1 A ball is rolling in a straight line such that its distance away from the starting point, s cm, can be modelled using the equation

$$s = at + \frac{b}{\sqrt{t+4}} + c,$$

where t is the time taken in seconds, and a, b and c are real constants.

The ball is at the starting point when $t = 0$, and moved 10 cm in the first 5 seconds. It moved another 9 cm in the next 16 seconds. Find the ball's distance away from the starting point when $t = 50$. [4]

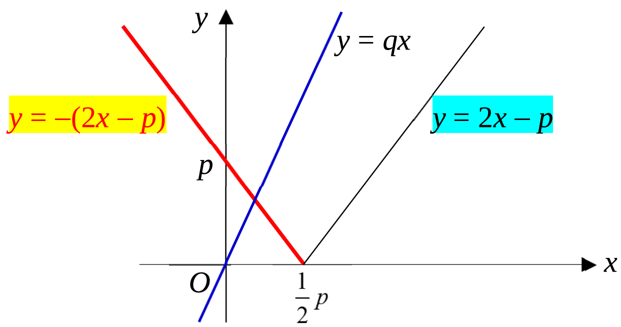
	[Solutions]	Remarks
	When $t = 0$, $s = 0$, $\quad 0 = \frac{1}{2}b + c \quad \cdots\cdots\cdots\cdots(1)$ When $t = 5$, $s = 10$, $\quad 10 = 5a + \frac{1}{3}b + c \quad \cdots\cdots(2)$ When $t = 21$, $s = 19$, $\quad 19 = 21a + \frac{1}{5}b + c \quad \cdots\cdots(3)$ Solving (1), (2) & (3) using GC, $a = 0.08333$ or $\frac{1}{12}$ $b = -57.5$ or $-\frac{115}{2}$ $c = 28.75$ or $\frac{115}{4}$ $\therefore\ s = \frac{1}{12}t - \frac{115}{2\sqrt{t+4}} + \frac{115}{4}$ When $t = 50$, $s = \frac{1}{12}(50) - \frac{115}{2\sqrt{50+4}} + \frac{115}{4} = 25.092$ Thus the distance away from starting point is 25.1 cm (3 s.f.)	- **3 unknowns** need **3 equations** to solve. To simplify equations to as shown. - To read key word such as "another", "next" and "starting point". - To watch presentation of labeling "when…." and to number all equations - Use GC to solve! - Always read back to see objective in this case find s when t=50.

2 On a single diagram, sketch the graphs of $y=|2x-p|$ and $y=qx$ where the following conditions are satisfied, indicating the axial intercepts.

- p and q are constants, $p>1$ and $q>0$, and
- the graphs have only one point of intersection. [2]

(a) State the least value of q. [1]

(b) Solve the inequality $|2x-p|>qx$, leaving your answer in terms of p and q. [2]

	[Solutions]	**Remarks**
		Useful Tips: When dealing with modulus curve, it is advisable to label the positive/negative equations on the diagram. Note that $\lvert 2x-p\rvert=\begin{cases}2x-p, & x\ge\frac{p}{2}\\-(2x-p), & x<\frac{p}{2}\end{cases}$ When sketching the graphs, ensure that • **x and y intercepts** are clearly indicated • the graph of $y=\lvert 2x-p\rvert$ is **symmetrical** about $x=\frac{p}{2}$ • the line $y=qx$ should be steeper than $y=2x-p$ in order to have only one intersection point • the equation of each graph must be clearly labelled
(a)	For the graphs to have only one point of intersection, the line $y=qx$ has the **same or a greater gradient** than the line $y=2x-p$, i.e $q\ge 2$. Least value of q = 2	
(b)	At the intersection point, **Use graph sketched earlier!** $-(2x-p)=qx$ $(2+q)x=p$ $x=\dfrac{p}{q+2}$ For $\lvert 2x-p\rvert>qx$, $x<\dfrac{p}{q+2}$	Note: When solving inequalities using graphical method, we should always attempt to first find intersection points (if any). **From (a),** intersection point occurs at $x<\frac{p}{2}$. Hence when finding the intersection point, we should equate $y=-(2x-p)$ with $y=qx$ instead of $y=\lvert 2x-p\rvert$ with $y=qx$.

	Alternative method for finding intersection point $\lvert 2x-p \rvert = qx$ $2x-p=qx$ or $2x-p=-qx$ $2x-qx=p$ $\quad$ $2x+qx=p$ $(2-q)x=p$ $\quad$ $(2+q)x=p$ $x=\dfrac{p}{2-q}$ $\quad$ $x=\dfrac{p}{2+q}$ (rejected $\because \dfrac{p}{2-q}<0$ since $q \geq 2$)	For students who used the alternative method, do pay attention to the correct reasons for rejecting the other answer.

3 Find

(a) $\int \tan^2(x-1)\,\mathrm{d}x$, [2]

(b) $\int \sin^{-1}2x\ \mathrm{d}x$. [3]

	[Solutions]		Remarks
(a)	$\int \tan^2(x-1)\,\mathrm{d}x$ $=\int \sec^2(x-1)-1\,\mathrm{d}x$ $=\tan(x-1)-x+c$	$\int \sin^2(ax+b)\,\mathrm{d}x$ $\int \cos^2(ax+b)\,\mathrm{d}x$ $\int \mathrm{cosec}^2(ax+b)\,\mathrm{d}x$ $\int \sec^2(ax+b)\,\mathrm{d}x$ $\int \cot^2(ax+b)\,\mathrm{d}x$	- Students to note how to handle trigo with power 2 and linear angle! 1) $\sec^2$, cosec^2 : **can integrate directly**! 2) $\tan^2$, $\cot^2$: use trigo identity to convert to (1)! 3) $\sin^2$, $\cos^2$: use double angle formula - Students must remember the trigo identity formula correctly!
(b)	$\int 1\cdot\sin^{-1}2x\ \mathrm{d}x$ **I C − ∫ I D** $=(x)\sin^{-1}2x-\int (x)\cdot\frac{2}{\sqrt{1-(2x)^2}}\,\mathrm{d}x$ $=x\sin^{-1}2x+\frac{1}{4}\int (-8x)(1-4x^2)^{-\frac{1}{2}}\,\mathrm{d}x$ $=x\sin^{-1}2x+\frac{1}{4}\times\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}}+c$ $=x\sin^{-1}2x+\frac{1}{2}\sqrt{1-4x^2}+c$		- Students pls note that : "Inverse $\neq$ Reciprocal" - Do not mix up $\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x$ with $\int \sin^{-1}x\,\mathrm{d}x$ where we need to use by part for integration! - Must remember chain rule for differentiation and note the angle for trigo when applying the formula "$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(2x)=\frac{2}{\sqrt{1-(2x)^2}}$" - Students to remember the following formula *for* $n\neq-1$, identify carefully what is the f(x) that will affect the $\mathrm{f}'(x)$ and to divide by the new power: $\int \mathrm{f}'(x)[\mathrm{f}(x)]^n = \frac{[\mathrm{f}(x)]^{n+1}}{n+1}+c$

4 **Do not use a calculator in answering this question.**

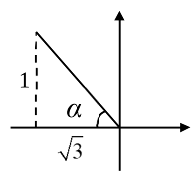
(a) It is given $w = -\sqrt{3} + \mathrm{i}$.

(i) Find arg w. [1]

(ii) Express $\mathrm{i}w^8$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(b) **(i)** It is given that $(1 + a\mathrm{i})^2 = -3 - 4\mathrm{i}$. Find the value of the real constant a. [2]

(ii) Hence solve the equation $2z^2 + (-3 + 2\mathrm{i})z + (1 - \mathrm{i}) = 0$. [3]

	[Solutions]	**Remarks**
(a)(i)	$w = -\sqrt{3} + \mathrm{i}$ (2^{nd} quadrant) $\tan\alpha = \left\|\frac{y}{x}\right\| = \left\|\frac{1}{-\sqrt{3}}\right\|$ $\Rightarrow \alpha = \frac{\pi}{6}$  arg $w = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$	*As calculators are not allowed, detailed workings should be given.* Note that $\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ gives $-\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{5\pi}{6}, \ldots\ldots$ The principal argument is the angle such that $-\pi < \theta \le \pi$.
(ii)	$\|w\| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ $\boxed{\|\mathrm{i}w^8\| = \|\mathrm{i}\|\|w\|^8} = 1 \times 2^8 = 256$ $\boxed{\arg(\mathrm{i}w^8) = \arg(\mathrm{i}) + 8\arg w}$ $= \frac{\pi}{2} + 8\left(\frac{5}{6}\pi\right)$ $= \frac{43}{6}\pi = 6\pi + \frac{7}{6}\pi \notin (-\pi, \pi]$ **Principal argument** $= -\frac{5}{6}\pi$ Thus $\mathrm{i}w^8 = 256\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}}$	$\|\mathrm{i}\| = 1$ $\arg(\mathrm{i}) = \frac{\pi}{2}$ Note that $\frac{43}{6}\pi \neq -\frac{5}{6}\pi$ r is the modulus and must be a positive real number. - What about $-\mathrm{i}w^8$?
	Alternatively, from **(i)**, $w = 2\mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}$ $\therefore w^8 = \left(2\mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}\right)^8 = 2^8\mathrm{e}^{\frac{40}{6}\pi\mathrm{i}} = 256\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}$ $\mathrm{i}w^8 = \mathrm{e}^{\frac{\pi}{2}\mathrm{i}} \times 256\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}$ $= 256\mathrm{e}^{\left(\frac{\pi}{2} + \frac{2\pi}{3}\right)\mathrm{i}} = 256\mathrm{e}^{\frac{7}{6}\pi\mathrm{i}}$ $= 256\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}}$	Note that $\mathrm{i} = (1)\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ - What is the geometrical relationship between i and iz?

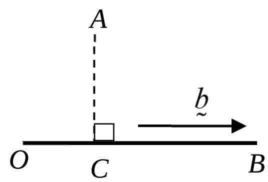
<table>
<tr><td>(b)(i)</td><td>$(1+a\mathrm{i})^2=-3-4\mathrm{i}$
$1-a^2+2a\mathrm{i}=-3-4\mathrm{i}$
Comparing imaginary parts,
$2a=-4$
$a=-2$
[Check: real parts $=1-a^2=1-2^2=-3$]</td><td></td></tr>
<tr><td>(ii)</td><td>$2z^2+(-3+2\mathrm{i})z+(1-\mathrm{i})=0$
Using the quadratic formula,
$$z=\frac{3-2\mathrm{i}\pm\sqrt{(-3+2\mathrm{i})^2-4(2)(1-\mathrm{i})}}{2(2)}$$
$$=\frac{3-2\mathrm{i}\pm\sqrt{-3-4\mathrm{i}}}{4}$$
$$=\frac{3-2\mathrm{i}\pm(1-2\mathrm{i})}{4}$$ Use earlier result in (i)!
$$=1-\mathrm{i}\quad\text{or}\quad\frac{1}{2}$$</td><td>Hence means to use the result in part (i)
$(1-2\mathrm{i})^2=-3-4\mathrm{i}$
$\Rightarrow\sqrt{-3-4\mathrm{i}}=\pm(1-2\mathrm{i})$
to obtain the roots of the equation.</td></tr>
</table>

5 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. C is the point on line OB such that AC is perpendicular to OB.

(a) **By using a suitable scalar product**, or otherwise, show that $\overrightarrow{OC} = \dfrac{(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{b}|^2}\mathbf{b}$. [3]

(b) Give a geometrical interpretation of $\dfrac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|}$. [1]

(c) It is given that $\mathbf{a} = \begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ h \\ 0 \end{pmatrix}$. Given also that the length of the line segment AB is 5 units and angle AOB is an obtuse angle, find the exact value of h. [4]

	[Solutions]	Remarks
(a)	Since C is a point on line OB, $\overrightarrow{OC} = \lambda \underset{\sim}{b}$ for some $\lambda \in \mathbb{R}$  AC is perpendicular to OB $\overrightarrow{AC}\cdot\underset{\sim}{b} = 0$ $(\lambda\underset{\sim}{b} - \underset{\sim}{a})\cdot\underset{\sim}{b} = 0$ $\lambda\underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ $\lambda\lvert\underset{\sim}{b}\rvert^2 = \underset{\sim}{a}\cdot\underset{\sim}{b}$ $\lambda = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert^2}$ Thus $\overrightarrow{OC} = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert^2}\underset{\sim}{b}$ (shown)	It is important to not just read the question but also to process the information provided. A good practice is to annotate on the question what each key piece of information translates to as you read the question e.g. …C is the point on line OB… (jot down $\overrightarrow{OC} = \lambda\underset{\sim}{b}$) …AC is perpendicular to OB… (jot down $\overrightarrow{AC}\cdot\overrightarrow{OB} = 0$)
(b)	$\overrightarrow{OC} = \left(\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}\right)\dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$ $\lvert\overrightarrow{OC}\rvert = \left\lvert\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}\right\rvert\left\lvert\dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}\right\rvert = \dfrac{\lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert}{\lvert\underset{\sim}{b}\rvert}(1) = \dfrac{\lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert}{\lvert\underset{\sim}{b}\rvert}$ This is the length of projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. OR this is the length OC.	Need to be careful of the term used. Length of projection not ~~line~~ of projection

(c)

Step 1: Determining possible *h* values

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ h \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ h-3 \\ 1 \end{pmatrix}$$

Given: $\left|\overrightarrow{AB}\right| = \sqrt{16 + (h-3)^2 + 1} = 5$

$$(h-3)^2 = 25 - 17$$

$$h = 3 \pm 2\sqrt{2}$$

Step 2: Determining the correct *h* value

Approach 1A

Given: $\angle AOB$ is an obtuse angle

$$\overrightarrow{OA} \cdot \overrightarrow{OB} < 0$$

$$\begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ h \\ 0 \end{pmatrix} = -3 + 3h < 0 \;\Rightarrow\; h < 1$$

Thus $h = 3 - 2\sqrt{2}$

Approach 1B

For $h = 3 + 2\sqrt{2}$

$$\begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ h \\ 0 \end{pmatrix} = -3 + 3h = -3 + 3\left(3 + 2\sqrt{2}\right) = 6 + 6\sqrt{2} > 0,$$

while for $h = 3 - 2\sqrt{2}$

$$\begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ h \\ 0 \end{pmatrix} = -3 + 3h = -3 + 3\left(3 - 2\sqrt{2}\right) = 6 - 6\sqrt{2} < 0$$

$\therefore h = 3 - 2\sqrt{2}$

Cambridge questions are very precise in their phrasing. When asked for exact value (singular), it literally means one value.

This means further working is required to reject the non-applicable value.

For angle between two vectors, $\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

For approach 1A, it is sufficient to compare both *h* values and state the conclusion

For approach 1B, as the substitution of *h* values require further evaluation to determine the sign of the dot product, it is insufficient to just prove either $h = 3 + 2\sqrt{2}$ leads to positive dot product or $h = 3 - 2\sqrt{2}$ leads to negative dot product.

Both *h* values must be substituted to completely prove their validity or invalidity.

Approach 2A

$\angle AOB$ is an obtuse angle.

$\Rightarrow \cos\theta < 0$

For $h = 3 + 2\sqrt{2}$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{\begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3+2\sqrt{2} \\ 0 \end{pmatrix}}{\sqrt{(-3)^2 + 3^2 + (-1)^2}\sqrt{1^2 + \left(3+2\sqrt{2}\right)^2 + 0^2}}$$

$$= 0.562 > 0$$

For $h = 3 - 2\sqrt{2}$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{\begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3-2\sqrt{2} \\ 0 \end{pmatrix}}{\sqrt{(-3)^2 + 3^2 + (-1)^2}\sqrt{1^2 + \left(3-2\sqrt{2}\right)^2 + 0^2}}$$

$$= -0.562 < 0$$

$\therefore h = 3 - 2\sqrt{2}$

Approach 2B

Further process the $\cos\theta$ values to obtain the angle for direct comparison.

You may use the idea of the four quadrants to remember the signs of trigonometric ratios for different angles.

For angle between two vectors, $\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

It is not $\cos\theta = \frac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$\cos\theta = \frac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{|\underset{\sim}{a}||\underset{\sim}{b}|}$ applies only to

- acute angle between 2 lines,
- acute angle between 2 planes
- acute angle between a line and a plane.

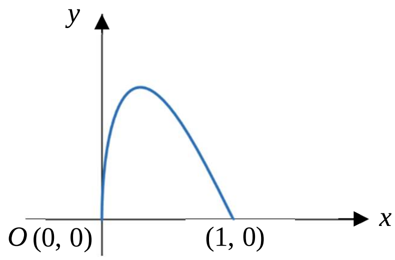
6 The curve C is defined by the parametric equations

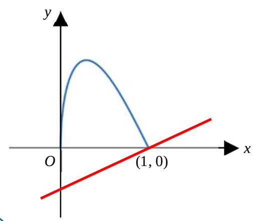
$$x = 1 - \cos t,\quad y = \sin 2t,\quad \text{where } 0 \le t \le \frac{\pi}{2}.$$

(a) Sketch C, giving the exact coordinates of the points where C meets the x-axis. [1]

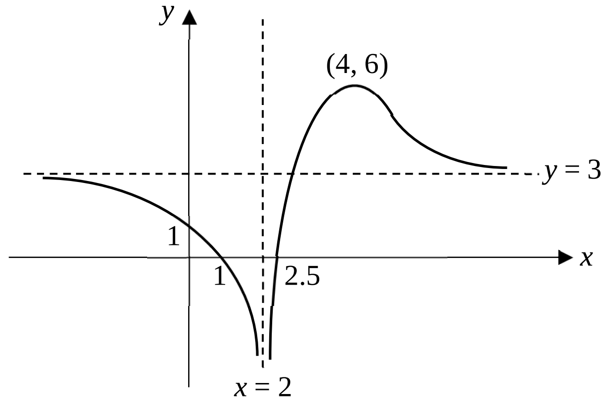
(b) The normal to C at the point where $t = \frac{\pi}{2}$ cuts the y-axis at D. Show that the y-coordinate of D is $-\frac{1}{2}$. [4]

(c) Find the exact area of the region bounded by C, the normal in part **(b)** and the y-axis. [5]

	[Solutions]	**Remarks**
(a)		Always check (and double-check) that you have **set the correct range of the parameter *t*.** Remember that the default setting after resetting the GC is $0 \le t \le 2\pi$. Note the question requirement of giving **coordinates**.
(b)	$x = 1 - \cos t \;\Rightarrow\; \frac{dx}{dt} = \sin t$ $y = \sin 2t \;\Rightarrow\; \frac{dy}{dt} = 2\cos 2t$ $\frac{dy}{dx} = \frac{2\cos 2t}{\sin t}$ When $t = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{2\cos \pi}{\sin \frac{\pi}{2}} = -2$, $x = 1, y = 0$ Hence gradient of normal = $\frac{-1}{-2} = \frac{1}{2}$ Equation of normal: $y = \frac{1}{2}(x-1)$ When $x = 0$, $y = -\frac{1}{2}$	Evaluate the value of $\frac{dy}{dx}$ first, instead of writing expressions in terms of $\frac{2\cos 2t}{\sin t}$

(c)	Area $=\left(\frac{1}{2}\right)\frac{1}{2}(1) \;+\; \int_0^1 y\,\mathrm{d}x$ $=\frac{1}{4}+\int_0^{\frac{\pi}{2}} \sin 2t \sin t\,\mathrm{d}t$ $=\frac{1}{4}-\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 3t-\cos t\,\mathrm{d}t$ $=\frac{1}{4}-\frac{1}{2}\left[\frac{\sin 3t}{3}-\sin t\right]_0^{\frac{\pi}{2}}$ $=\frac{1}{4}-\frac{1}{2}\left[\frac{-1}{3}-1\right]=\frac{11}{12}$ units^2  	1. Notice that one part of the region is a triangle. Use the formula for area of triangle. 2. You should always start with the expression $\int_0^1 y\,\mathrm{d}x$ before converting in terms of t. OR $\int \sin 2t \sin t\,\mathrm{d}t$ $=\int (2\sin t\cos t)\sin t\,\mathrm{d}t$ $=2\int \cos t(\sin t)^2\,\mathrm{d}t$ $=2\left[\frac{(\sin t)^3}{3}\right]+C$

7 **(a)** The diagram shows the curve with equation $y = \mathrm{f}(x)$. The curve crosses the x-axis at $x = 1$ and $x = 2.5$, crosses the y-axis at $y = 1$ and has a maximum point at $(4,\ 6)$. The equations of the asymptotes are $x = 2$ and $y = 3$. Sketch the graph of $y = \mathrm{f}'(x)$, giving the equations of asymptotes, coordinates of turning points and axial intercepts, where possible. [2]

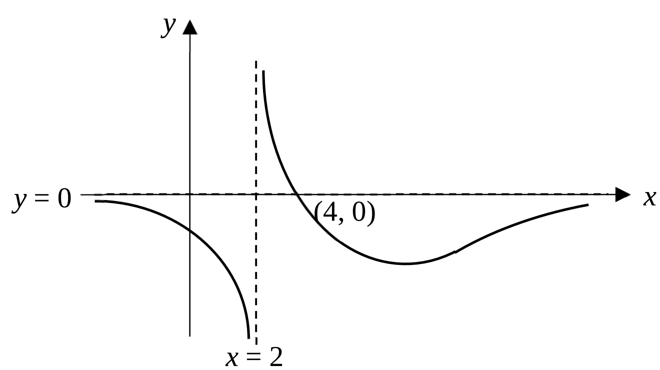


(b) The curve C has equation $y = \dfrac{x^2 + kx - 1}{x + 1}$, where k is a non-zero constant.

(i) Find the range of values of k for which C has no stationary points. [4]

(ii) Given that $y = x + 3$ is an asymptote of C, show that $k = 4$. [2]

(iii) State a sequence of transformations which transform the graph of $y = \dfrac{x}{4} - \dfrac{1}{x}$ onto the graph of $y = \dfrac{x^2 + 4x - 1}{x + 1}$. [3]

	[Solutions]	**Remarks**
(a)	$y = \mathrm{f}'(x)$  	Note that $y = 0$ is an asymptote that is not seen in the original graph. Note also that it is not possible to deduce the y-intercept of this graph.

(b)(i)	$y=\dfrac{x^2+kx-1}{x+1}=x+k-1-\dfrac{k}{x+1}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+1)(2x+k)-(x^2+kx-1)(1)}{(x+1)^2}=\dfrac{x^2+2x+k+1}{(x+1)^2}$ Let $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \;\Rightarrow\; x^2+2x+k+1=0$ For no stationary points, the quadratic equation has no real roots Discriminant = $2^2-4(1)(k+1)<0$ $k+1>1 \;\Rightarrow\; k>0$	To approach this question, instead of thinking "$\dfrac{\mathrm{d}y}{\mathrm{d}x}<0$ or $\dfrac{\mathrm{d}y}{\mathrm{d}x}>0$", think "$\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ has no real solution" Note: k = 0 is also a solution for this question but we do not mark students down for not mentioning it.
(ii)	Since $y = x + 3$ is an asymptote of the hyperbola, $y=\dfrac{x^2+kx-1}{x+1}=x+3+\dfrac{c}{x+1}=\dfrac{x^2+4x+3+c}{x+1}$ Comparing coefficients, $k = 4$ (and $c = -4$) (shown)	It is incorrect to state $x+3=\dfrac{x^2+kx-1}{x+1}$ (LHS is linear, RHS is not)
(iii)	$y=\dfrac{x^2+4x-1}{x+1}=x+3-\dfrac{4}{x+1}$ $y=\dfrac{x}{4}-\dfrac{1}{x}$ $\downarrow$ replace y by $\left(\dfrac{y}{4}\right)$ $y=4\left(\dfrac{x}{4}-\dfrac{1}{x}\right)=x-\dfrac{4}{x}$ $\downarrow$ replace x by $(x+1)$ $y=(x+1)-\dfrac{4}{(x+1)}$ $\downarrow$ replace y by $(y-2)$ $y=\left(x+1-\dfrac{4}{x+1}\right)+2=x+3-\dfrac{4}{x+1}$ The transformations are (in order): 1. A scaling parallel to the y-axis by factor 4. 2. A translation of 1 unit in the negative direction of x-axis. 3. A translation of 2 units in the positive direction of y-axis.	Use the idea of "replacement" to check your answers. Ensure that you use the correct terms and phrases for linear transformations. Cambridge has been particularly strict on this.

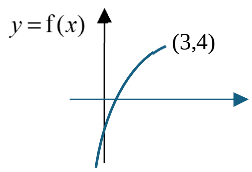
8 The function f and g are defined by

$$\mathrm{f}: x \mapsto -(x-4)^2+5, \quad x\in\mathbb{R},\ x\le 3,$$
$$\mathrm{g}: x \mapsto \mathrm{e}^{|x-3|}, \quad x\in\mathbb{R},\ x\le 10.$$

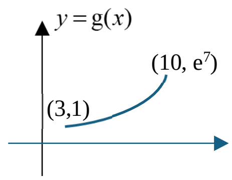
(a) Show that the composite function gf exists. [2]

(b) Find an expression for gf(x) and state the domain of gf. Hence find the value of x such that $x=(\mathrm{gf})^{-1}(1)$. [5]

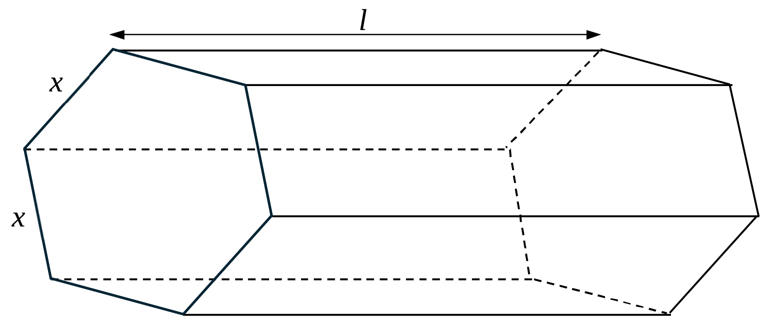
(c) The function g has an inverse if its domain is restricted to $\alpha \le x \le 10$. State the smallest possible value of α and find $\mathrm{g}^{-1}(x)$, stating its domain. [4]

	[Solutions]	**Remarks**
(a)	$R_f=(-\infty,4]$ $D_g=(-\infty,10]$  Since $R_f \subset D_g$, gf exists.	- When finding range, please remember to draw graph according to DOMAIN! - Please **DO NOT OVERWRITE** the ") and]" when wrote wrongly as cannot tell which is the one! You are to **CANCEL and REWRITE**! - Please also remember the condition for checking composite function exit. - Check use of notation / keyword "subset" ! Note "subset, $\subset$" is not the same as "element of, $\in$"
(b)	$\mathrm{gf}(x)=\mathrm{e}^{\lvert(-(x-4)^2+5)-3\rvert}=\mathrm{e}^{\lvert-(x-4)^2+2\rvert},\ x\le 3$ $x=(\mathrm{gf})^{-1}(1)$ $(\mathrm{gf})(x)=1$ Similar to how you do inverse trigo in secondary sch Eg $\sin^{-1}a=\theta \Rightarrow \sin\theta = a$ $\mathrm{e}^{\lvert(x-4)^2-2\rvert}=1$ $\ln \mathrm{e}^{\lvert(x-4)^2-2\rvert}=\ln 1$ $\lvert(x-4)^2-2\rvert=0$ $(x-4)^2=2$ Note: in this case the modulus is not ignored but is because $(x-4)^2-2=\pm 0=0$ $x-4=\pm\sqrt{2}$ $x=4\pm\sqrt{2}$ Since $x\le 3$, $x=4-\sqrt{2}$ i.e. $x=(\mathrm{gf})^{-1}(1)=4-\sqrt{2}$	- Please remember domain of composite function is domain of 1st function! ie $D_{gf}=D_f$. Note: $D_{gf}\ne R_f$! - **Please do not ignore/ remove modulus without proper justification!** - Please remember the '$\pm$' when taking square root! - Whenever got 2 answers, please check to see if reject

		any! In this the restriction of x is based on the domain of gf. - Students to be mindful to present answer clearly as there are too many x in the question, so pls ensure to write down the final line '$x=\left(\text{gf}\right)^{-1}(1)=4-\sqrt{2}$' to ensure this is answering the question
	Method 2: finding inverse function $y=\mathrm{e}^{\left\|-(x-4)^2+2\right\|}$ $\ln y=\left\|-(x-4)^2+2\right\|$ $\pm\ln y=-(x-4)^2+2$ $(x-4)^2=2\mp\ln y$ $x-4=\pm\sqrt{2\mp\ln y}$ $x=4\pm\sqrt{2\mp\ln y}$ since $x\le 3$, $x=4-\sqrt{2\mp\ln y}$ When $y=1$, $x=4-\sqrt{2}$ $\therefore x=\left(\text{gf}\right)^{-1}(1)=4-\sqrt{2}$ **IMPT: The definition of function in general: every input has only 1 output.** Hence CANNOT write $\left(\text{gf}\right)^{-1}\left(x\right)=4\pm\sqrt{2\mp\ln y}$ which will give 2 output because of the $\pm$	- Most students used this method but please be careful of how to remove modulus as well as the power 2! Will have 2 '$\pm$' in your working - Whenever have 2 answers, please check which to reject and **state reason clearly!** - Lastly, ensure to read question that they want it at y = 1, hence need to sub in and write according to question to ensure you are answering the question!
(c)	Smallest value of $\alpha=3$ (for g to be one-one)	- To find smallest α, students can draw graph using GC and find the turning point.

	Since $3 \le x \le 10, \lvert x-3 \rvert = x-3$ $\therefore y = e^{x-3}$ $\ln y = x-3$ $x = \ln y + 3$ $g^{-1}(x) = \ln x + 3$ $D_{g^{-1}} = R_g = [1,\ e^7]$ 	- To find g^{-1}, do not ignore the modulus! - **And to reject with reason!** - Students to note that if they can reject, pls do so at the right juncture because if they proceed to reject at later part, they will have to provide more reasoning! - Most students know that $D_{g^{-1}} = R_g$ but please remember in finding R_g, you must draw graph **according to domain.**

9



The figure above shows a metal rod with length l cm. The cross-section of the rod is a regular hexagon with sides of length x cm.

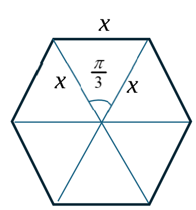
(a) A regular hexagon is made up of six identical triangles. Show that the area of the cross-section of the rod is $\dfrac{3\sqrt{3}x^2}{2}$ cm^2. [2]

(b) Suppose the rod has a fixed volume of C cm^3, show that the total surface area, S cm^2, of the rod may be expressed as $S = 3\sqrt{3}x^2 + \dfrac{4C}{\sqrt{3}x}$. [3]

(c) By using differentiation, find the value of x, in terms of C, which minimises S. [4]

Lucas heats up one of these metal rods. When heated, the metal rod expands uniformly such that it always retains its shape. At time t seconds, the length of each side of the hexagon is x cm, the length of the rod is l cm and the volume of the rod is V cm^3.

(d) Given that x and l are both increasing at a constant rate of 0.0025 cms^{-1}, find the rate of increase of V at the instant when $x = 2$ and $l = 5$. [2]

	[Solutions]	**Remarks**
(a)	Area $= \left(\frac{1}{2}\left(x^2\right)\sin\frac{\pi}{3}\right)\times 6$ $= \frac{3\sqrt{3}}{2}x^2$ 	Each of the 6 equilateral triangles **subtends** $\frac{360^\circ}{6} = 60^\circ$ **at the centre.** Write 60°, not 60. Use area of triangle $= \frac{1}{2}ab\sin C$
(b)	Volume of rod = cross sectional area $\times$length $C = \frac{3\sqrt{3}}{2}x^2\left(l\right)$ $l = \frac{2C}{3\sqrt{3}x^2}$	

<table>
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$$S = 2\left(\frac{3\sqrt{3}}{2}x^2\right) + 6xl$$

$$= 2\left(\frac{3\sqrt{3}}{2}x^2\right) + 6x\left(\frac{2C}{3\sqrt{3}x^2}\right)$$

$$= 3\sqrt{3}x^2 + \frac{4C}{\sqrt{3}x} \quad \text{(shown)}$$

</td>
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As the answer is given, detailed steps are required.

</td>
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<tr>
<td>

(c)

</td>
<td>

$$\frac{dS}{dx} = 6\sqrt{3}x - \frac{4C}{\sqrt{3}x^2}$$

At stationary points, let $\frac{dS}{dx} = 0$

$$6\sqrt{3}x = \frac{4C}{\sqrt{3}x^2}$$

$$18x^3 = 4C$$

$$x = \sqrt[3]{\frac{2C}{9}}$$

Using 2nd derivative test (preferred)

$$\frac{d^2S}{dx^2} = 6\sqrt{3} + \frac{8C}{\sqrt{3}x^3}$$

Since $x > 0$ and $C > 0$, $\frac{d^2S}{dx^2} > 0$

[or when $x = \sqrt[3]{\frac{2C}{9}}$,

$$\frac{d^2S}{dx^2} = 6\sqrt{3} + \frac{8C}{\sqrt{3}\left(\frac{2C}{9}\right)} = 18\sqrt{3} \text{ or } 31.2 > 0]$$

Hence S is minimum when $x = \sqrt[3]{\frac{2C}{9}}$.

OR Using 1st derivative test

$$\frac{dS}{dx} = 6\sqrt{3}x - \frac{4C}{\sqrt{3}x^2} = \frac{1}{\sqrt{3}}x\left(18 - \frac{4C}{x^3}\right)$$

<table>
<tr><td>x</td><td>$\left(\sqrt[3]{\frac{2C}{9}}\right)^-$
eg $0.60C$</td><td>$\left(\sqrt[3]{\frac{2C}{9}}\right)$</td><td>$\left(\sqrt[3]{\frac{2C}{9}}\right)^+$
eg $0.61C$</td></tr>
<tr><td>$\frac{dS}{dx}$</td><td>$-0.180C < 0$</td><td>0</td><td>$0.133C > 0$</td></tr>
<tr><td>slope</td><td>\</td><td>—</td><td>/</td></tr>
</table>

Hence S is minimum when $x = \sqrt[3]{\frac{2C}{9}}$.

</td>
<td>

$$\frac{d}{dx}\left(\frac{4C}{\sqrt{3}x}\right) = \left(\frac{4C}{\sqrt{3}}\right)\frac{d}{dx}\left(x^{-1}\right)$$

$$= -\left(\frac{4C}{\sqrt{3}}\right)\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{4C}{\sqrt{3}x^2}\right) = \left(\frac{4C}{\sqrt{3}}\right)\frac{d}{dx}\left(x^{-2}\right)$$

$$= -2\left(\frac{4C}{\sqrt{3}}\right)\left(\frac{1}{x^3}\right)$$

Do not write $x = \sqrt[3]{\frac{2C}{9}}$ is a minimum point/value ✗

Give **values** of $\frac{dS}{dx}$

</td>
</tr>
</table>

(d)	$V=\frac{3\sqrt{3}}{2}x^2l$ **Differentiate wrt *t*,** $\frac{dV}{dt}=\left(\frac{3\sqrt{3}}{2}\right)\left[l\left(2x\frac{dx}{dt}\right)+x^2\frac{dl}{dt}\right]$ When $x=2$ and $l=5$ and $\frac{dx}{dt}=\frac{dl}{dt}=0.0025$ $\frac{dV}{dt}=\left(\frac{3\sqrt{3}}{2}\right)\left[5\left(2(2)(0.0025)\right)+2^2(0.0025)\right]$ $=0.159$ Thus the rate of increase of V is 0.159 cm^3s^{-1}. **Alternatively,** $V=\frac{3\sqrt{3}}{2}x^2l$ **Differentiate wrt *x*,** $\frac{dV}{dx}=\left(3\sqrt{3}x\right)l\ +\ \left(\frac{3\sqrt{3}x^2}{2}\right)\frac{dl}{dx}$ $\frac{dl}{dx}=\left(\frac{dl}{dt}\right)/\left(\frac{dx}{dt}\right)=\frac{0.0025}{0.0025}=1$ $\frac{dV}{dt}=\frac{dV}{dx}\times\frac{dx}{dt}$ $=\left[\left(3\sqrt{3}(2)\right)(4)\ +\ \left(\frac{3\sqrt{3}(2)^2}{2}\right)(1)\right]\times 0.0025$ $=0.156$ Thus the rate of increase of V is 0.159 cm^3s^{-1}.	**Since *l* is not a constant, we need to use product rule to find $\frac{dV}{dt}$** **Since *l* is not a constant, we need to use product rule to find $\frac{dV}{dx}$**

10 Anand writes a computer programme to simulate a population of organisms in a controlled environment. It is assumed that none of the organisms die or leave the environment within the duration of a simulation.

(a) In Simulation A, 200 organisms are introduced to the environment on Day 1. At the start of each subsequent day, 48 more organisms are introduced to the environment. Find the first day when the number of organisms in the environment exceeds 2025 at the end of that day. [2]

(b) In Simulation B, 15 organisms are introduced to the environment on Day 1. At the start of each subsequent day, each organism in the environment spawns two more organisms of the same type, i.e there are 45 organisms at the end of Day 2. Find the number of organisms in the environment at the end of Day 20. [2]

(c) In Simulation C, 5 organisms are introduced to the environment on Day 1. At the start of each subsequent day, the organisms in the environment will spawn in either one of the following ways.

I: Each organism will spawn three more organisms of the same type.

II: Each organism will spawn five more organisms of the same type.

On Day 2 to Day 9, the organisms undergo process **I** on *m* days and process **II** on the other days. Given that there are 1,105,920 organisms at the end of Day 9, find the value of *m*. [2]

Anand then adjusts the programme such that the simulation would allow for organisms to die at certain junctures.

(d) In Simulation D, 100 organisms are introduced to the environment at the start of Day 1. At the end of each day, 10% of the total population in the environment would die. At the start of Day 2 and each subsequent day, 20 organisms are introduced to the environment.

(i) Find an expression for the population size, *P*, in the environment at the start of Day *n*, after the organisms have been introduced. Leave your answer in the form $s-t\left(r^{n-1}\right)$, where *s* and *t* are positive integers and *r* is a real number. [4]

(ii) Describe what happens to the population size in the environment in the long term. [1]

(iii) Explain why the conclusion in **(ii)** does not depend on the population size in the environment on Day 1. [1]

<table>
<tr><th></th><th>[Solutions]</th><th>Remarks</th></tr>
<tr><td>(a)</td><td>
Let A_n be the existing number of organisms under Stimulation A on Day n.
<table>
<tr><th>Day</th><th>Existing Number</th></tr>
<tr><td>1</td><td>200</td></tr>
<tr><td>2</td><td>200 + 48 = 200 + 1(48)</td></tr>
<tr><td>3</td><td>(200 + 48) + 48 = 200 + 2(48)</td></tr>
<tr><td>4</td><td>((200 + 48) + 48) + 48 = 200 + 3(48)</td></tr>
<tr><td>…</td><td>…</td></tr>
<tr><td>n</td><td>200 + $(n – 1)$(48)</td></tr>
</table>
$A_n = 200 + (n - 1)(48) \geq 2025$ (A.P.)
$n \geq 39.02$
Thus there are at least 2025 organisms in the environment on Day 40.
</td><td>
Tabulating the values to derive a trend for the information given is helpful in assessing whether the nth term or the sum to nth term is the logical value required.

When tabulating values, the focus is on deriving the trend. Do not over evaluate.

Evaluating the expression to obtain the final value may hinder the derivation of the correct trend.
</td></tr>
<tr><td>(b)</td><td>
Let B_n be the number of organisms under Stimulation B on Day n.
<table>
<tr><th>Day</th><th>Current</th><th>Spawned</th><th>Total</th></tr>
<tr><td>1</td><td>15</td><td>0</td><td>15</td></tr>
<tr><td>2</td><td>15</td><td>15(2)</td><td>15 + 15(2)
= 15(1 + 2)
$= 15(3)^1$</td></tr>
<tr><td>3</td><td>$15(3)^1$</td><td>15(3) × 2</td><td>15(3) + 15(3) × 2
= [15(3)](1 + 2)
$= 15(3)^2$</td></tr>
<tr><td>4</td><td>$15(3)^2$</td><td>$15(3)^2$</td><td>$15(3)^2 + 15(3)^2 \times 2$
$= [15(3)^2](1 + 2)$
$= 15(3)^3$</td></tr>
<tr><td>…</td><td>…</td><td>…</td><td>…</td></tr>
<tr><td>n</td><td>$15(3)^{n-2}$</td><td>$15(3)^{n-2} \times 2$</td><td>$15(3)^{n-2} + 15(3)^{n-2} \times 2$
$= [15(3)^{n-2}](1 + 2)$
$= 15(3)^{n-1}$</td></tr>
</table>
$B_{20} = 15(3)^{19} = 1.74 \times 10^{10}$ (G.P.)
</td><td>
It is important to read the question carefully.

The phrase "there are 45 organisms at the end of Day 2" provides important information that the original 15 organisms whereby each spawned 2 others the next day yielded a total population of 15 × 3 = 45 organisms.

This indicates the parent organism is part of the population i.e after the daily spawning process, the population is tripled.
</td></tr>
</table>

(c)

$5\times4^m\times6^{8-m} = 1105920$

Using GC, $m = 5$

OR By factorisation, $1105920 = 5\times4^5\times6^3$ $\therefore$ $m = 5$

Note:

<u>Commutative Property of Multiplication</u>

You would have probably learned it formally in the lower secondary levels.

Essentially, we can say that

$5\times4\times6 = 5\times6\times4$.

i.e. the order of multiplication does not matter, which we all know.

The implication here is that the order of spawning via process I or II does not matter and since it does not matter, all we need to know that in the 8 days running from days 2 to 9, m days involve spawning by process I and 8 – m days will involve spawning by process II, leading to the final population being computed by $5\times4^m\times6^{8-m}$.

Extending from part **(b)**, the spawning 3 more organisms will means the population is multiplied by 4 times.

Likewise, spawning 4 more organisms will indicate the population is multiplied by 5 times.

Looking beyond the question, the important learning point to take away to scrutinize the information provided carefully as it may have a downstream impact towards the understanding of the different parts of the question.

Notice that once part (b) was incorrectly understood, part (c) would have likely used the wrong values of 3 and 5 instead of the required 4 and 6.

(d) (i)

n	Organisms at start of Day n (units)	Organisms at end of Day n (units)
1	100	(0.9)100
2	(0.9)100 + 20	$(0.9)^2(100) + (0.9)(20)$
3	$(0.9)^2(100) + (0.9)(20) + 20$ $= (0.9)^2(100) + (0.9)^1(20) + (0.9)^0(20)$	$(0.9)^3100 + (0.9)^2(20) + (0.9)(20)$
4	$(0.9)^3(100) + (0.9)^2(20) + (0.9)^1(20) + (0.9)^0(20)$	:
:	:	:
n	$(0.9)^{n-1}(100) + (0.9)^{n-2}(20) + (0.9)^{n-3}(20) + \ldots + 20(0.9)^0$	:

At start of Day n, after the organisms are introduced, the population size

$= (0.9)^{n-1}(100) + (0.9)^{n-2}(20) + (0.9)^{n-3}(20) + \ldots + 20$

$$= (0.9)^{n-1}(100) + 20\left(\frac{1-(0.9)^{n-1}}{1-0.9}\right)$$

$= (0.9)^{n-1}(100) + 200(1-(0.9)^{n-1})$
$= 200 - 100(0.9)^{n-1}$

Comments

1. When tabulating a fairly complicated series, it is always important not to over evaluate. The focus is to identify a trend in the tabulated expression and not a number pattern arising from evaluated values.

2. Group like terms together to form a series. To this end, some rules of thumb which may be useful are:
   - Terms with the same constant value multiplied to the same ratio that changes exponentially likely forms a GP e.g. in this question we have:

     $$(0.9)^{n-2}(20) + (0.9)^{n-3}(20) + \ldots + 20(0.9)^{0}$$

     where 20 is the same constant value and 0.9 is the multiplier ratio that changes exponentially.

     Note that when the same constant value has no ratio multiplied to it, in this case 20, we can append the multiplier term $(\text{ratio})^0$, in this case 0.9^0 to ensure that the number of terms in the GP can be counted correctly, in this case from 0 to $n-2$, there will be $(n-2)-0+1 = n-1$ terms [the idea of no. of terms = upper limit – lower limit + 1]

   - A standalone constant raised to an exponent or a standalone constant multiplied to a ratio raised to an exponent is likely a power series e.g. in this question $100(0.9)^{n-1}$ or say 87^n for another unrelated instance.

   - Terms with the same constant being add progressively will likely form an AP.

3. It is good practice to put your working for identifying the trend in a table for proper organisation.

4. To determine the n^{th} expression correctly, the trick is to first observe the latest terms of each group to see what is the relation to n e.g. in this question

Day	Organisms at start of Day n (units)
4	$(0.9)^3(100) + (0.9)^2(20) + (0.9)^1(20) + (0.9)^0(20)$

For the power series $(0.9)^3(100)$, 0.9 is raised to the power $3 = 4 - 1$. So it follows that in the n^{th} term, the expected component will be $(0.9)^{n-1}(100)$

Likewise, for the terms comprising of $(0.9)^2(20) + (0.9)^1(20) + (0.9)^0(20)$, the highest power of 0.9 is 2. Hence for the n^{th} term, the expected expression will be $(0.9)^{n-2}(20) + \ldots + (0.9)^0(20)$.

(ii)	As $n \to \infty$, $(0.9)^{n-1} \to 0$ and thus $P \to 200$. In the long run, the population size approaches 200.	It is important not to provide the conclusion directly but instead provide a term-wise trend leading to the final conclusion to ensure clarity. Make sure that the conclusion is logical with reference to the context of the question e.g. in this case, the population cannot be a negative value.
(iii)	If the starting population was S instead of 100, the population at the end of Day n would be $(0.9)^{n-1}(S) + (0.9)^{n-2}(20) + (0.9)^{n-3}(20) + \dots + 20$ $= (0.9)^{n-1}(S) + 20\left(\dfrac{1-(0.9)^{n-1}}{1-0.9}\right)$ $= (0.9)^{n-1}(S) + 200(1-(0.9)^{n-1})$ $= (0.9)^{n-1}(S) + 200 - 200(0.9)^{n-1}$ $= 200 + (S-200)(0.9)^{n-1}$ Since $(0.9)^{n-1} \to 0$ as $n \to \infty$ for all values of S, the population size would still approach 200.	The 100 in the final expression $200 - 100(0.9)^{n-1}$ does not represented the initial population. The misconception arises as the value of 100 coincided with the initial population of the question. If we trace the working, we will realise that 100 herein isn't the initial population.

11 A metal ball is released from the surface of the liquid in a tall cylinder. The ball falls vertically through the liquid and the distance, x cm, that the ball has fallen in time t seconds is measured. The speed of the ball at time t seconds is v cms^{-1}. The ball is released in a manner such that $x = 0$ and $v = 0$ when $t = 0$.

(a) The motion of the ball is modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t} - 10 = 0.$$

It is given that $v = \dfrac{\mathrm{d}x}{\mathrm{d}t}$.

(i) Show that the differential equation can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - \frac{1}{2}v.$$ [1]

(ii) Using the differential equation in **(a)(i)**, find v in terms of t. **Hence** find x in terms of t. [6]

(a)(i)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t} - 10 = 0$ --- (1) $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ Differentiating w.r.t t: $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ Substituting into (1), $\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{2}v - 10 = 0$ $\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - \frac{1}{2}v$ (shown)	Students are reminded to show all their workings clearly for shown question, in particular how $\frac{\mathrm{d}v}{\mathrm{d}t}$ is obtained.

(ii)	$\frac{dv}{dt} = 10 - \frac{1}{2}v = \frac{20-v}{2}$ $\boxed{\text{Simplify into a single fraction first.}}$ $\int \frac{1}{20-v}\,dv = \int \frac{1}{2}\,dt$ $-\ln\lvert 20-v\rvert = \frac{1}{2}t + c$ $\lvert 20-v\rvert = e^{-\frac{1}{2}t-c}$ $20 - v = \pm e^{-c} e^{-\frac{1}{2}t}$ $20 - v = Ae^{-\frac{1}{2}t}$, where $A = \pm e^{-c}$ Alternatively, $\frac{dv}{dt} = 10 - \frac{1}{2}v$ $\int \frac{1}{10-\frac{1}{2}v}\,dv = \int 1\,dt$ $-2\ln\lvert 20-v\rvert = t + c$ $\ln\lvert 20-v\rvert = -\frac{1}{2}t - \frac{1}{2}c$ $\lvert 20-v\rvert = e^{-\frac{1}{2}t-\frac{1}{2}c}$ $20 - v = \pm e^{-\frac{1}{2}t-\frac{1}{2}c} = \pm e^{-\frac{1}{2}t} e^{-\frac{1}{2}c}$ $20 - v = Ae^{-\frac{1}{2}t}$, $A = \pm e^{-\frac{1}{2}c}$ Given: When $t = 0$, $v = 0$: $20 = A$ Thus $v = 20 - 20e^{-\frac{1}{2}t}$ Substituting $v = \frac{dx}{dt}$, $\frac{dx}{dt} = 20 - 20e^{-\frac{1}{2}t}$ $x = \int 20 - 20e^{-\frac{1}{2}t}\,dt$ $x = 20t - 20\left(\frac{e^{-\frac{1}{2}t}}{-\frac{1}{2}}\right) + d$ $x = 20t + 40e^{-\frac{1}{2}t} + d$ When $t = 0$, $x = 0$: $0 = 40 + d \Rightarrow d = -40$ Thus $x = 20t + 40e^{-\frac{1}{2}t} - 40$	Students are reminded not to memorise solution but to seek an understanding on how each step is obtained. Remove modulus first before finding the value of A. Recall: $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$ Need to sub in given conditions to find value of A and express v in terms of t as stated in the question Note that the relation Distance = speed × time is used only when speed is a constant. Here, the speed is not a constant! Sub in given conditions to find d!

(b) The metal ball is now released in another tall cylinder filled with a different liquid. However, for this liquid, the motion of the ball is modelled by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - k^2v^2 \text{, where } k \text{ is a positive constant.}$$

It is given that $x = 0$ and $v = 0$ when $t = 0$.

(i) Find v in terms of t and k. [5]

(ii) When the ball falls through this liquid, its speed will approach its "terminal speed" which is the speed it will attain after a long time. Find the ball's terminal speed in terms of k. You must show sufficient working to justify your answer. [2]

(b)(i)	$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - k^2v^2$ $\int \frac{1}{\left(\sqrt{10}\right)^2 - (kv)^2}\,\mathrm{d}v = \int 1\,\mathrm{d}t$ $\frac{1}{k}\int \frac{k}{\left(\sqrt{10}\right)^2 - (kv)^2}\,\mathrm{d}v = \int 1\,\mathrm{d}t$ Put in modulus! $\frac{1}{2k\sqrt{10}}\ln\left\|\frac{\sqrt{10}+kv}{\sqrt{10}-kv}\right\| = t + f$ $\ln\left\|\frac{\sqrt{10}+kv}{\sqrt{10}-kv}\right\| = 2k\sqrt{10}t + 2k\sqrt{10}f$ $\frac{\sqrt{10}+kv}{\sqrt{10}-kv} = \pm e^{2k\sqrt{10}f}e^{2k\sqrt{10}t} = Be^{2k\sqrt{10}t}$, $B = \pm e^{2k\sqrt{10}f}$	Recall: $\int \frac{f'(x)}{a^2 - [f(x)]^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left\|\frac{a+f(x)}{a-f(x)}\right\| + C$ **Note:** There's no modulus in the formula in MF26 as it is stated that $\|x\| < a$ which makes $\frac{a+x}{a-x} > 0$. f(x): $\frac{1}{a^2-x^2}$; $\int$ f(x) dx: $\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$; $(\|x\| < a)$ However, in this question, we do not know whether $\frac{\sqrt{10}+kv}{\sqrt{10}-kv}$ is positive or negative. Hence we need to put modulus for the ln function to be defined!
	When $t = 0$, $v = 0$: $\frac{\sqrt{10}}{\sqrt{10}} = 1 = B$ $\sqrt{10} + kv = e^{2k\sqrt{10}t}\left(\sqrt{10} - kv\right)$ $kv\left(1 + e^{2k\sqrt{10}t}\right) = \sqrt{10}\left(e^{2k\sqrt{10}t} - 1\right)$ $v = \frac{\sqrt{10}}{k}\left(\frac{e^{2k\sqrt{10}t} - 1}{e^{2k\sqrt{10}t} + 1}\right)$	Find v in terms of t and k!

(ii)

Method 1

$$v=\frac{\sqrt{10}}{k}\left(\frac{e^{2k\sqrt{10}t}-1}{e^{2k\sqrt{10}t}+1}\right)$$

Do not write $\frac{\infty}{\infty}$!

As $t\to\infty$, $e^{2k\sqrt{10}t}\to\infty$,

$$\frac{e^{2k\sqrt{10}t}-1}{e^{2k\sqrt{10}t}+1}\to 1$$

Essential workings

and thus $v\to\frac{\sqrt{10}}{k}$

Hence the ball's terminal speed is

$$\frac{\sqrt{10}}{k}\ \text{cms}^{-1}$$

Method 2

$$v=\frac{\sqrt{10}}{k}\left(\frac{e^{2k\sqrt{10}t}-1}{e^{2k\sqrt{10}t}+1}\right)\times\frac{e^{-2k\sqrt{10}t}}{e^{-2k\sqrt{10}t}}$$

$$=\frac{\sqrt{10}}{k}\left(\frac{1-e^{-2k\sqrt{10}t}}{1+e^{-2k\sqrt{10}t}}\right)$$

As $t\to\infty$, $e^{-2\sqrt{10}t}\to 0$

and thus $v\to\frac{\sqrt{10}}{k}\left(\frac{1-0}{1+0}\right)=\frac{\sqrt{10}}{k}$

Essential workings

Hence the ball's terminal speed is

$$\frac{\sqrt{10}}{k}\ \text{cms}^{-1}$$

Method 3

$$v=\frac{\sqrt{10}}{k}\left(\frac{e^{2\sqrt{10}t}-1}{e^{2\sqrt{10}t}+1}\right)$$

$$=\frac{\sqrt{10}}{k}\left(1-\frac{2}{e^{2\sqrt{10}t}+1}\right)\text{(by long division)}$$

As $t\to\infty$, $e^{2\sqrt{10}t}\to\infty$,

$$\frac{2}{e^{2\sqrt{10}t}+1}\to 0$$

Essential workings

and thus $v\to\frac{\sqrt{10}}{k}$

Method 4

At terminal speed,

$$\frac{dv}{dt}=10-k^2v^2=0$$

	$v^2 = \frac{10}{k^2}$ $v = \pm\frac{\sqrt{10}}{k}$ Since speed, $v > 0$, the ball's terminal speed is $\frac{\sqrt{10}}{k}$ cms^{-1}	