RAFFLES INSTITUTION
RAFFLES PROGRAMME 2023
YEAR 4 MATHEMATICS
TOPIC 2: REMAINDER & FACTOR THEOREMS AND PARTIAL FRACTIONS (MATH 1)

WORKSHEET 4

Name: ( ) Class: 4 ( ) Date:

<u>WORKSHEET 4: PARTIAL FRACTIONS</u>
think! Add Math Textbook A Chapter 4 p.70

KEY UNDERSTANDING(S)

Students will understand that

- Rational functions (algebraic fractions) can be expressed as partial fractions.
- Partial fraction decomposition, which breaks down an algebraic fraction into simpler partial fractions, is useful in the **integration** process.

LEARNER OUTCOMES

At the end of this worksheet, students will be able to

- Express proper algebraic fractions in partial fractions using an appropriate form based on what the denominator contains.
  - (i) Distinct linear factors $(ax+b)(cx+d)$
  - (ii) Repeated linear factors $(ax+b)^2$
  - (iii) Quadratic factor x^2+c^2 (cannot be factorized)
- Identify improper fractions and express it as a sum of a polynomial and a proper algebraic fraction first.

(1) INTRODUCTION

<u>(1.1) Proper and Improper Algebraic Fractions</u>

An algebraic fraction $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$, where $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ are polynomials such that $\mathrm{Q}(x) \neq 0$, is called a rational function.

The table shows some examples of proper and improper algebraic fractions.

Proper algebraic fractions	$\frac{1}{5-x}$, $\frac{8-x}{x^2-2x+3}$, $-\frac{4-2x+3x^4}{2x^5+7}$
Improper algebraic fractions	$-\frac{x^3+8}{4x^2-x+6}$, $\frac{x^4-x^3+2x}{9-5x}$, $\frac{x}{x+2}$, $\frac{6x^2+x+9}{(3x-1)(x+2)}$

Is $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ proper or improper if:

(a) degree of $\mathrm{P}(x)$ < degree of $\mathrm{Q}(x)$?

(b) degree of $\mathrm{P}(x)$ > degree of $\mathrm{Q}(x)$?

(c) degree of $\mathrm{P}(x)$ = degree of $\mathrm{Q}(x)$?

(1.2) What are Partial Fractions?

Early in Algebra, you learn how to combine "simple" algebraic fractions into single algebraic fraction.

For example,

$$\frac{2}{x-2}+\frac{3}{x+1}=\frac{2(x+1)+3(x-2)}{(x-2)(x+1)}$$

$$=\frac{5x-4}{(x-2)(x+1)}$$

The **Method of Partial Fractions** does the opposite. It dissects a single algebraic fraction into a sum of single proper fractions. While this is a little more complicated than going the other direction, it is also more useful. Applications of the method of partial fractions include:

- Integration of rational functions in Calculus
- Integration of the secant function in Mercator map projection (widely used in navigation)
- Fractional radioactive decay law and Bateman equations in nuclear engineering
- Minimum payments due on credit card bills

(2) PROPER FRACTION WITH DISTINCT LINEAR FACTORS IN DENOMINATOR

We first consider proper algebraic fraction whose denominator can be factorised completely into *n* distinct linear factors.

Rule 1:
For every linear factor of the form $(ax+b)$ in the denominator, there will be a component of the form $\frac{A}{ax+b}$, where A is a constant.

EG 1 Express $\frac{9x-5}{(x-3)(2x+5)}$ in partial fractions.

Steps:

1. Express the fraction as a sum of its components in the form $\frac{A}{ax+b}$ for each distinct linear factor in the denominator.
2. Find the values of unknown constants by "removing" the denominators on both sides and solve the identity using the methods we learnt in Worksheet 1.
3. Express the original algebraic fraction in partial fractions.

Computational Thinking - Opportunities for Algorithmic Thinking

- *Write out the general approach as a sequence of steps*
- *Work out the rules for partial fractions*

EG 2 Express each of the following in partial fractions.

(a) $\dfrac{2x+1}{(x-1)(x+2)(x+3)}$ (b) $\dfrac{1}{1-x-2x^2}$

Note:

1. ***Remember to factorise the denominator completely.***
2. ***Do not expand the denominator in the answer.***

<u>EG 3</u> *think! Add Math Textbook A p.71 Practice Now 13 Q2*

(i) Factorise completely the cubic polynomial $2x^3+3x^2-17x+12$.

(ii) Express $\dfrac{7x^2-25x+8}{2x^3+3x^2-17x+12}$ as a sum of 3 partial fractions.

HOMEWORK 1

LEVEL 2

1. Express each of the following in partial fractions.

   (a) $\dfrac{8x+51}{9-64x^2}$ (b) $\dfrac{7}{2x^2+3x}$

[Ans: (a) $\dfrac{8}{3+8x}+\dfrac{9}{3-8x}$ (b) $\dfrac{7}{3x}-\dfrac{14}{3(2x+3)}$]

(3) COVER-UP METHOD

The cover-up method is a faster technique in finding unknown constants in partial fractions. We can only apply this method when the denominator is a product of **linear factors**.

For example, if the denominator has three distinct linear factors,
we have

$$\frac{\mathrm{f}(x)}{(x-a)(x-b)(x-c)} \equiv \frac{A}{x-a}+\frac{B}{x-b}-\frac{C}{x-c}$$

Then by **cover-up method**, A can be computed by covering up the term $(x-a)$ in the denominator on the LHS and substituting $x=a$ in the remaining expression:

$$A=\frac{\mathrm{f}(a)}{(a-b)(a-c)}$$

This works because the computation is equivalent to multiplying the expression throughout by the term $(x-a)$ and then making the substitution $x=a$.

Similarly, by substituting $x=b$ and $x=c$, we can compute B and C:

$$B=\frac{\mathrm{f}(b)}{(b-a)(b-c)} \quad , \quad C=\frac{\mathrm{f}(c)}{(c-a)(c-b)}$$

Using **EG 3**:

$$\frac{7x^2-25x+8}{(x-1)(x+4)(2x-3)} \equiv \frac{A}{x-1}+\frac{B}{x+4}-\frac{C}{2x-3}$$

By cover-up method:

Sub. $x=1$: $\quad A=\dfrac{7(1)^2-25(1)+8}{(1+4)(2-3)}=\dfrac{-10}{5(-1)}=2$

Sub. $x=-4$: $\quad B=\dfrac{7(-4)^2-25(-4)+8}{(-4-1)(-8-3)}=\dfrac{220}{-5(-11)}=4$

Sub. $x=\dfrac{3}{2}$: $\quad C=\dfrac{7\left(\frac{3}{2}\right)^2-25\left(\frac{3}{2}\right)+8}{\left(\frac{3}{2}-1\right)\left(\frac{3}{2}+4\right)}=\dfrac{-\frac{55}{4}}{\frac{1}{2}\left(\frac{11}{2}\right)}=-5$

Hence $\dfrac{7x^2-25x+8}{2x^3+3x^2-17x+12} \equiv \dfrac{2}{x-1}+\dfrac{4}{x+4}-\dfrac{5}{2x-3}$

EG 4 Express each of the following in partial fractions using the cover-up method.

(a) $$\frac{1}{(5x-1)(4x+1)}$$

(b) $$\frac{10x-11}{2x^2-3x-5}$$

<u>EG 5</u> (i) Express $\dfrac{1}{x(x+2)}$ in partial fractions.

(ii) Hence, find the exact value of $\dfrac{1}{1\times 3}+\dfrac{1}{2\times 4}+\dfrac{1}{3\times 5}+\ ...\ +\dfrac{1}{18\times 20}$.

HOMEWORK 2

LEVEL 1

1. Express each of the following in partial fractions.

(a) $\dfrac{3x-4}{(x+2)(2x-1)}$ (b) $\dfrac{3}{(2x-1)(x+2)}$ (c) $\dfrac{4x+1}{x^2+3x-4}$

[Ans: (a) $\dfrac{2}{x+2}-\dfrac{1}{2x-1}$ (b) $\dfrac{6}{5(2x-1)}-\dfrac{3}{5(x+2)}$ (c) $\dfrac{3}{x+4}+\dfrac{1}{x-1}$]

LEVEL 2

1. Express each of the following in partial fractions.

(a) $\dfrac{4x-3}{2x^2-5x-3}$ (b) $\dfrac{2}{4x^3-3x^2-x}$

[Ans: (a) $\dfrac{10}{7(2x+1)}+\dfrac{9}{7(x-3)}$ (b) $-\dfrac{2}{x}+\dfrac{32}{5(4x+1)}+\dfrac{2}{5(x-1)}$]

2. Express $\dfrac{2x^2 - x + 3}{x^3 - 2x^2 - x + 2}$ in partial fractions.

[Ans: $\dfrac{1}{x+1} - \dfrac{2}{x-1} + \dfrac{3}{x-2}$]

(4) PROPER FRACTION WITH REPEATED LINEAR FACTORS IN DENOMINATOR

Consider the following example where two algebraic fractions are combined into a single algebraic fraction:

$$\frac{3}{2x-5}-\frac{2}{(2x-5)^2}=\frac{3(2x-5)-2}{(2x-5)^2}$$

$$=\frac{6x-17}{(2x-5)^2}$$

To express $\frac{6x-17}{(2x-5)^2}$ in partial fractions, the form to be used is $\frac{A}{2x-5}+\frac{B}{(2x-5)^2}$, where there is a partial fraction component for each subsequent lower power of the repeated linear factor.

Rule 2:
For every repeated linear factor of the form $(ax+b)^n$ in the denominator, there will be components of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots+\frac{A_n}{(ax+b)^n}$$

where $A_1, A_2, A_3, \ldots, A_n$ are constants.

EG 6 Express $\frac{4x^2+5x-32}{(x-11)(x+2)^2}$ in partial fractions.

Note:
For repeated linear factors, the cover-up method can only solve for unknown constant where the repeated factor is of the highest degree. Do you know why?

<u>EG 7</u> Express $\dfrac{39x^2-35x+11}{\left(6x^2+7x-3\right)\left(3x-1\right)}$ in partial fractions.

HOMEWORK 3

LEVEL 1

1. Express each of the following in partial fractions.

(a) $\dfrac{2}{(x+1)(x-1)^2}$ (b) $\dfrac{11-x-x^2}{(x+2)(x-1)^2}$ (c) $\dfrac{x^2-4x-8}{(x+2)^2(x+1)}$

[Ans: (a) $\dfrac{1}{2(x+1)}-\dfrac{1}{2(x-1)}+\dfrac{1}{(x-1)^2}$ (b) $\dfrac{1}{x+2}-\dfrac{2}{x-1}+\dfrac{3}{(x-1)^2}$ (c) $\dfrac{4}{x+2}-\dfrac{4}{(x+2)^2}-\dfrac{3}{x+1}$]

LEVEL 2

1. Express each of the following in partial fractions.

(a) $\dfrac{1}{s^2(s-1)^2}$ (b) $\dfrac{x^2+1}{(x^2+3x+2)(x+2)}$ (c) $\dfrac{7-2x}{x^3-3x^2+4}$

[Ans: (a) $\frac{2}{s}+\frac{1}{s^2}-\frac{2}{s-1}+\frac{1}{(s-1)^2}$ (b) $\frac{2}{x+1}-\frac{1}{x+2}-\frac{5}{(x+2)^2}$ (c) $\frac{1}{x+1}-\frac{1}{x-2}+\frac{1}{(x-2)^2}$]

(5) PROPER FRACTION WITH QUADRATIC FACTOR IN DENOMINATOR

Irreducible quadratic factors are quadratic factors that when set equal to zero only have *complex* roots. As a result, they cannot be reduced into factors containing only real numbers, hence the name **irreducible**.

Examples include x^2+1, x^2+4 or indeed x^2+c^2 for any real number c, x^2+x+1 (use the quadratic formula to see the roots) and $2x^2-x+1$.

<u>Rule 3</u>:
For every irreducible quadratic factor of the form $\left(x^2+c^2\right)$ in the denominator, there will be a component of the form $\dfrac{Ax+B}{x^2+c^2}$, where A and B are constants.

Is it possible for the numerator to be a constant? If yes, why do we write $\dfrac{Ax+B}{x^2+c^2}$ *instead of* $\dfrac{B}{x^2+c^2}$*?*

<u>EG 8</u> Express each of the following in partial fractions.

(a) $\dfrac{x^3+x-1}{x^2+x^4}$ (b) $\dfrac{1}{(x-1)^2\left(x^2+1\right)}$

HOMEWORK 4

LEVEL 1

1. Express each of the following in partial fractions.

(a) $\dfrac{5x}{(1+2x)(x^2+1)}$ (b) $\dfrac{5x^2}{(1+x^2)(x-2)}$ (c) $\dfrac{2x-1}{(x-1)(x^2+1)}$

[Ans: (a) $-\dfrac{2}{1+2x}+\dfrac{x+2}{x^2+1}$ (b) $\dfrac{x+2}{1+x^2}+\dfrac{4}{x-2}$ (c) $\dfrac{1}{2(x-1)}+\dfrac{3-x}{2(x^2+1)}$]

LEVEL 2

1. Express each of the following in partial fractions.

(a) $\dfrac{2+5x+15x^2}{(2-x)(1+2x^2)}$ (b) $\dfrac{x^2+15}{(x+3)^2(x^2+3)}$

[Ans: (a) $\dfrac{8}{2-x}+\dfrac{x-3}{1+2x^2}$ (b) $\dfrac{1}{2(x+3)}+\dfrac{2}{(x+3)^2}+\dfrac{1-x}{2(x^2+3)}$]

(6) IMPROPER ALGEBRAIC FRACTIONS

An algebraic fraction $\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is **improper** if **degree of** $\mathbf{P}(x) \geq$ **degree of** $\mathbf{Q}(x)$.

If an algebraic fraction is improper, we must express it as a sum of a polynomial and a proper fraction first, before expressing the proper fraction in partial fractions. To do this, we use **long division** to divide the numerator of the improper fraction by the denominator.

EG 9 Express $\dfrac{2x^2-7x-1}{x^2-x-2}$ in partial fractions.

Method 1 (use long division to find the quotient and the remainder)

Method 2 (write down the quotient by observation)

Note: Method 2 is only possible if you can read the quotient directly without division.

EG 10 Express each of the following in partial fractions.

(a) $\dfrac{5x^3-3x^2+6x-5}{(x-1)(x^2+2)}$ (b) $\dfrac{x^3+2x^2-x+1}{(x-1)(x+2)}$

HOMEWORK 5

LEVEL 1

1. Find the values of A, B and C if $\dfrac{2x^2+x+1}{(x+1)(x-2)} \equiv A + \dfrac{B}{x+1} + \dfrac{C}{x-2}$.

[Ans: $A = 2, B = -\dfrac{2}{3}, C = \dfrac{11}{3}$]

2. Express each of the following in partial fractions.

(a) $\dfrac{x^2+3x}{x^2-4}$ (b) $\dfrac{2x^3-3x^2+5x+4}{(2x+1)(x-2)}$ (c) $\dfrac{x^3-2}{(x+3)(x-1)}$

[Ans: (a) $1+\dfrac{1}{2(x+2)}+\dfrac{5}{2(x-2)}$ (b) $x-\dfrac{1}{5(2x+1)}+\dfrac{18}{5(x-2)}$ (c) $x-2+\dfrac{29}{4(x+3)}-\dfrac{1}{4(x-1)}$]

LEVEL 2

1. Express $\dfrac{4x^3+x^2-15x+21}{(x+2)(x-1)^2}$ in partial fractions.

[Ans: $4+\dfrac{23}{9(x+2)}-\dfrac{14}{9(x-1)}+\dfrac{11}{3(x-1)^2}$]

2. Express $\dfrac{x^3+2x-1}{2x^2-3x-2}$ in partial fractions.

[Ans: $\dfrac{1}{2}x+\dfrac{3}{4}+\dfrac{17}{20(2x+1)}+\dfrac{11}{5(x-2)}$]

LEVEL 3

*1. Express $\dfrac{2}{n(n+1)(n+2)}$ in partial fractions. Hence, evaluate

$$\frac{2}{1\times2\times3}+\frac{2}{2\times3\times4}+\frac{2}{3\times4\times5}+\ldots+\frac{2}{2005\times2006\times2007}+\frac{2}{2006\times2007\times2008}$$

[Ans: $\dfrac{2015027}{4030056}$]

(7) SUMMARY

Decomposition into partial fractions

Step 1: If $\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is improper, express it as a sum of a polynomial and a proper algebraic fraction first.

Step 2: Factorise the denominator of the proper algebraic fraction if possible.

Step 3: Express the proper algebraic fraction in partial fractions according to Case 1, 2 or 3.

Case	Denominator contains	Proper Fraction	Partial Fractions Form
1	Distinct linear factors $(ax+b)(cx+d)$	$\dfrac{px+q}{(ax+b)(cx+d)}$	$\dfrac{A}{ax+b}+\dfrac{B}{cx+d}$
2	Repeated linear factors $(ax+b)^2$	$\dfrac{px+q}{(ax+b)^2}$	$\dfrac{A}{ax+b}+\dfrac{B}{(ax+b)^2}$
3	Quadratic factor x^2+c^2 (cannot be factorized)	$\dfrac{px^2+qx+r}{(ax+b)(x^2+c^2)}$	$\dfrac{A}{ax+b}+\dfrac{Bx+C}{x^2+c^2}$

Step 4: Solve for unknown constants by substituting suitable values of x or comparing coefficients of the terms.

(8) FOR YOUR INTEREST

(8.1) PARTIAL FRACTION CALCULATOR

You can use this to verify your answers for partial fraction decomposition. Scan the QR code to start exploring.

https://www.wolframalpha.com/calculators/partial-fraction-calculator

(8.2) PROPER FRACTION WITH REPEATED QUADRATIC FACTORS IN DENOMINATOR

Rule:

For every repeated irreducible quadratic factor of the form $(x^2+c^2)^n$, $n \geq 2$, in the denominator, there will be components of the form

$$\frac{A_1x+B_1}{x^2+c^2}+\frac{A_2x+B_2}{(x^2+c^2)^2}+\frac{A_3x+B_3}{(x^2+c^2)^3}+\ldots+\frac{A_nx+B_n}{(x^2+c^2)^n}$$

where $A_1, A_2, A_3, \ldots, A_n$ and $B_1, B_2, B_3, \ldots, B_n$ are constants.

EG Express $\dfrac{4}{(x+1)(x^2+1)^2}$ in partial fractions.

Let $\dfrac{4}{(x+1)(x^2+1)^2} \equiv \dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}$

By cover-up method:

Sub $x=-1$: $A=\dfrac{4}{(1+1)^2}=1$

$$\frac{4}{(x+1)(x^2+1)^2} \equiv \frac{1}{x+1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

$$4 \equiv (x^2+1)^2+(Bx+C)(x+1)(x^2+1)+(Dx+E)(x+1)$$

Compare coefficients of x^4: $1+B=0$
$B=-1$

Compare coefficients of x^3: $B+C=0$
$C=1$

Compare coefficients of x^2: $2+B+C+D=0$
$D=-2$

Compare constant terms: $1+C+E=4$
$E=2$

$$\therefore \frac{4}{(x+1)(x^2+1)^2} \equiv \frac{1}{x+1}+\frac{1-x}{x^2+1}+\frac{2-2x}{(x^2+1)^2}$$

PRACTICE QUESTIONS

Express each of the following in partial fractions.

(a) $\dfrac{-3x^4+4x^3-5}{\left(x^2+1\right)^2\left(x+1\right)^2}$ [Ans: $\dfrac{4x-2}{\left(x^2+1\right)^2}-\dfrac{3}{\left(x+1\right)^2}$]

(b) $\dfrac{x^3}{\left(x-1\right)\left(x^2+2\right)^2}$ [Ans: $\dfrac{1}{9\left(x-1\right)}+\dfrac{8-x}{9\left(x^2+2\right)}+\dfrac{2x-4}{3\left(x^2+2\right)^2}$]

(c) $\dfrac{2x+1}{x^4+x^3+x+1}$ [Ans: $\dfrac{1}{3\left(x+1\right)}-\dfrac{1}{3\left(x+1\right)^2}+\dfrac{3-x}{3\left(x^2-x+1\right)}$]
