(\frac{7}{8}+\frac{9}{4}\ln 2\right)$ $=\frac{7}{18}+\ln 2$ $=\frac{1}{18}(7+18\ln 2)$ (Shown) $a=7 \quad b=18$
4	The fourth term in the binomial expansion of $\left(x-\frac{2}{x^2}\right)^n$, where n is a positive integer, is a constant a. **(a)** Show that $n=9$ and hence find the value of a.
(a)	$\left(x-\frac{2}{x^2}\right)^n$ $T_{r+1}=\binom{n}{r}(x)^{n-r}\left(-\frac{2}{x^2}\right)^r$ $=\binom{n}{r}(-2)^r(x)^{n-3r}$ Fourth term is when r = 3 $x^{n-3(3)}=x^0$ $n-9=0$ $n=9$ (Shown)

	$a=\binom{9}{3}(-2)^3$ $=-672$
(b)	Find the coefficient of x^6 in the expansion of $\left(x-\frac{2}{x^2}\right)^9(1+3x)^4$.
	$\left(x-\frac{2}{x^2}\right)^9(1+3x)^4$ $=\left(x^9+9x^8\left(-\frac{2}{x^2}\right)+\binom{9}{2}x^7\left(-\frac{2}{x^2}\right)^2-672+...\right)\left(1+4(3x)+\binom{4}{2}(3x)^2+\binom{4}{3}(3x)^3+...\right)$ $=\left(x^9-18x^6+144x^3-672+..\right)\left(1+12x+54x^2+108x^3+...\right)$ Term in x^6 = $=-18x^6+\left(144x^3\right)\left(108x^3\right)$ Therefore, coefficient of x^6 $=-18+15552$ $=15534$
5 (a)	The equation of a quadratic curve is $y=-3x^2+4x-5$. The line $y=mx-2$ is a tangent to the curve at the point Q where $m > 0$. Find the value of the constant m and hence find the coordinates of Q.
	$-3x^2+4x-5=mx-2$ $3x^2+(m-4)x+3=0$ Since line is a tangent to the curve, $D = 0$ $(m-4)^2-4(3)(3)=0$ $m^2-8m+16-36=0$ $m^2-8m-20=0$ $(m-10)(m+2)=0$ $m=10$ or $m=-2$ Since $m > 0$, $m=10$

	$-3x^2+4x-5=10x-2$ $3x^2+(10-4)x+3=0$ $3x^2+6x+3=0$ $x^2+2x+1=0$ $(x+1)^2=0$ $x=-1$ Sub $x=-1$ into $y=10x-2$ $y=10(-1)-2$ $=-12$ Therefore, coordinates of Q is $(-1,-12)$.
(b)	Find the range of values of *p* such that the graph $y = px^2-5x+9p$ lies entirely below the *x*-axis.
	$y = px^2-5x+9p$ Since the graph lies entirely below the *x*-axis, there is no real roots, *p* < 0 and $D<0$ $(-5)^2-4(p)(9p)<0$ $25-36p^2<0$ $(5-6p)(5+6p)<0$  $p<-\frac{5}{6}$ *or* $p>\frac{5}{6}$ Since *p* < 0, reject $p>\frac{5}{6}$ $\therefore p<-\frac{5}{6}$

6	Mark wants to fence out a triangular plot of land for his garden as shown in the diagram. He also intends to build fences along *BC* to form two different plots of land to plant different vegetables such that they form a pair of similar triangles *ABC* and *ADE*. Point *B* lies on the straight line *AD* such that $AB = 15$ m and $BD = 9$ m. *AE* is perpendicular to *ED* and angle $ADE = \theta$ where $0° \le \theta \le 90°$. 
(i)	Show that *P* m, the perimeter of the plot of land *BCED*, is given by $P = 9\sin\theta + 39\cos\theta + 9$.
	$CE = 9\sin\theta$ $BC = 15\cos\theta$ $ED = 9\cos\theta + 15\cos\theta$ $= 24\cos\theta$ $P = 24\cos\theta + 9\sin\theta + 15\cos\theta + 9$ $P = 9\sin\theta + 39\cos\theta + 9$ (shown)
(b)	Express *P* in the form $9 + R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.
	$P = 9\sin\theta + 39\cos\theta + 9$ $R = \sqrt{9^2 + 39^2}$ $= \sqrt{1602}$ $= 40.025$ $\alpha = \tan^{-1}\frac{39}{9}$ $\alpha = 77.005°$ $P = 9 + \sqrt{1602}\sin(\theta + 77.0°)$ or $P = 9 + 40.025\sin(\theta + 77.005°)$ $P = 9 + 40.0\sin(\theta + 77.0°)$

(c)	Find the value of P and the corresponding value of θ if Mark will like the plot of land $BCED$ to be as large as possible.
	$P = 9 + 40.025\sin\left(\theta + 77.005°\right)$ Maximum value of P is when $\sin\left(\theta + 77.005°\right) = 1$ $P = 9 + 40.025$ $= 49.025$ $= 49.0\ m\ (3\ \textbf{s.f})$ $\sin\left(\theta + 77.005°\right) = 1$ $\theta + 77.005° = 90°$ $\theta = 12.995°$ $\theta = 13.0°\ \ (3\ \textbf{s.f})$ Corresponding value of $\theta = 13.0°$
7	The population of wild Red Pandas has been steadily decreasing over the years, facing the risk of extinction. The table shows the estimated population of wild Red Pandas from 2016 to 2020 where year 2016 is taken to be $t = 1$ and so on. A wildlife expert believed that these figures can be modelled using the formula $P = P_0 e^{-kt}$, where P_0 and k are constants.
(i)	Using the grid below, plot ln P against t and draw a straight line graph.

Year	2016	2017	2018	2019	2020
t	1	2	3	4	5
P	9900	7200	4800	3200	2300

Year	2016	2017	2018	2019	2020
t	1	2	3	4	5
P	9900	7200	4800	3200	2300
ln *P*	9.20	8.88	8.48	8.07	7.74

$P = P_0 e^{-kt}$

$\ln P = -kt + \ln P_0$

Plot $\ln P$ against t (Graph behind)

(ii)	Use your graph to estimate the value of P_0 and of k.
	$P = P_0 e^{-kt}$ $\ln P = -kt + \ln P_0$ $\ln P_0 = 9.55$

<table>
<tr><td></td><td>$P_0 = 14045$
$P_0 = 14000 \ (3\,\text{s.f})$

Gradient = $\dfrac{7.74 - 9.20}{5 - 1}$
= -0.365
Therefore, k = 0.365</td></tr>
<tr><td>(iii)</td><td>The Wildlife Expert uses this model to estimate the population of Red Pandas in 2030. Find the value of this estimation, correct to the nearest whole number, and explain if the estimation obtained reliable.</td></tr>
<tr><td></td><td>Evidence:
$P = P_0 e^{-kt}$
$P = 14045e^{-0.365t}$
When t = 15,
$P = 14045e^{-0.365(15)}$
$= 58.851$
$= 59$ (nearest whole number)
Conclusion & Concept:
No, because the value is extrapolated where the linear relationship may no longer hold.</td></tr>
<tr><td></td><td></td></tr>
<tr><td>8</td><td>A particle traveling in a straight line passes through O with a speed of 5 m/s. The acceleration a m/s^2, of the particle, t s after passing through O, is given by $a = -3e^{-0.5t}$.
The particle comes to instantaneous rest at the point X.
(i) Find the time taken for the particle to reach X.</td></tr>
<tr><td>(i)</td><td>$a = -3e^{-0.5t}$
$v = \int -3e^{-0.5t}\ dt$
$= \dfrac{3}{0.5} e^{-0.5t} + c$
$v = 6e^{-0.5t} + c$

When t = 0, v = 5,
$5 = 6e^0 + c$
$c = -1$
$v = 6e^{-0.5t} - 1$

At instantaneous rest, v = 0,</td></tr>
</table>

	$6e^{-0.5t} - 1 = 0$ $e^{-0.5t} = \frac{1}{6}$ $-0.5t = \ln\frac{1}{6}$ $t = 3.5835$ $t = 3.58\text{ s}$
(ii)	Calculate the distance *OX*.
	$s = \int \left(6e^{-0.5t} - 1\right) dt$ $s = -12e^{-0.5t} - t + d$ When $t = 0$, $s = 0$, $0 = -12e^{0} - 0 + d$ $d = 12$ $s = -12e^{-0.5t} - t + 12$ At $t = 3.5835$, $s = -12e^{-0.5(3.5835)} - 3.5835 + 12$ $= 6.41648$ $= 6.42$ Distance of *OX* = 6.42 m
(iii)	Show that the particle is again at *O* at some instant during the twelfth second after passing through *O*.
	When $t = 11$, $s = -12e^{-0.5(11)} - 11 + 12$ $= 0.95096$ When $t = 12$, $s = -12e^{-0.5(12)} - 12 + 12$ $= -0.029745$ Since the displacement of the particle changes from a positive value at $t = 11$ to a negative value at $t = 12$, the particle passes through *O* at some instant during the twelfth second.

9	The equation of a circle is $(x-4r)^2+(y+3r)^2=kr^2$ where r and k are positive constants. It is given that k = 9.
(a)	Explain why the x-axis is a tangent to the circle.
	Evidence: $(x-4r)^2+(y+3r)^2=9r^2$ Centre of circle : $(4r,-3r)$ Radius of circle = $3r$ **Concept:** The y-coordinate of the centre of the circle is $3r$ units below the x-axis, and the radius is $3r$ units. **Conclusion:** Therefore, the x-axis is a tangent to the circle. Or **Evidence:** Subst y = 0 into $(x-4r)^2+(y+3r)^2=9r^2$ $(x-4r)^2+(0+3r)^2=9r^2$ $x^2-8r+16r^2=0$ $D=(-8r)^2-4(16r^2)$ $=64r^2-64r^2$ $=0$ **Concept:** Since discriminant = 0, the equation has real and equal roots. The circle touches the line y = 0 at only 1 point. **Conclusion:** Therefore, the x-axis is a tangent to the circle
(b)	Find, in terms of r, the coordinates of the points on the circle at which the normal to the circle which passes through the centre of the circle, is perpendicular to the y-axis.
	Normal perpendicular to y-axis means tangent is parallel to y-axis Let the points be A and B. y-coordinate of A and B = $-3r$ x-coordinate of A = $4r$ - $3r$ = r x-coordinate of B = $4r$ + $3r$ = $7r$ Therefore, the points are $(r,-3r)$ and $(7r,-3r)$

	It is now given that $k = 25$.
(c)	Verify that the circle passes through the origin O.
	$(x-4r)^2+(y+3r)^2=25r^2$ $(x-4r)^2+(y+3r)^2=(5r)^2$ Distance from the centre to the origin $=\sqrt{(4r-0)^2+(3r-0)^2}$ $=\sqrt{25r^2}$ $=5r$ = Radius of circle Therefore, the circle passes through the origin **Or** Subst (0, 0) into $(x-4r)^2+(y+3r)^2=25r^2$ *LHS* $=(0-4r)^2+(0+3r)^2$ $=25r^2$ $=RHS$ Therefore, the circle passes through the origin
(d)	Given that OD is the diameter of the circle, find in terms of r, the equation of the tangent to the circle at D and hence, find the coordinates of the point at which this tangent meets the x-axis.
	Let coordinates of D be (x, y) Midpoint of OD = $(4r,-3r)$ $\left(\frac{x+0}{2},\frac{y+0}{2}\right)=(4r,-3r)$ $\frac{x}{2}=4r$ and $\frac{y}{2}=-3r$ $x=8r$ $x=-6r$ $\therefore D(8r,-6r)$ Gradient of OD $=\frac{-6r-0}{8r-0}=-\frac{3}{4}$ Therefore, gradient of tangent at D = $\frac{4}{3}$ Equation of tangent at $D(8r,-6r)$

	$y+6r=\frac{4}{3}(x-8r)$ $y=\frac{4}{3}x-\frac{50}{3}r$ When $y = 0$, $0=\frac{4}{3}x-\frac{50}{3}r$ $\frac{4}{3}x=\frac{50}{3}r$ $x=\frac{25}{2}r$ Coordinates of the point = $\left(\frac{25}{2}r,0\right)$
10	The diagram shows the curve of $y=3\cos 2x+3x$ for $0\le x\le\frac{\pi}{2}$ radians. The point Q is the minimum point of the curve and OQ is a straight line. Show that the area of the shaded region is $\frac{1}{16}\left(12+5\pi\sqrt{3}\right)$ units^2.
	$y=3\cos 2x+3x$ ---- (1) $\frac{\mathrm{d}y}{\mathrm{d}x}=-6\sin 2x+3$ For stationary pt, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $-6\sin 2x+3=0$ $\sin 2x=\frac{1}{2}$ $2x=\frac{\pi}{6},\frac{5\pi}{6}$ $x=\frac{\pi}{12},\frac{5\pi}{12}$ At min point Q, $x=\frac{5\pi}{12}$ since $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-12\cos 2\left(\frac{5\pi}{12}\right)=10.392>0$ Sub $x=\frac{5\pi}{12}$ into (1)

	$y=3\cos 2\left(\frac{5\pi}{12}\right)+3\left(\frac{5\pi}{12}\right)$ $=-\frac{3\sqrt{3}}{2}+\frac{15\pi}{12}$ Area under curve $=\int_0^{\frac{5\pi}{12}} 3\cos 2x+3x\ dx$ $=\left[\frac{3}{2}\sin 2x+\frac{3}{2}x^2\right]_0^{\frac{5\pi}{12}}$ $=\left[\frac{3}{2}\sin 2\left(\frac{5\pi}{12}\right)+\frac{3}{2}\left(\frac{5\pi}{12}\right)^2-0\right]$ $=\frac{3}{2}\left(\frac{1}{2}\right)+\frac{3}{2}\left(\frac{25\pi^2}{144}\right)$ $=\frac{3}{4}+\frac{75\pi^2}{288}$ Area of shaded region $=\frac{3}{4}+\frac{75\pi^2}{288}-\frac{1}{2}\times\frac{5\pi}{12}\left(\frac{15\pi}{12}-\frac{3\sqrt{3}}{2}\right)$ $=\frac{3}{4}+\frac{75\pi^2}{288}-\left(\frac{75\pi^2}{288}-\frac{5\pi\sqrt{3}}{16}\right)$ $=\frac{3}{4}+\frac{5\pi\sqrt{3}}{16}$ $=\frac{1}{16}\left(12+5\pi\sqrt{3}\right)$ units^2 (Shown)

Q7(i)