Topic 10 *Oscillations*

Guiding Questions

- *What are the characteristics of periodic motion? How can we study and describe such motion?*
- *How can circular motion be related to simple harmonic motion?*
- *How do we analyse simple harmonic motion?*

Content

- Simple harmonic motion
- Energy in simple harmonic motion
- Damped and forced oscillations, resonance

Learning Outcomes

Candidates should be able to:

(a) describe simple examples of free oscillations
(b) investigate the motion of an oscillator using experimental and graphical methods
(c) show an understanding of and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency
(d) recall and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion
(e) recognise and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$
(f) recognise and use the equations $v = v_0 \cos \omega t$ and $v = \pm\omega\sqrt{\left(x_0^{\ 2} - x^2\right)}$
(g) describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion
(h) describe the interchange between kinetic and potential energy during simple harmonic motion
(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and to the importance of critical damping in cases such as a car suspension system
(j) describe practical examples of forced oscillations and resonance
(k) describe graphically how the amplitude of a forced oscillation changes with driving frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and the sharpness of the resonance
(l) show an appreciation that there are some circumstances in which resonance is useful and other circumstances in which resonance should be avoided.

Introduction

An **oscillation** is a periodic to-and-fro motion of an object between two limits.
Periodic means the oscillations repeat themselves.

There are three types of oscillations:
(1) free oscillation, (2) damped oscillation, & (3) forced oscillation

(a) describe simple examples of free oscillations

Free Oscillations (oscillating system is isolated)

When an object undergoes **free oscillation**, it oscillates with no energy gain or loss. The object oscillates with constant amplitude, as there are no external force acting on it.

Examples of free oscillations include, when friction and air resistance are negligible,

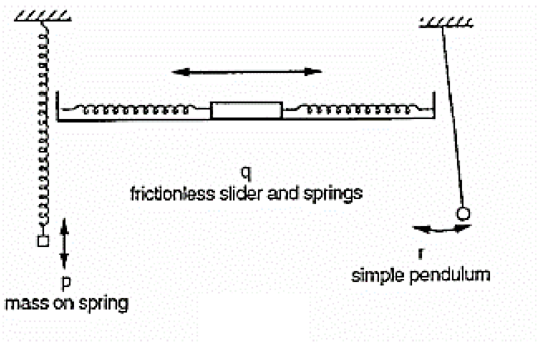
(1) a simple vertical (vibrating) spring-mass system,
(2) a simple horizontal spring-mass system, and
(3) a simple (swinging) pendulum

It can be shown (in the Annex) that the natural frequency of p, q and r are

$f_p = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, $f_q = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$ and $f_r = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$

respectively

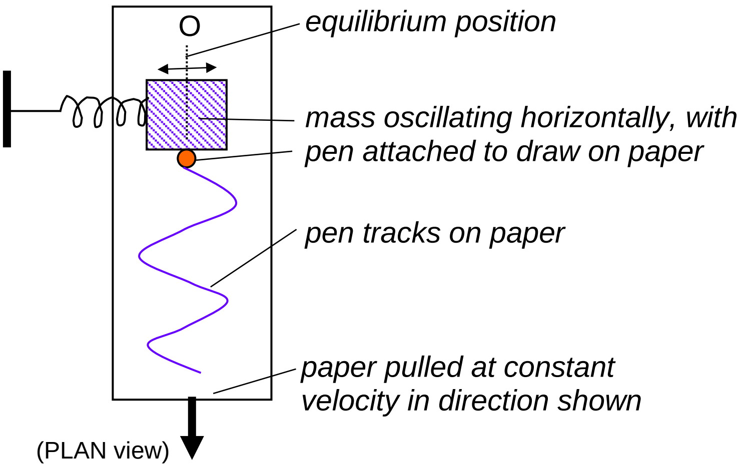
where the spring constant of the spring is k, the length of the pendulum is l and the mass of the oscillator is m.



(b) investigate the motion of an oscillator using experimental and graphical methods

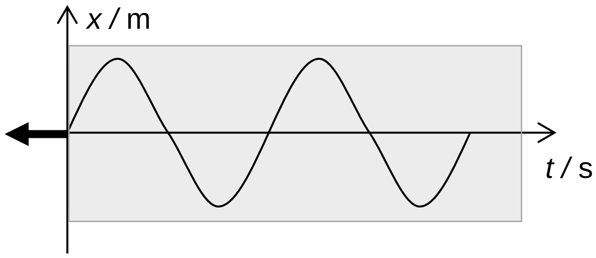
Experimental Investigation

We can plot the displacement-time graphs for oscillators. One possible experimental method is illustrated below.



(PLAN view)

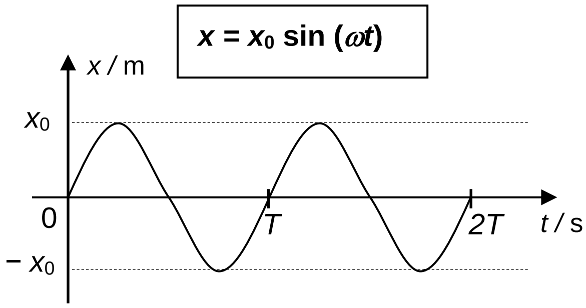
The displacement of the oscillating body varies with time sinusoidally.

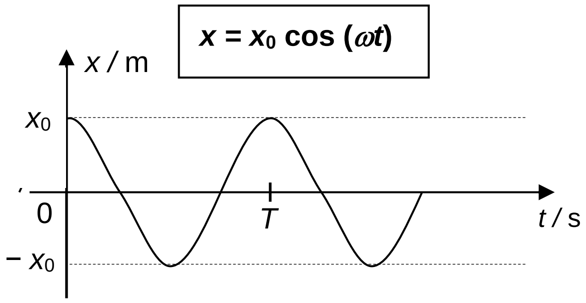


Note:
- In the absence of air resistance and friction, a displacement-time graph is sinusoidal. This is a characteristic of an important type of oscillation called **simple harmonic motion.**
- The mass does not actually move along the sinusoidal curve. It always moves back and forth over the same path (periodic).
- Graph is sinusoidal, but need not always be a sine curve (i.e. oscillator which is at equilibrium position at t = 0). It can also be a cosine curve (i.e. oscillator at maximum displacement at t = 0) or any other sinusoidal curve, depending on the initial conditions.

(c) show an understanding of and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency

Since an oscillatory motion can be graphically represented by a sinusoidal representation, we can mathematically describe its displacement (x) with respect to time (t) as the general equation $x = x_0 \sin(\omega t + \phi)$, and usually we are concerned with the particular solutions in which $\phi = 0^\circ$ or $\phi = 90^\circ$, depending on where the object starts oscillation at t = 0:



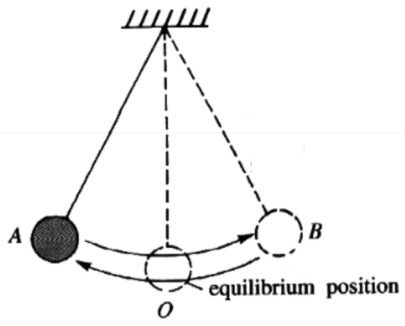


For $\boldsymbol{x = x_0}$ **sin** $\boldsymbol{(\omega t)}$, at t = 0 (starting time), the value of x = 0. i.e. the object passes the equilibrium position at t = 0.

For $\boldsymbol{x = x_0}$ **cos** $\boldsymbol{(\omega t)}$, at t = 0 (starting time), the value of $x = x_0$. i.e. the object is at the extreme position at t = 0.

In the table that follows, we will define the terms that is important in the topic of "Oscillations":

Physical Quantity	Definition	SI Unit (symbol)	Type
amplitude, x_0	The maximum displacement of the oscillating object from the equilibrium position.	metre (m)	scalar
displacement, x	The distance of the oscillating object from its equilibrium position in a stated direction.	metre (m)	vector
period, T	The time taken for one complete oscillation.	second (s)	scalar
frequency, f	The number of complete to-and-fro cycles per unit time made by the oscillating object. $f = \frac{1}{T}$	hertz (Hz)	scalar

angular frequency, ω	Refers to the constant which characterizes the particular simple harmonic oscillator and is related to its natural frequency given by $2\pi f$ Note that the displacement x must return to its original value after one period T of the motion, i.e. $x(t) = x(t+T)$ $x_0 \cos\omega t = x_0 \cos\omega(t+T) = x_0 \cos(\omega t + \omega T)$ $\cos\omega t = \cos(\omega t + \omega T)$ Since $\cos\theta = \cos(\theta + 2\pi)$, we have $\boxed{\omega = \frac{2\pi}{T} = 2\pi f}$	radian per second (rad s^{-1})	scalar
phase	An angle, in either degrees or radians, which gives a measure of the fraction of a cycle that has been completed by an oscillating particle or by a wave. e.g. at the stage when ¼ cycle completed, phase = π/2 rad at the stage when ½ cycle completed, phase = π rad at the stage when ¾ cycle completed, phase = 3π/2 rad	radian (rad)	scalar
 *Simple pendulum.*	For instance, in a simple pendulum, ONE complete oscillation occurs when it moves from position *A* to *O* to *B*, then back to *O* and *A*. When the pendulum reaches position *O* from *A*, we say it has completed ¼ of a cycle. Hence, phase = ¼ × 2π = π/2 rad at this stage of the cycle.		
phase difference between two oscillations	*Phase difference* between two oscillations is a measure of <u>how much</u> one oscillation is <u>out of step</u> with another. If two oscillations are in step with one another, they are said to be <u>in phase</u> with one another OR phase difference = zero. Oscillations are said to be in <u>anti-phase</u> OR phase difference = π rad when the displacement of one oscillation reaches a positive maximum value at the same instant as the other reaches a negative maximum (i.e. the displacements are opposite to each other).	radian (rad)	scalar

Example 1

A particle undergoes S.H.M. in which its displacement is given by

$$x = (0.050\text{ m})\cos(4\pi t)$$

(a) What is the amplitude of the motion?

(b) What is the period of the motion?

(c) What is the displacement of the particle when t = 0.3 s?

(d) What is the time when the displacement of the particle is 0.025 m?

Solution:

(a) Amplitude, x_0 = 0.050 m

(b) Angular frequency, $\omega = 4\pi \Rightarrow 2\pi / T = 4\pi \Rightarrow T = 0.50\text{ s}$

(c) $x = (0.050\text{ m})\cos(4\pi t) = (0.050\text{ m})\cos[4\pi(0.30\text{ s})] = (0.050\text{ m})\cos(216^\circ) = -0.040\text{ m}$

(d) $x = (0.050\text{ m})\cos(4\pi t) \Rightarrow 0.025 = (0.050\text{ m})\cos(4\pi t) \Rightarrow \cos(4\pi t) = 0.50 \Rightarrow t = 0.083\text{ s}$

Example 2

The rise and fall of water in a harbour is simple harmonic. The depth varies between 1.0 m at low tide and 3.0 m at high tide. The time between successive high tides is 12 hours.

A boat, which requires a minimum depth of water of 2.5 m, approaches the harbour when the water depth is 2.0 m. How long will the boat have to wait before entering?

A 0.50 hours **B** 1.0 hours **C** 1.5 hours **D** 2.0 hours

The depth of the water can be represented by

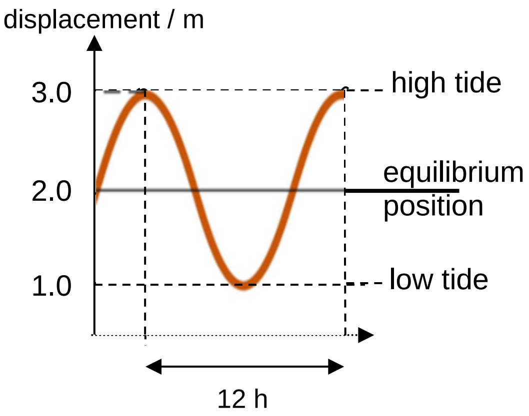
$$x = 2.0 + (1.0)\sin\left(\frac{2\pi}{12}\right)t$$

At $x = 2.5$ m,

$$2.5 = 2.0 + (1.0)\sin\left(\frac{2\pi}{12}\right)t$$

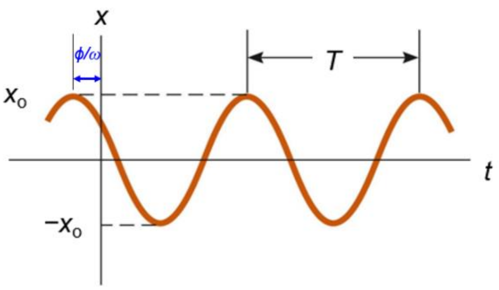
$$\sin\left(\frac{2\pi}{12}\right)t = 0.50 \Rightarrow \left(\frac{2\pi}{12}\right)t = \frac{\pi}{6},\ \frac{5\pi}{6}$$

$t = 1.0$ hr, 5.0 hr



Ans: [B]

More on phase and phase difference



In general, $x = x_0 \cos(\omega t + \phi)$

The phase of the motion of a particle is the quantity $(\omega t + \phi)$.

x_0 and ϕ are determined uniquely by the position and velocity of the particle at t = 0. E.g. If the particle is at $x = x_0$ at t = 0, then ϕ = 0.

A graph of $x = x_0 \cos(\omega t + \phi)$ is the graph of $x_0 \cos \omega t$ displaced *to the left* by a time interval $\frac{\phi}{\omega}$.

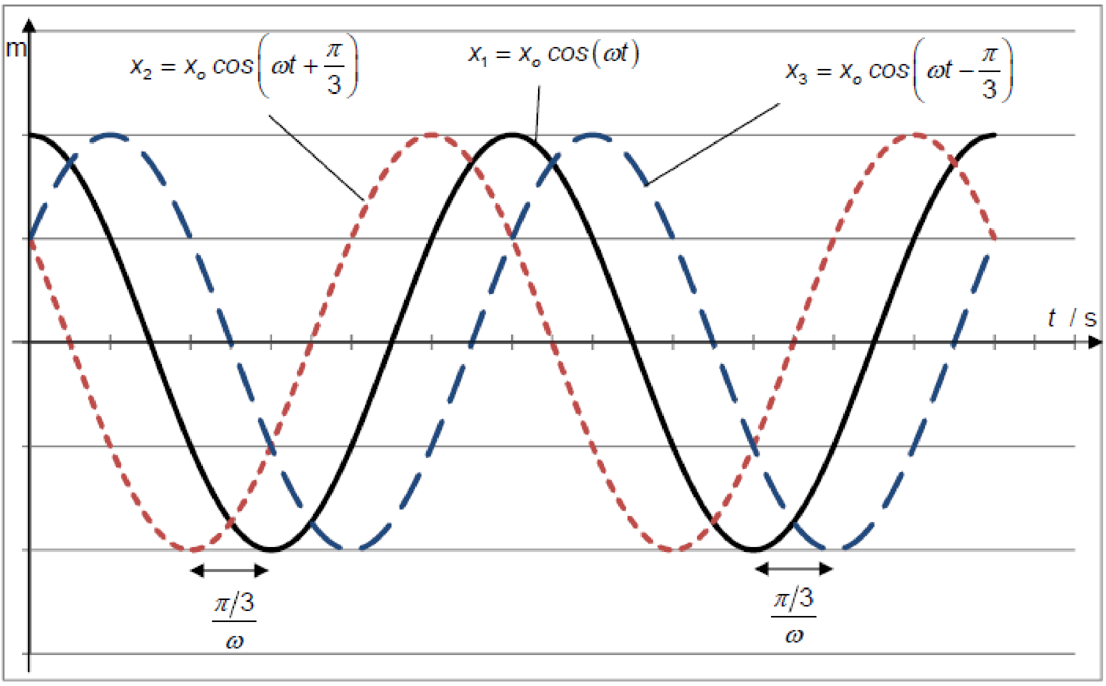
The motion described by $x = x_0 \cos(\omega t + \phi)$ is *not in phase* with that described by $x_0 \cos \omega t$.

It is out of phase by angle ϕ (radian) or time $\frac{\phi}{\omega}$. The plus sign indicates that this motion *leads* by time $\frac{\phi}{\omega}$ and so the graph is displaced to the left.

If the motion was described by $x = x_0 \cos(\omega t - \phi)$, the graph would be displaced *to the right*. This motion would be said to *lag* the motion of $x_0 \cos \omega t$.

Example 3

In the diagram below, what are the *phase difference* ϕ and the *time difference* Δt between x_2 and x_3? Hence, or otherwise, deduce a relationship between ϕ, Δt and T, the common period of x_1, x_2 and x_3.



x_2 and x_3 is out of phase by $\phi = \frac{2\pi}{3}$ radian and time $\Delta t = \frac{2\pi/3}{\omega}$.

$$\boxed{\frac{\Delta t}{T} = \frac{\phi}{2\pi}}$$

(d) recognise and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion

(e) recognise and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$

We will study a special type of oscillatory motion known as simple harmonic motion.

Simple harmonic motion is defined as the oscillatory motion of a particle whose acceleration is directly proportional to its displacement from a fixed point and this acceleration is always in opposite direction to its displacement.

Thus

$$a = -\omega^2 x$$

where x represents the displacement from the equilibrium position and
ω^2 is the constant of proportionality

This is the defining equation for simple harmonic motion.

The constant is a squared quantity, because this will ensure that the constant is always positive. This is so that the minus sign in the equation is always preserved. It has special significance, because it tells us that the acceleration a is always opposite to the direction of displacement x.

Note that the acceleration of a body undergoing simple harmonic motion is always directed towards the equilibrium position – the position at which no *net* force acts on the oscillating object.

Show that the equations $x = x_0 \sin \omega t$ and $x = x_0 \cos \omega t$ *are valid solutions to the defining equation* $a = -\omega^2 x$.

Assuming that $x = x_0 \sin \omega t$

Hence, velocity, $v = \dfrac{dx}{dt}$

$= x_0 \omega \cos \omega t$

Acceleration, $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

$= - x_0 \omega^2 \sin \omega t$

By comparison, $a = -\omega^2 [\, x_0 \sin \omega t \,]$

$a = -\omega^2 x$

If instead we started from $x = x_0 \cos \omega t$

Hence, velocity, $v = \dfrac{dx}{dt}$

$= - x_0 \omega \sin \omega t$

Acceleration, $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

$= - x_0 \omega^2 \cos \omega t$

By comparison, $a = -\omega^2 [\, x_0 \cos \omega t \,]$

$a = -\omega^2 x$

Hence we have shown that both are valid solutions to the defining equation for simple harmonic motion.

(f) recognise and use the equations $v = v_0 \cos \omega t$ and $v = \pm\omega\sqrt{\left(x_0^{\,2} - x^2\right)}$

We have seen that one solution to $a = -\omega^2 x$, for the case t = 0 when x = 0, is given by

$$\boxed{x = x_0 \sin \omega t}$$

To get velocity-time, we have

$$v = \frac{dx}{dt} = \frac{d}{dt}(x_0 \sin \omega t) = x_0 \omega \cos \omega t$$

$$\boxed{v = v_0 \cos \omega t}$$

where $v_0 = x_0\omega$

To get acceleration-time, we have

$$a = \frac{dv}{dt} = \frac{d}{dt}(x_0 \omega \cos \omega t) = -\omega^2 x_0 \sin \omega t = -\omega^2 x$$

$$\boxed{a = -a_0 \sin \omega t}$$

where $a_0 = x_0 \omega^2$

We can also get the equations for velocity and acceleration against displacement as

$$\boxed{v = \pm\omega\sqrt{\left(x_0^{\,2} - x^2\right)}}$$

(see annex for derivation)

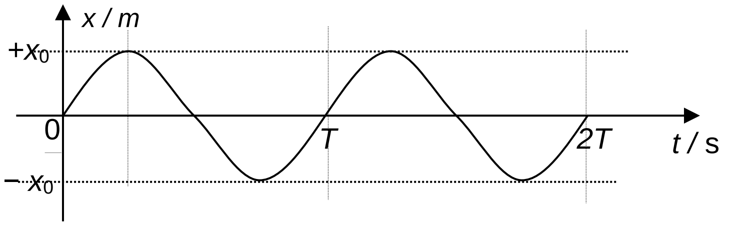
and acceleration

$$\boxed{a = -\omega^2 x}$$

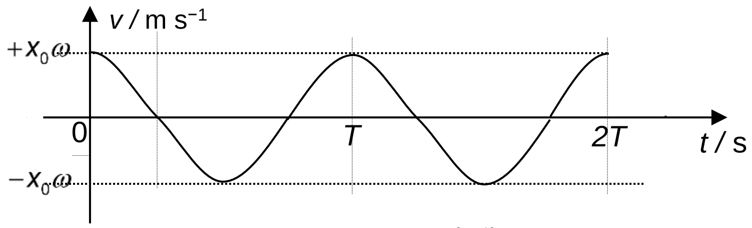
which is the defining equation for simple harmonic motion.

(g) describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion

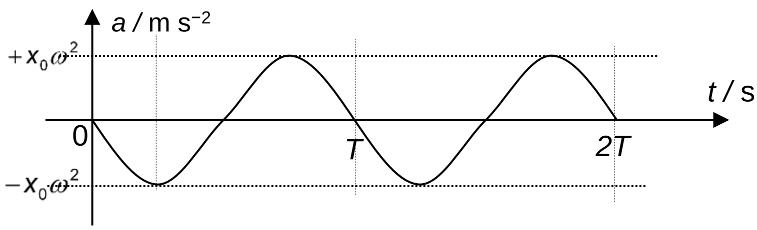
Below are the graphs to describe the equations given in page 8:



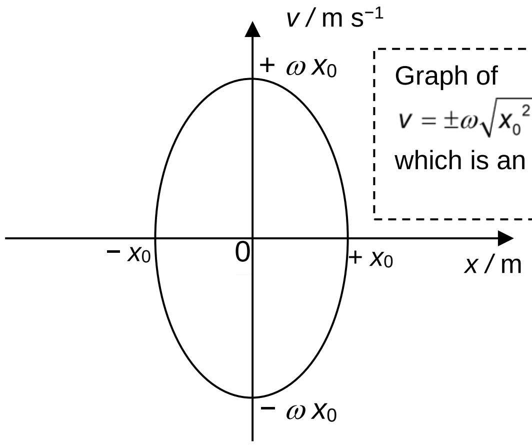
Graph of $x = x_0 \sin(\omega t)$



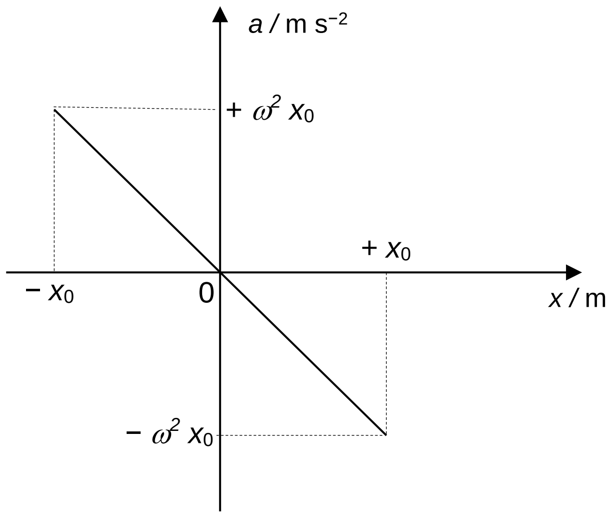
Graph of $v = x_0\omega\cos(\omega t)$



Graph of $a = -x_0\omega^2 \sin(\omega t)$



Graph of $v = \pm\omega\sqrt{x_0^2 - x^2}$, which is an ellipse.



It is important to note the changes (both *magnitude* and *direction*) in displacement, velocity and acceleration of an oscillator executing simple harmonic motion, especially when the oscillator is at its equilibrium position and at its maximum displacement.

Example 4

Base on the graphs above, what are the *phase difference* and the *time difference* between (i) *v* and *x*, (ii) *v* and *a*, and (iii) *a* and *x*?

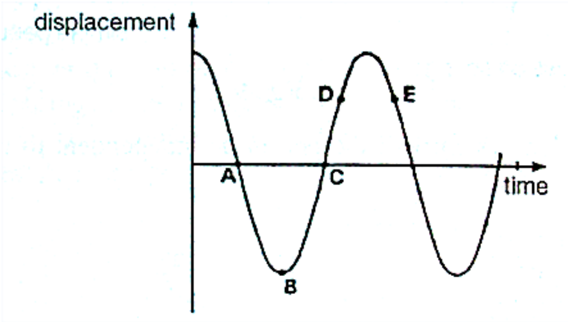
(i) The phase difference is $\pi/2$ rad, the time difference is $T/4$, and *v* leads *x*.

(ii) The phase difference is $\pi/2$ rad, the time difference is $T/4$, and *a* leads *v*.

(iii) The phase difference is π rad, the time difference is $T/2$, and *x* and *a* are in 'antiphase'.

Example 5 [N85/I/7]

The diagram shows a displacement-time graph of a body performing simple harmonic motion.



At which one of the points, **A**, **B**, **C**, **D** or **E**, is the body travelling and accelerating in the same direction?

Answer: [E]

Direction of "Travelling" means the direction of "velocity".

Paraphrase - "At which point is the velocity & acceleration in the same direction?"

At **A**: equilibrium position, $a = 0$ but velocity is maximum

At **B**: extreme end, $v = 0$, but a is maximum

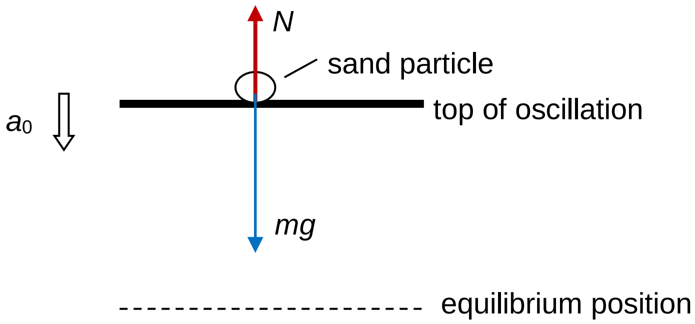
At **C**: same as A, but direction of velocity is opposite

At **D**: travelling towards positive extreme end, but acceleration is pointing in the opposite direction

At **E**: travelling towards equilibrium position, while acceleration is also pointing towards equilibrium position

Example 6

A horizontal plate is vibrating vertically with simple harmonic motion at a frequency of 20 Hz as shown in the diagram.



What is the maximum amplitude of vibration so that fine sand on the plate always remains in contact with it?

The restoring force on the sand is the resultant force acting on it. It acts towards the equilibrium position. Therefore

$$N - mg = ma$$

$$N - mg = m(-\omega^2 x)$$

$$N = mg - m\omega^2 x$$

For the sand to remain in contact with the plate, the normal contact force N should be more than 0:

$$N > 0$$

$$mg - m\omega^2 x > 0$$

$$x < \frac{g}{\omega^2}$$

$$x < \frac{g}{(2\pi f)^2} = \frac{9.81}{(2\pi(20))^2} = 0.00062 \text{ m}$$

Hence the maximum value of x is 0.62 mm.

The amplitude of vibration must be at most 0.62 mm for the sand to always remain in contact with the plate.

Energy in Simple Harmonic Motion

(h) describe the interchange between kinetic and potential energy during simple harmonic motion.

The principle of conservation of mechanical energy states that the *total* energy in an *isolated* system is *constant*.

For an oscillator in S.H.M, its *total* energy is the *sum* of its *kinetic* and *potential* energy.

$$E_{\text{SHM, total}} = E_{\text{kinetic}} + E_{\text{potential}}$$

If $x = x_0 \sin \omega t$,
in terms of *t*

$$E_{\text{kinetic}} = \frac{1}{2}mv^2 = \underline{\frac{1}{2}m\omega^2 {x_0}^2 \cos^2(\omega t)}$$

or in terms of *x*,

$$E_{\text{kinetic}} = \underline{\frac{1}{2}m\omega^2 \left({x_0}^2 - x^2\right)}$$

The maximum kinetic energy of an oscillator in S.H.M is given by

$$E_{\text{kinetic, max}} = \frac{1}{2}m\omega^2 {x_0}^2$$

and it occurs when the oscillator is at its equilibrium position i.e. at $x = 0$ or when $t = 0$.

When the kinetic energy of an oscillating object is maximum, its potential energy must be minimum, since the total energy is constant.

If we define $E_{\text{potential, min}} = 0$, then

$$E_{\text{SHM, total}} = E_{\text{kinetic, max}} = E_{\text{potential, max}}$$

thus in terms of *t*,

$$\frac{1}{2}m\omega^2 {x_0}^2 = \frac{1}{2}m\omega^2 {x_0}^2 \cos^2(\omega t) + E_{\text{potential}}$$

$$E_{\text{potential}} = \underline{\frac{1}{2}m\omega^2 {x_0}^2 \sin^2(\omega t)}$$

or in terms of *x*,

$$\frac{1}{2}m\omega^2 {x_0}^2 = \frac{1}{2}m\omega^2 \left({x_0}^2 - x^2\right) + E_{\text{potential}}$$

$$E_{\text{potential}} = \underline{\frac{1}{2}m\omega^2 x^2}$$

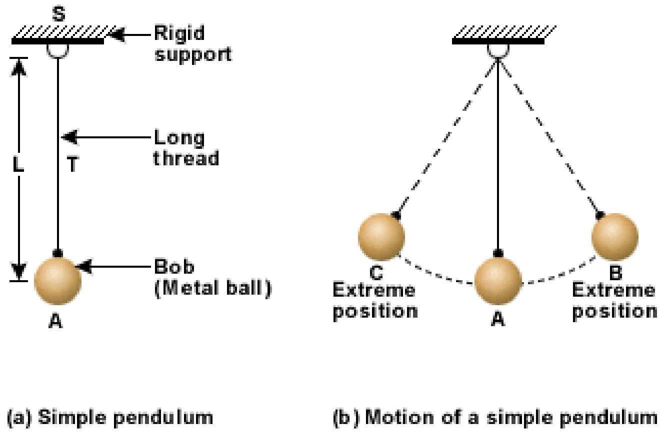
Energy changes in simple harmonic motion – a simple pendulum as an example

Consider one cycle of the motion of a simple pendulum in the absence of energy losses to the surroundings:

A ➔ B ➔ A ➔ C ➔ A

At position B and C, whereby the displacement of the bob is maximum, its potential energy is maximum and its kinetic energy is minimum.

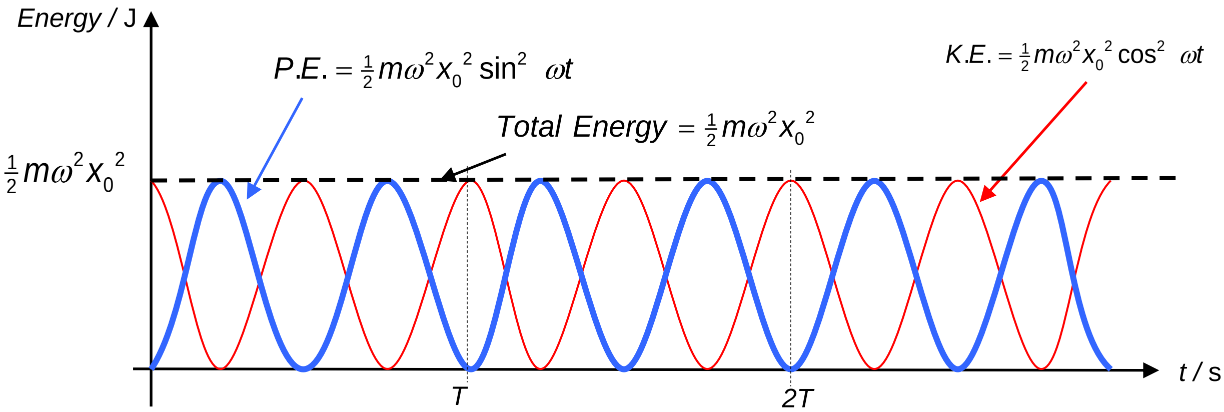
At position A, the bob's equilibrium position, its potential energy is minimum and its kinetic energy is maximum.



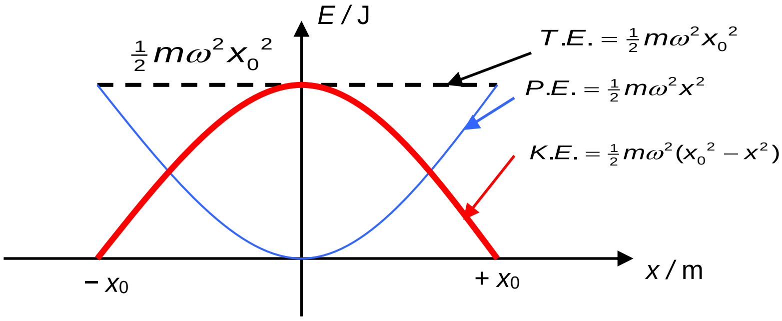
A cycle in the motion of a simple pendulum

In between A and B or A and C, the bob has a combination of potential and kinetic energy.

At any instant of time, the total energy of the bob is a constant.



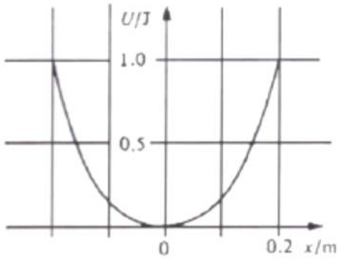
Energy-time graph for an oscillator in simple harmonic motion.
These set of graphs are only true if $x = 0$ when $t = 0$ and $E_{\text{potential, min}} = 0$



Energy-displacement graph for an oscillator in S.H.M
Note the range of values of x and E for which the graph is valid

Example 7: [N82/II/9]

A particle of mass 4 kg moves with simple harmonic motion and its potential energy U varies with position x as shown in the diagram.



What is the period of oscillation of the mass?

A $\frac{2\pi}{25}$ s **B** $\frac{\pi\sqrt{2}}{5}$ s **C** $\frac{8\pi}{25}$ s **D** $\frac{2\pi\sqrt{2}}{5}$ s

At $x = 0.2$ m, $U = 1.0$ J

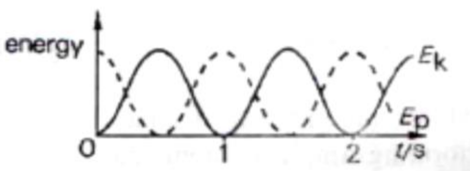
$$U = \frac{1}{2}m\omega^2x^2 \Rightarrow 1.0 = \frac{1}{2}(4)\left(\frac{2\pi}{T}\right)^2(0.2)^2 \Rightarrow T = \frac{2\pi\sqrt{2}}{5} \text{ s}$$

Ans: [D]

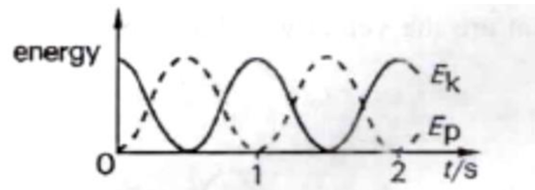
Example 8: [N92/I/9](Modified)

The bob of a simple pendulum of period 2 s is given a small displacement and then released at time $t = 0$. Which diagram shows the variation with time of the bob's kinetic energy E_k and its potential energy E_p?

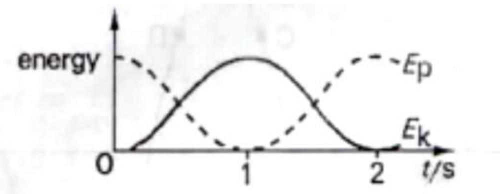
A



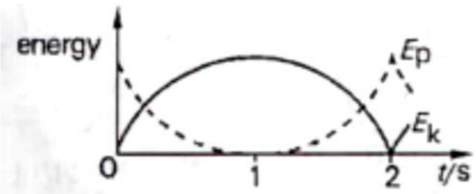
B



C



D



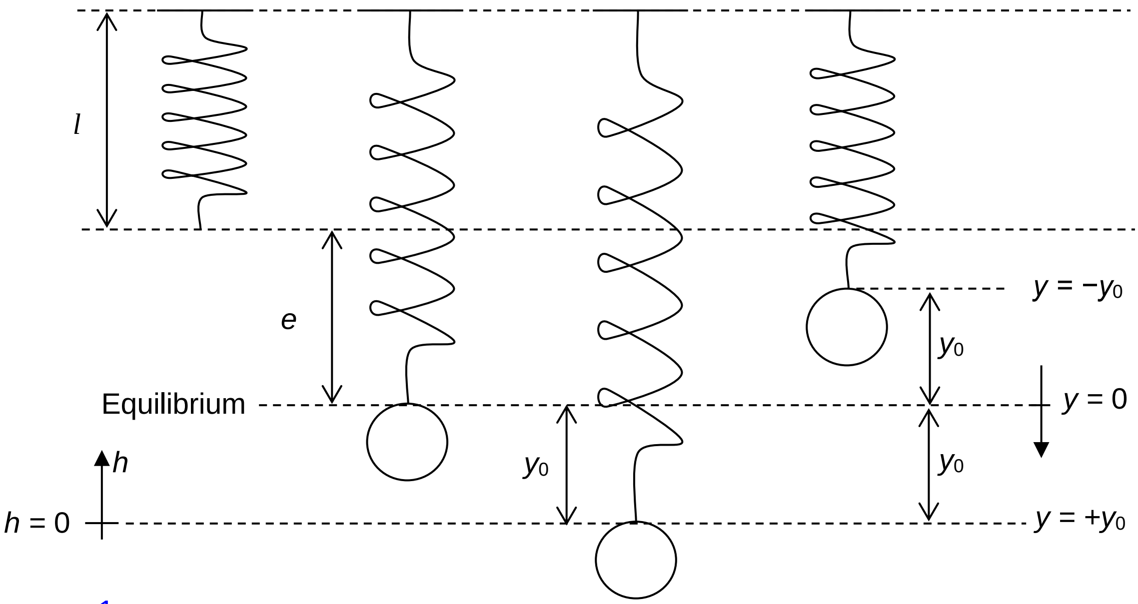
At $t = 0$, the pendulum starts to travel from the extreme end ($E_k = 0$) (either Option **A** or **C**)
In one oscillation ($t = 2$ s), the pendulum will travel pass the equilibrium position twice at max E_k

Ans: [A]

Energy of vertical spring-mass system

A vertical oscillating spring-mass system interchanges three types of energies, namely kinetic energy, gravitational potential energy and elastic potential energy.

Refer to the diagram below. Assume that the spring obeys Hooke's law, and $y_0 \le e$.



$KE = \frac{1}{2}mv^2$ where $v = \pm\omega\sqrt{\left(y_0^2 - y^2\right)}$ and y is the displacement from the equilibrium level.

$GPE = mgh$ where $h = 0$ is (assumed arbitrarily) the level when the object is at the lowest position and $(h = y_0 - y)$.

$EPE = \frac{1}{2}k(e+y)^2$ where e is the extension of the spring when the object is at the equilibrium position.

$TE = KE + GPE + EPE$

Position of object	KE	GPE	EPE	TE
At displacement y	$\frac{1}{2}m\omega^2\left(y_0^2 - y^2\right)$	$mg(y_0 - y)$	$\frac{1}{2}k(e+y)^2$	KE + GPE + EPE
At the highest point, $y = -y_0$	0	$2mgy_0$	$\frac{1}{2}k(e-y_0)^2$	
At the equilibrium position, $y = 0$	$\frac{1}{2}m\omega^2 y_0^2$	mgy_0	$\frac{1}{2}ke^2$	
At the lowest point, $y = +y_0$	0	0	$\frac{1}{2}k(e+y_0)^2$	

If zero GPE level is half way between natural length and equilibrium point, then $GPE = -mg\left(y + \frac{e}{2}\right)$, and the <u>total potential energy</u> is given by

$$\text{Total PE} = GPE + EPE = -mg\left(y + \frac{e}{2}\right) + \frac{1}{2}k(e+y)^2 = \frac{1}{2}ky^2 \text{ because } ke = mg.$$

(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and to the importance of critical damping in applications such as a car suspension system

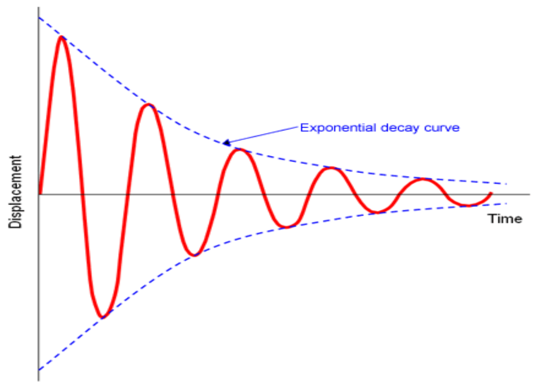
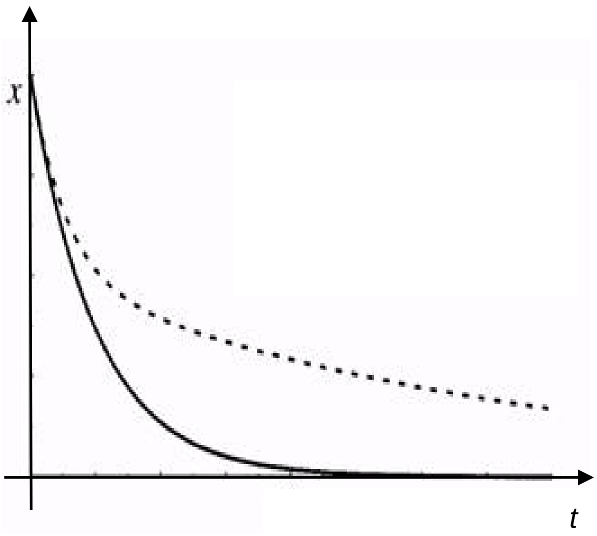
<u>Damped Oscillations</u>

In the previous sections, we discussed systems oscillating in the absence of *dissipative forces e.g. drag forces.*

Damping is a process where energy is taken from an oscillating system as a result of dissipative forces.

Damped oscillation occurs when there is a <u>continuous dissipation of energy to the surroundings</u> such that the total energy in the system decreases with time, hence the <u>amplitude of the motion progressively decreases with time</u>.

The 3 degrees of damping

<u>Light damping</u> The object undergoes a number of complete oscillations with the amplitude of vibration decreasing exponentially with time. An example is a vertical simple spring-mass system in air or in a liquid of low viscosity.	
<u>Critical damping</u> No oscillations occur. The displacement is brought to zero in the <u>shortest possible time</u>. An example is a passenger car suspension system which reduces discomfort on bumpy roads. ***<u>Heavy damping</u>*** No oscillations occur about the equilibrium position when the damping force increases beyond the point of critical damping. The system takes a long time to return to the equilibrium position compared to the critically damped system. An example is a door closer with hydraulic (liquid-filled) damper that closes the door automatically but very slowly.	 Displacement time graph for a (a) Heavy damping system (dotted line) (b) Critical damping system (solid line)

Applications of damping

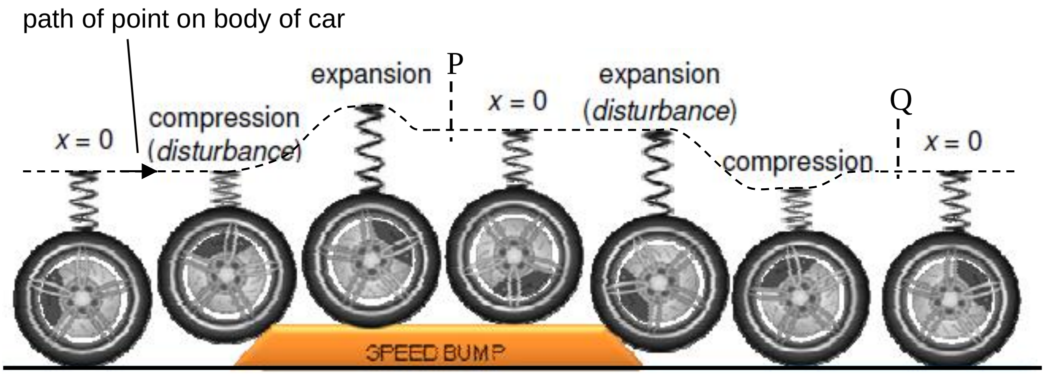
(1) Car Suspension System

The degree of damping a mechanical system is important. The suspension system of a car should ensure a comfortable ride for passengers when the car moves on a bumpy road.

If the suspension is only lightly damped, passengers would be thrown up and down since the suspension system would take some time to stop oscillating. This is what happened when the suspension system is faulty. The setting up of vibration could make control difficult or cause damage to the car.

A good suspension system is one which is critically damped. Shock absorbers on a car critically damp the suspension of the vehicle.

The diagram below shows that by the time the car has reached P, the shock absorbing system is ready for the drop in road surface. After Q, it is ready for another bump.



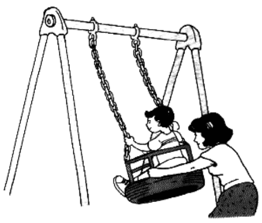
A heavily damped shock absorbing system would still have a compressed spring by the time P is reached and so would not be able to respond to the sudden drop in road surface. So long as there are bumps on a road then these must have an effect on a passenger in a car. The shock-absorbing system can only reduce the forces applied. It cannot eliminate them because, clearly, in the above diagram, the passenger must rise and drop eventually by the height of the bump.

(2) Analogue voltmeters and ammeters

Instruments such as analogue voltmeters and ammeters are also designed to be critically damped so that the pointer comes quickly to the correct position without oscillating in the shortest possible time.

(j) describe practical examples of forced oscillations and resonance

Forced Oscillations



Forced oscillations are caused by the continual input of energy by external applied force to an oscillating system to compensate the loss due to damping in order to maintain the amplitude of the oscillation.

The system then oscillates at the frequency of the external periodic force.

Resonance

- When the driving frequency (f) equals the natural frequency of a system (f_0), the amplitude of oscillation will be a maximum. This is called resonance.

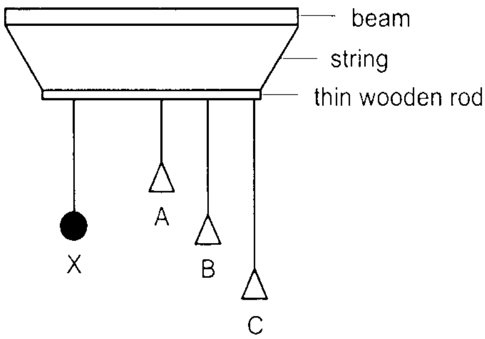
Resonance occurs when the resulting amplitude of the system becomes a maximum when the driving frequency of external driving force equals to natural frequency of the system. At resonance, there is a maximum transfer of energy from the driving system to the driven system.

Practical example of forced oscillations and resonance

Barton's Pendulums

A, B and C are three pendulums, each with a light paper cone for a bob. X is another pendulum with a heavy metal bob. Pendulum X is displaced perpendicularly to the plane containing the bobs at rest and then released.

Pendulum X provides the driving force. The kinetic energy of X is transmitted along the wooden rod to the other pendulums, which subject them to a periodic force from X.
Damping in this case is light because air resistance does not provide a large viscous force.



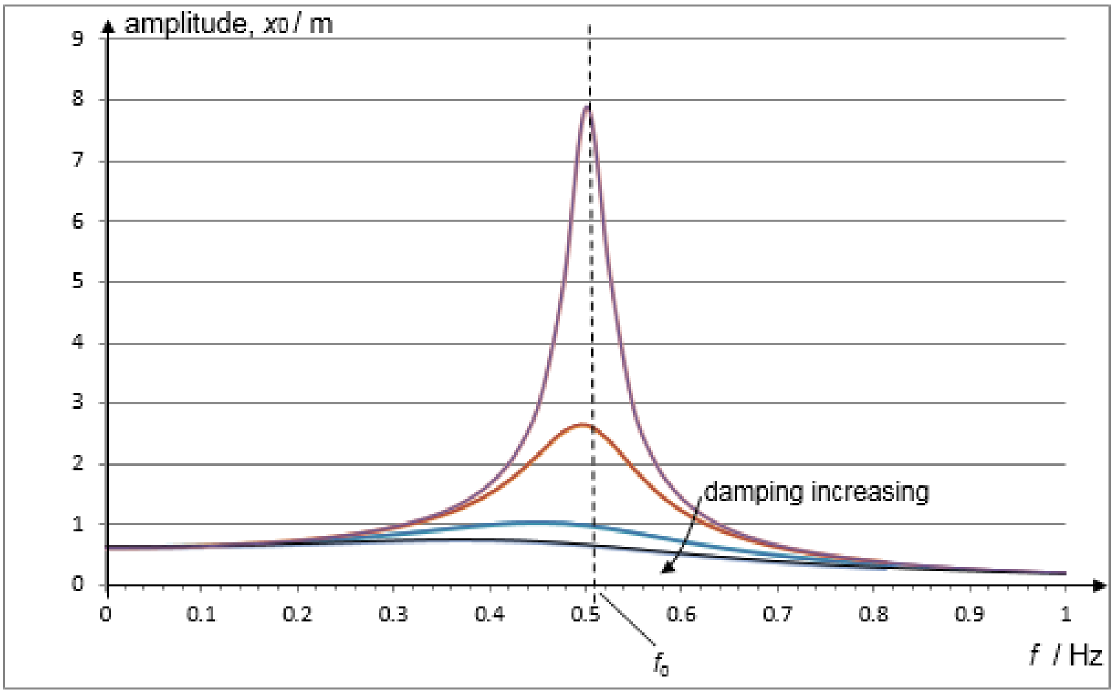
A modified "Barton's Pendulums" experimental setup with 4 pendulums

The motion settles down after a short time and all the pendulums oscillate with very nearly the same frequency as the driver but with different amplitudes. This is forced oscillation.
The frequency of X is close to the natural frequency of pendulum B because they have the same length. As a result, resonance will take place for B and it oscillates with the greatest amplitude. Pendulums A and C will oscillate with smaller amplitude because their natural frequencies are different from the driving frequency of X.

(k) describe graphically how the amplitude of a forced oscillation changes with driving frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and sharpness of the resonance

Frequency Response Graph

The graph below shows how the amplitude x_0 of a forced oscillation depends on the driving frequency *f* when the damping of the system is light and heavy.



For a forced oscillation, when conditions are *steady*, one observes the following:

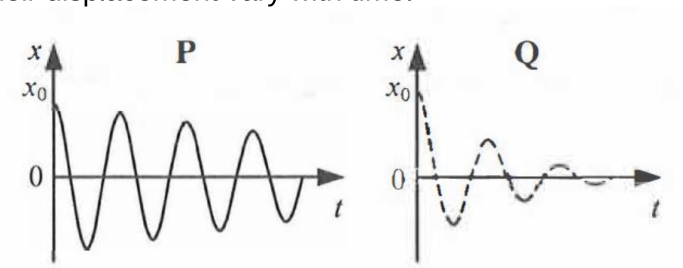
- The amplitude of a forced oscillation depends upon:
  1. the damping of the system,
  2. the relative values of the driving frequency *f* and the natural frequency f_0 of free oscillation (i.e. how far *f* is from f_0).
- The vibrations with largest amplitude (i.e. resonance) occur when *f* is equal to f_0.

As damping increases,

- the response is <u>less sharp</u> (lower and flatter peak and smaller amplitudes at all frequencies) and
- the maximum amplitude is reached when the driving frequency is <u>slightly less than</u> the natural frequency

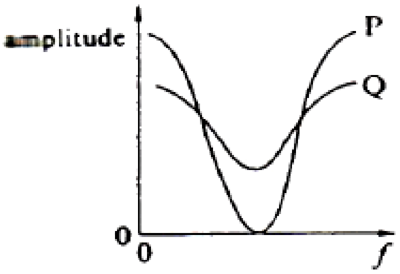
<u>Example 9: [J99/I/9] [N93/I/7] [J86/I/8]</u>

Two objects P and Q are given an initial displacement x_0 and then released. The graphs below show how their displacement vary with time.

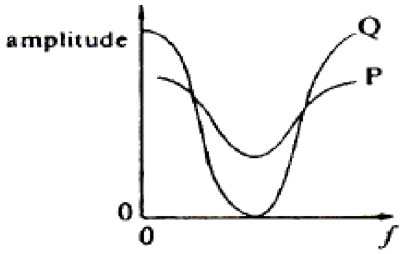


P and Q are then subjected to a driving force of constant amplitude and of variable frequency *f*. Which graph best represents the way in which the amplitudes of P and Q vary with *f*?

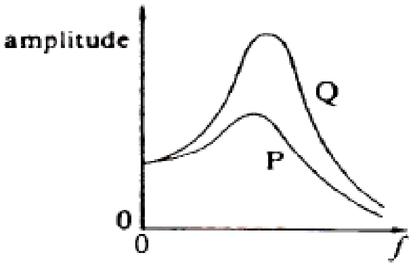
A



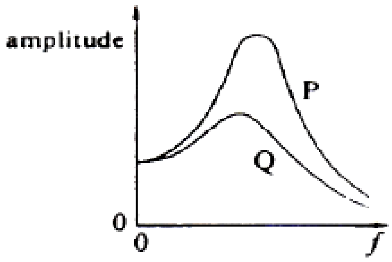
B



C



D



From the *x*-*t* graphs of P & Q, P has less damping compared to Q.

Hence, at resonance, P will have a larger amplitude (less damping), while Q will peak at a frequency less than P (more damping).

Ans : [D]

(l) show an appreciation that there are some circumstances in which resonance is useful and other circumstances in which resonance should be avoided

Useful Applications of Resonance

a. Radio - Electrical resonance occurs when a radio circuit is tuned by making its natural frequency for electrical oscillations equal to that of the incoming signals. (*RLC* Circuits – H3 Physics)

b. Musical instruments - Resonance in musical instruments creates a richer audible sound of the selected note

Undesirable Effects of Resonance

a. Shattering of glass - With the correct pitch (frequency) and sufficient amplification (energy), it is possible for the human voice (sound waves) to break glass. Obviously, this is very dangerous and should NOT be tried out at home.

In experiments, often an amplifier and speaker are used to achieve the required volume (>100 dB). This works even better if the sound is concentrated on one area by putting the glass behind a wooden screen containing a hole. The sound is directed through this hole and on to the glass.

The glass should be affected by damping as little as possible. A wine glass is a good choice because it stands on a stem. This reduces the amount of damping that could be caused by a glass sitting directly on a surface such as a pint glass due to the smaller surface area of contact. The glass should also be empty because any liquid present will absorb the vibration of the glass.

Lastly, the walls of the glass should be as thin as possible and does not contain impurities so that it is structurally weaker. Imperfections in the glass such as a small crack will increase the likelihood of the glass shattering.

Once a suitable wine glass is found, to determine the resonant frequency of the wine glass, 'ping' the wine glass and listen to the sound. The wine glass will vibrate at approximately its natural frequency in the case of light damping (e.g. simply air resistance) and give out a specific tone.

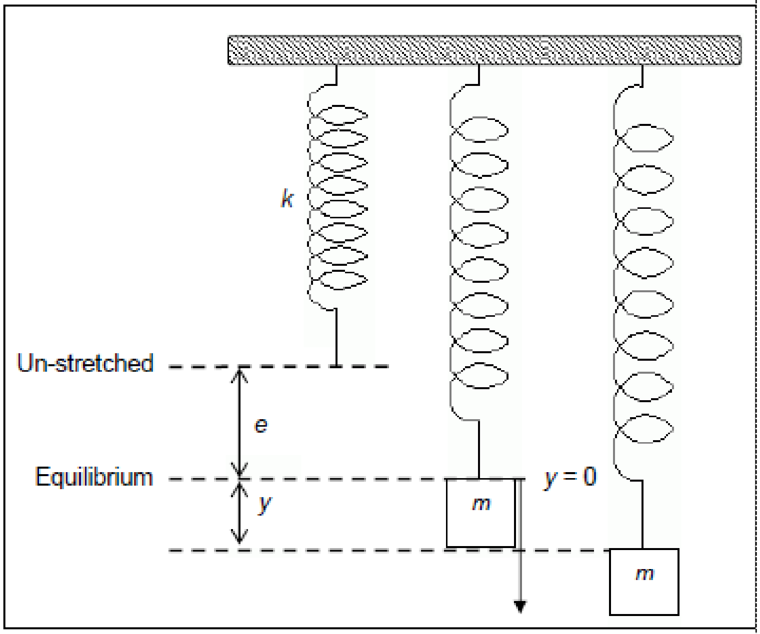
(Left): Pint glass
(Centre): Wine glass
(Right): A glass shattering

b. Resonance in rigid structures - Unwanted vibrations may act on rigid structures such as buildings and bridges causing resonance which may be destructive, unpleasant or simply inconvenient. Sources of such unwanted vibrations include earthquakes, mechanical human sources such as masses of people walking, wind.

Annex

(A) Show that the natural frequency of a vertical spring-mass system f with mass m and effective spring constant k is $f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

Solution:



A vertically suspended spring of negligible mass and spring constant k is stretched by an amount e when a block of mass m is hung on it. The block is given an additional distance y (positive direction downward) and then released.

The initial static equilibrium is characterized by a balance between the spring force and the block's weight:

$$mg = ke$$

Once the block is released, the restoring force is

$$F_{restoring} = -k(e+y)+mg$$

Newton's 2nd law is given by:

$$F_{restoring} = ma$$

$$-k(e+y)+mg = ma$$

$$-ky = ma$$

$$\boxed{a=-\frac{k}{m}y}$$

Compare this with the standard equation for S.H.M. $a=-\omega^2 x$, we get

$$\boxed{\omega^2=\frac{k}{m}}$$

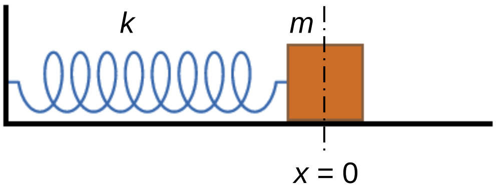
Thus, the natural frequency of a vertical spring-mass system is given by

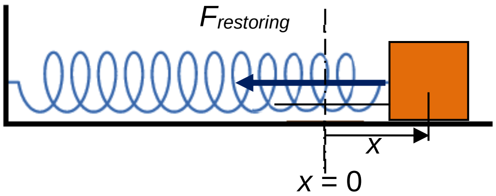
$$\boxed{f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

(B) Show that the natural frequency of a horizontal spring-mass system f with mass m and effective spring constant k is $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

Solution:

Consider a block of mass m attached to the end of a spring of negligible mass and spring constant k, with the block free to move on a horizontal frictionless surface. When the spring is neither stretched nor compressed it is at its equilibrium position as shown.





The mass is displaced a distance of x m to the right.

The restoring force exerted by the spring on the mass is

$$F_{restoring} = -kx$$

It is the resultant force acting on the mass, hence by Newton's 2nd law of motion:

$$F_{restoring} = ma$$

$$-kx = ma$$

$$\boxed{a = -\frac{k}{m}x}$$

Compare this with the standard equation for S.H.M. $a = -\omega^2 x$, we get

$$\boxed{\omega^2 = \frac{k}{m}}$$

Thus, the natural frequency of a horizontal spring-mass system is given by

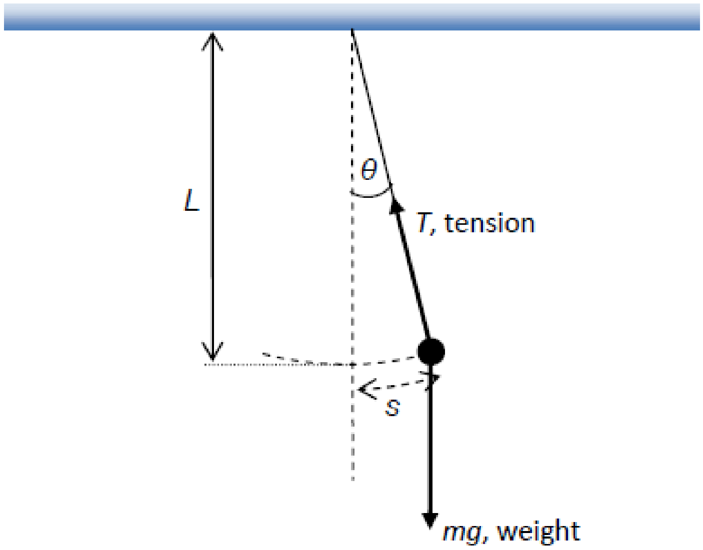
$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

similar to that of the vertical spring-mass system!

(C) Show that the natural frequency of a simple pendulum f with mass m and length L is $f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$.

Solution:

The forces acting on the bob are the tension T in the string and the weight mg of the bob of mass m. When the bob is displaced by a small angle θ (<10°), it is displaced by a distance $s = L\theta$ to the right.

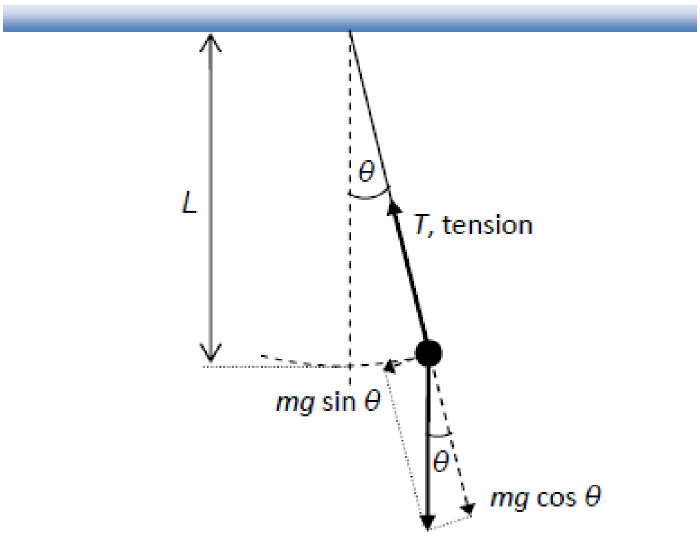


The restoring force is the resolved component of mg tangential to the path:

$$F_{restoring} = -mg\sin\theta$$

Since θ is very small, $\sin\theta \approx \theta$ and

$$F_{restoring} \approx -mg\theta = -mg\frac{s}{L}$$



Taking rightward direction as positive, Newton's 2nd law:

$$F_{restoring} = ma$$

$$-mg\frac{s}{L} = ma$$

$$\boxed{a = -\frac{g}{L}s}$$

Compare this with the standard equation for S.H.M. $a = -\omega^2 x$, we get

$$\boxed{\omega^2 = \frac{g}{L}}$$

Thus, the natural frequency of a simple pendulum is given by

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}}$$

(D) Equation for velocity

For $x = x_0 \sin \omega t$

velocity, $v = \dfrac{dx}{dt}$

$= x_0\omega \cos\omega t$

Squaring both sides:

$v^2 = x_0^2\omega^2 \cos^2\omega t$

$\dfrac{v^2}{\omega^2} = x_0^2\cos^2\omega t$ ….. (1)

However, since

$x = x_0 \sin \omega t$

$\Rightarrow$ $x^2 = x_0^2 \sin^2 \omega t$ ….. (2)

Hence, (1) + (2):

$\dfrac{v^2}{\omega^2} + x^2 = x_0^2\left[\cos^2\omega t + \sin^2\omega t\right]$

$\Rightarrow$ $\dfrac{v^2}{\omega^2} + x^2 = x_0^2$

$\therefore$ $v^2 = \omega^2 [x_0^2 - x^2]$

$\Rightarrow$ $v = \pm\omega\sqrt{x_0^2 - x^2}$

For $x = x_0 \cos \omega t$

velocity, $v = \dfrac{dx}{dt}$

$= - x_0\omega \sin \omega t$

Squaring both sides:

$v^2 = x_0^2\omega^2 \sin^2\omega t$

$\dfrac{v^2}{\omega^2} = x_0^2 \sin^2 \omega t$ ….. (1)

However, since

$x = x_0 \cos \omega t$

$\Rightarrow$ $x^2 = x_0^2 \cos^2 \omega t$ ….. (2)

Hence, (1) + (2):

$\dfrac{v^2}{\omega^2} + x^2 = x_0^2\left[\cos^2\omega t + \sin^2\omega t\right]$

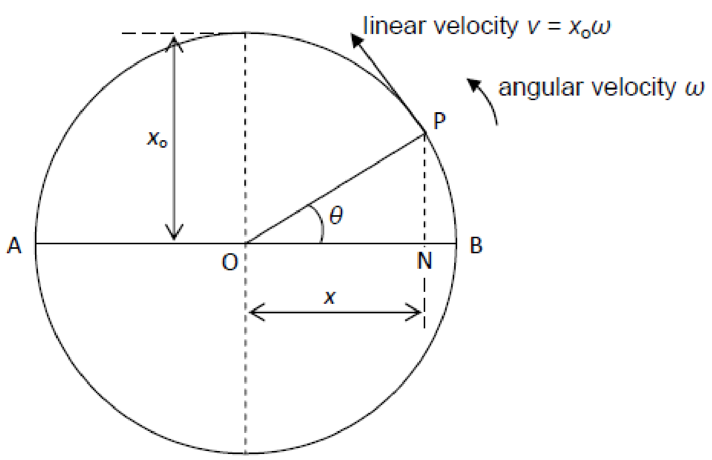
$\Rightarrow$ $\dfrac{v^2}{\omega^2} + x^2 = x_0^2$

$\therefore$ $v^2 = \omega^2 [x_0^2 - x^2]$

$\Rightarrow$ $v = \pm\omega\sqrt{x_0^2 - x^2}$

Note that the $\pm$ sign indicate that there are two velocities with the same magnitude, which correspond to the two cases when the body is at the same position but moving in **opposite** directions.

(E) Uniform circular motion and S.H.M.



If a point P moves in a circle of radius x_o at a steady angular velocity ω. N is the foot of the perpendicular from P to the diameter AOB of the circle. As P moves steadily round the circle, N moves to and fro along AOB.

The acceleration of P is $x_o\omega^2$, directed towards O.

Assuming that t = 0 when θ = 0 (i.e. t = 0 when the point N is at B).
After a time t,

$$\theta = \omega t$$

$$x = x_0 \cos\theta = x_0 \cos\omega t$$

The acceleration of N is the component of the acceleration of P parallel to AB:

$$a = -x_0\omega^2 \cos\theta$$

The negative sign indicates that the acceleration is directed towards O.
We can write

$$\begin{aligned} a &= -x_0\omega^2 \cos\theta \\ &= -x_0\omega^2 \cos\omega t \\ &= -\omega^2 \left(x_0 \cos\omega t\right) \\ &= -\omega^2 x \end{aligned}$$

Thus the motion of N is S.H.M. Its acceleration is zero at O and a maximum at A or B.
The period of N (time taken for N to go from A to B and back again) is given by

$$T = \frac{2\pi}{\omega}$$

The model shows that when a point moves in a circle with a steady speed, the projection of that point on to a diameter of the circle moves with S.H.M. Uniform circular motion projected perpendicularly on to any straight line produces S.H.M.

~ THE END ~

Definition List

Oscillation	An oscillation is a periodic to-and-fro motion of an object between two limits.
Free oscillations	Oscillating system where there is no energy gain or loss (no external force acting on the system).
Forced oscillations (2005)	Forced oscillations are caused by continual input of energy by external applied force to an oscillating system to compensate the loss due to damping in order to *maintain* the amplitude of the oscillation.
Damped oscillations	Oscillation in which there is a continuous dissipation of energy to the surroundings such that the total energy in the system decreases with time, hence the amplitude of the motion progressively decreases with time.
Equilibrium position	Equilibrium position (or neutral position) is the position at which no *net* force acts on the oscillating mass.
Simple Harmonic motion (SHM) (2003, 2006, 2007, 2008, 2009, 2011)	Simple harmonic motion is defined as oscillatory motion of a particle whose acceleration is directly proportional to its displacement from a fixed point and this acceleration is always in opposite direction to its displacement.
Angular frequency (ω) (2008)	*Angular frequency* (ω) of an oscillation refers to the constant which characterizes the particular simple harmonic oscillator and is related to its natural frequency f given by $\omega = 2\pi f$. Unit: radians per second (rad s^{-1}).
Frequency (f) (2008)	Frequency is the number of complete to-and-fro cycles per unit time made by the oscillating object. Unit: Hertz.
Period (T)	Period (T) is the time taken for one complete oscillation. Unit: second.
Phase	An angle in either degrees or radians which gives a measure of the fraction of a cycle that has been completed by an oscillating particle or by a wave.
Phase difference (2010)	Phase difference is a measure of how much one wave is out of step with another. It is measured in either degrees or radians.
Displacement	Displacement is the distance of the oscillating mass from its equilibrium position at any instant in a stated direction.
Amplitude	Amplitude is the maximum displacement of the oscillating mass from the equilibrium position.
Damping	Damping is a process where energy is taken from an oscillating system as a result of dissipative forces.
Natural Frequency	A system that is free to vibrate will vibrate at its natural frequency when no external force acting on it. Natural frequency depends on its dimensions and nature of material.
Resonance (2005)	Resonance occurs when the resulting amplitude of the system becomes a maximum when the driving frequency of external driving force equals to natural frequency of the system. At resonance, there is a maximum transfer of energy from the driving system to the driven system.
Shape of frequency response graph of damped oscillation (2005)	• Lower peak • Flatter peak • Peak occurs at lower frequency than (shifts leftwards slightly from) natural frequency • Smaller amplitude at *all* frequencies

Tutorial Questions

1. The displacement of a particle P which moves with simple harmonic motion can be described by the expression

$$x = (0.05)\sin 8\pi t$$

where x is in metres and t in seconds.

(a) What is the amplitude of the motion? [0.05 m]

(b) What is the frequency of the motion? [4.0 Hz]

(c) The particle starts its motion at t = 0 s. How long does it take for one complete oscillation? [0.25 s]

(d) What is the velocity of the particle as it passes through its equilibrium position, and at the extreme end of the swing? [1.26 m s^{-1}, 0]

(e) What is the maximum acceleration of the particle during its motion? [31.6 m s^{-2}]

(f) Another particle Q also moves with simple harmonic motion of the same frequency. However, the motion of Q *lags* that of P by $\pi/2$ rad and the amplitude of Q is twice that of P. Draw, using the same axes, the displacement-time graphs for motions of P and Q. Write an equation to describe how the displacement of Q varies with time.

2. A pendulum bob has a speed of 0.264 m s^{-1} at the mid-point of its travel. It is oscillating along the arc of a circle of radius 0.962 m. What is the total angle through which it is oscillating? [9.8°]

3. Discuss the energy changes which take place when a mass suspended from a spring is pulled downwards and released, such that it oscillates vertically.

4. A light spring stretches 0.150 m when a 0.300 kg mass is hung from its lower end. The mass is pulled down 0.100 m below this equilibrium point and released. Determine

(a) the spring constant [19.6 N m^{-1}]

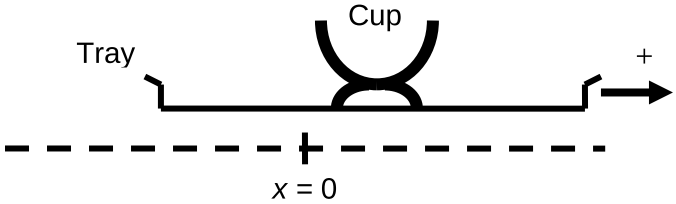
(b) the amplitude of the oscillation [0.100 m]

(c) the maximum velocity [0.808 m s^{-1}]

(d) the magnitude of velocity when the mass is 0.050 m from equilibrium [0.700 m s^{-1}]

(e) the magnitude of the maximum acceleration of the mass [6.53 m s^{-2}]

5. A tray, holding an empty cup, is moved horizontally back and forth in simple harmonic motion. At one instant of time, the tray is displaced to the right of the equilibrium position (x = 0) as indicated by the arrow shown in the figure below.



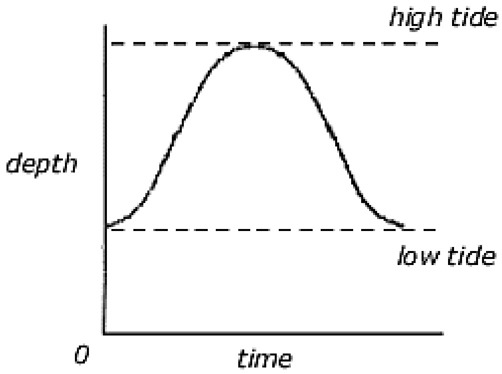
(a) Draw the frictional force F acting on the cup for the instant of time shown.

(b) Write an equation for F in terms of the mass m of the cup, the angular frequency ω of the motion, and the displacement x of the tray.

(c) Given that the maximum value of F is half the weight of the cup, explain why the cup will be observed to slip if the frequency of oscillation increases beyond a certain value.

(d) If the amplitude of the motion is 0.050 m, calculate the maximum possible frequency such that the cup does not slip. [1.58 Hz]

6. The rise and fall of water in a harbour is simple harmonic. The depth varies between 1.0 m at low tide and 3.0 m at high tide. The time between successive low tides is 12 hours.



A boat which requires a minimum depth of water of 1.5 m approaches the harbour at low tide. How long will the boat have to wait before entering? [2 hours]

7. A vertical peg is fixed to the rim of a horizontal turntable of radius 20 cm, rotating with a constant angular speed 3.5 rads^{-1}.

Parallel light is incident on the turntable so that the shadow of the peg is observed on a screen which is normal to the incident light.

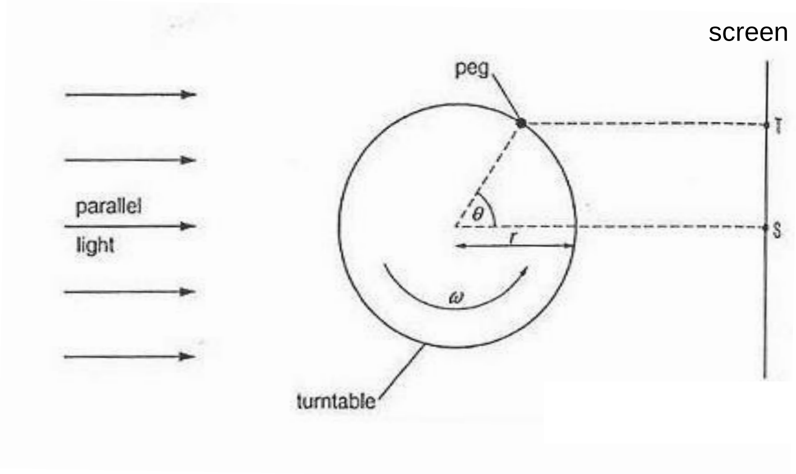
Calculate, for the motion of the shadow on the screen

(a) the amplitude [20 cm]

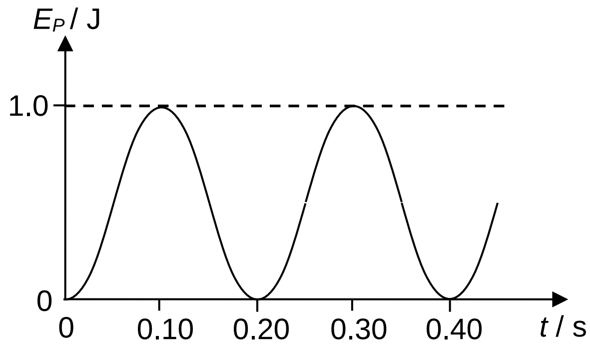
(b) the period [1.8 s]

(c) the speed of the shadow as it passes through S [70 cm s^{-1}]

(d) the magnitude of the acceleration of the shadow when the shadow is instantaneously at rest. [2.5 m s^{-2}]

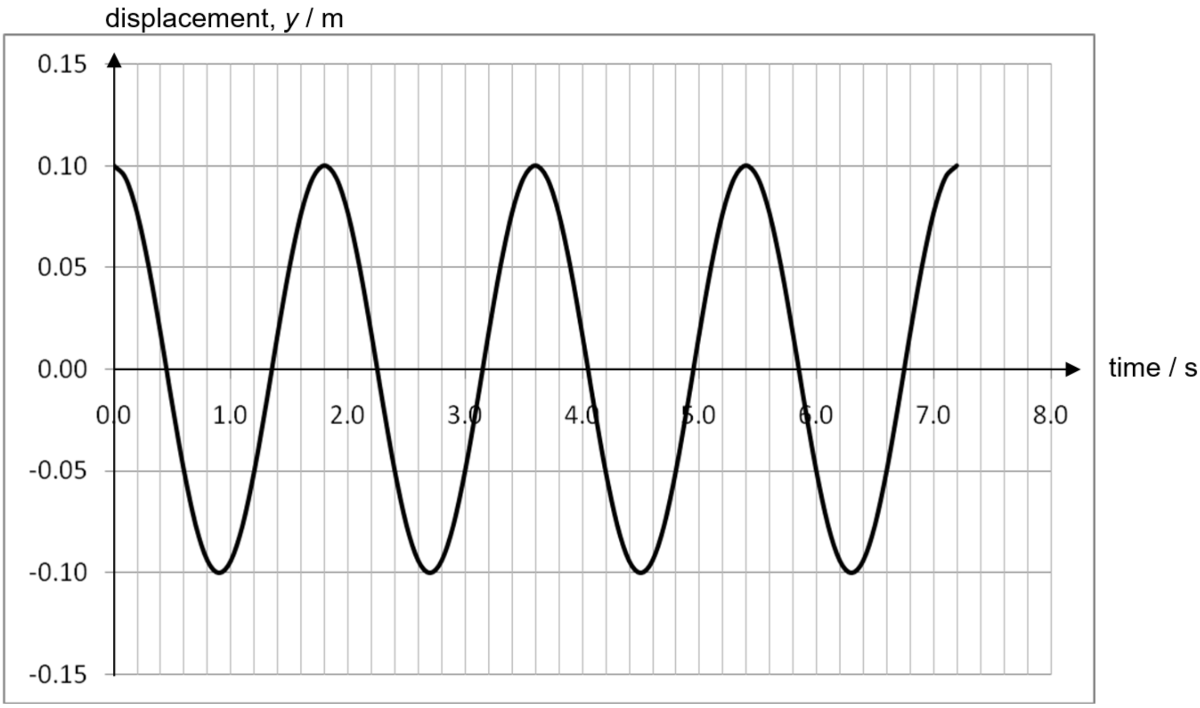


8. An object undergoes simple harmonic motion with amplitude of 0.30 cm. The graph shows the variation of its potential energy, E_P with time, t.



What is the maximum acceleration and mass of the object? [0.74 m s^{-2}, 900 kg]

9. An object undergoing a forced oscillation has displacement y, as shown.



Use the graph to determine, for this oscillation, its amplitude, period and angular frequency.

State, for each of the following, a time at which the oscillating object has

(a) maximum positive velocity
(b) maximum positive acceleration
(c) maximum negative acceleration
(d) maximum kinetic energy
(e) maximum potential energy

10. (a) A spring that has an unstretched length of 0.650 m is attached to a fixed point. A mass of 0.400 kg is attached to the spring and gently lowered until equilibrium is reached. The spring has then stretched elastically by a distance of 0.200 m.

Calculate, for the stretching of the spring,

(i) the loss in gravitational potential energy of the mass [0.7848 J]

(ii) the elastic potential energy gained by the spring. [0.3924 J]

(b) Explain why the two answers to (a) are different.

(c) The load on the spring is now set into simple harmonic motion of amplitude 0.200 m. Calculate

(i) the resultant force on the load at the lowest point of its movement, [3.924 N]

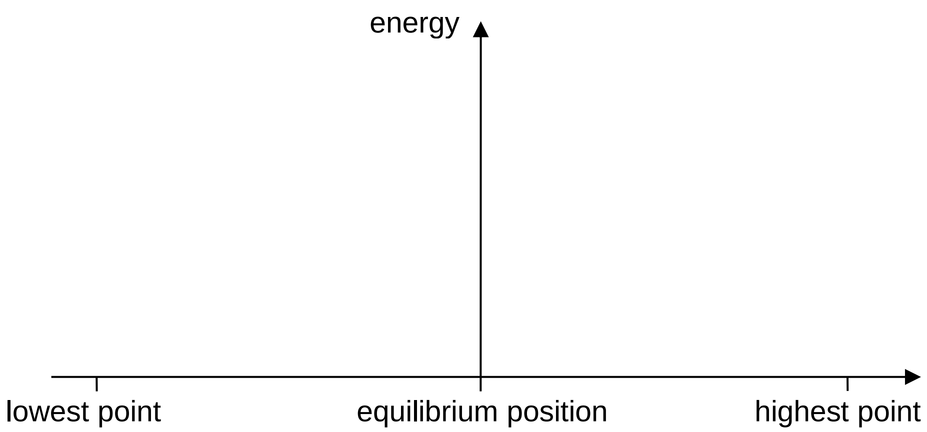
(ii) the angular frequency of the oscillation, [7.00 rad s^{-1}]

(iii) the maximum speed of the mass. [1.40 m s^{-1}]

(d) The figure below is a table of the energies of the simple harmonic motion. Complete the table.

	gravitational potential energy / J	elastic potential energy / J	kinetic energy / J	total energy / J
lowest point	0			
equilibrium position				
highest point				

(e) Copy and enlarge the axes of the figure below, sketch five graphs to show the shape of the variation with position of the five energies. Label each graph.



11. A vertical spring supports a mass, as shown in Fig.(a).

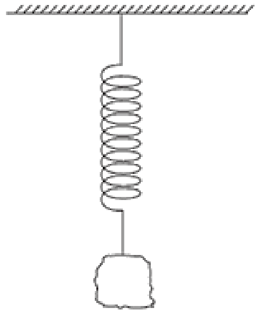


Fig. (a)

The mass is displaced vertically then released. The variation with time t of the displacement y from its mean position is shown.

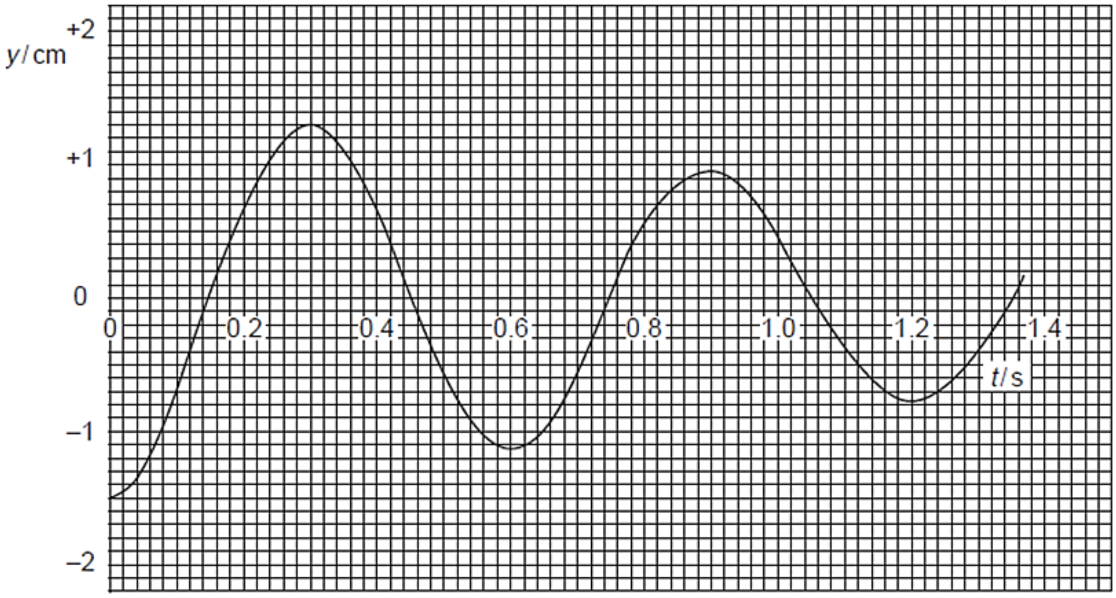


Fig. (b)

A student claims that the motion of the mass may be represented by the equation

$$y = y_0 \sin \omega t.$$

(a) Give two reasons why the use of this equation is inappropriate.

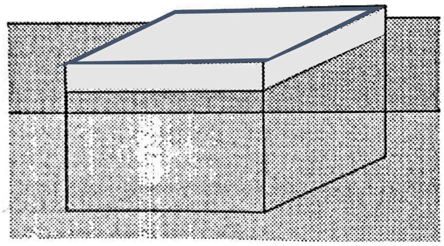
(b) Determine the angular frequency ω of the oscillatons. [10.5 rad s^{-1}]

(c) The mass is a lump of plasticine. The plasticine is now flattened so that its surface area is increased. The mass of the lump remains constant and the large surface area is horizontal.

The plasticine is displaced downwards by 1.5 cm and then released. On Fig. (b), sketch a graph to show the subsequent oscillations of the plasticine.

12. A block of wood of mass m floats in still water as shown. When the block is pushed down into the water, without totally submerging it, and is then released, it bobs up and down in the water with a frequency f given by the expression:

$$f = \frac{1}{2\pi}\sqrt{\frac{28}{m}}$$



where f is measured in Hz and m in kg.

Surface water waves of frequency 3.0 Hz are incident on the block. These cause resonance in the up-and-down motion of the block.

(a) Explain what is meant by the term *resonance*.

(b) Calculate the mass of the block. [0.079 kg]

(c) Describe and explain what happens to the amplitude of the vertical oscillations of the block after the following changes are made independently

(i) water waves of larger amplitude are incident on the block,

(ii) the frequency the water waves decreases,

(iii) the block has absorbed some water.

13. **Data-based Question**

We know that slow oscillations of a rolling ship can produce sea-sickness, although the origin of car-sickness is less clear. Machine operators are subject to more rapid vibrations; the pneumatic road drill is an extreme example.

Some effects of oscillations of various frequencies are shown in Fig. 6.1.

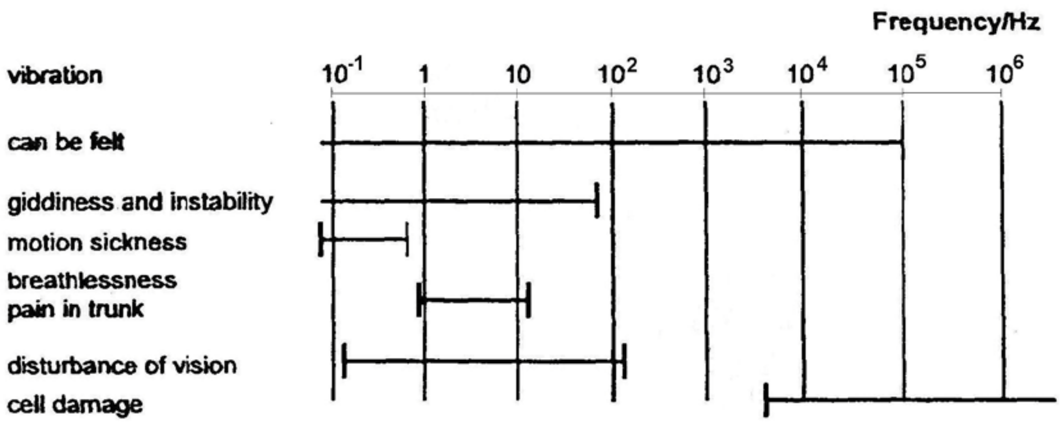


Fig. 6.1

Most serious effects are due to resonance – when the natural frequency of oscillation of some part of the body is equal to the frequency of a vibrating platform with the person on it. Fig. 6.2 shows the motion of the abdomen wall at various frequencies.

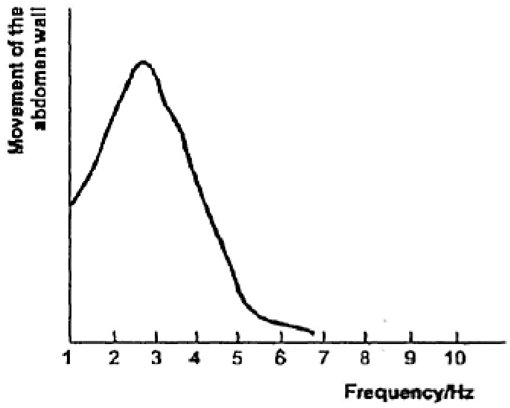


Fig. 6.2

The tolerance of human beings to vibration varies with frequency. You can see this in Fig. 6.3, which is a graph showing the results of a study of human vibration tolerance. Such studies are of special importance in designing aircraft and space probes. Human vibration engineering is also important in designing hand-operated machine tools. The use of such a tool for intricate work would be very difficult if it is vibrated at a resonant frequency of the hand-arm system.

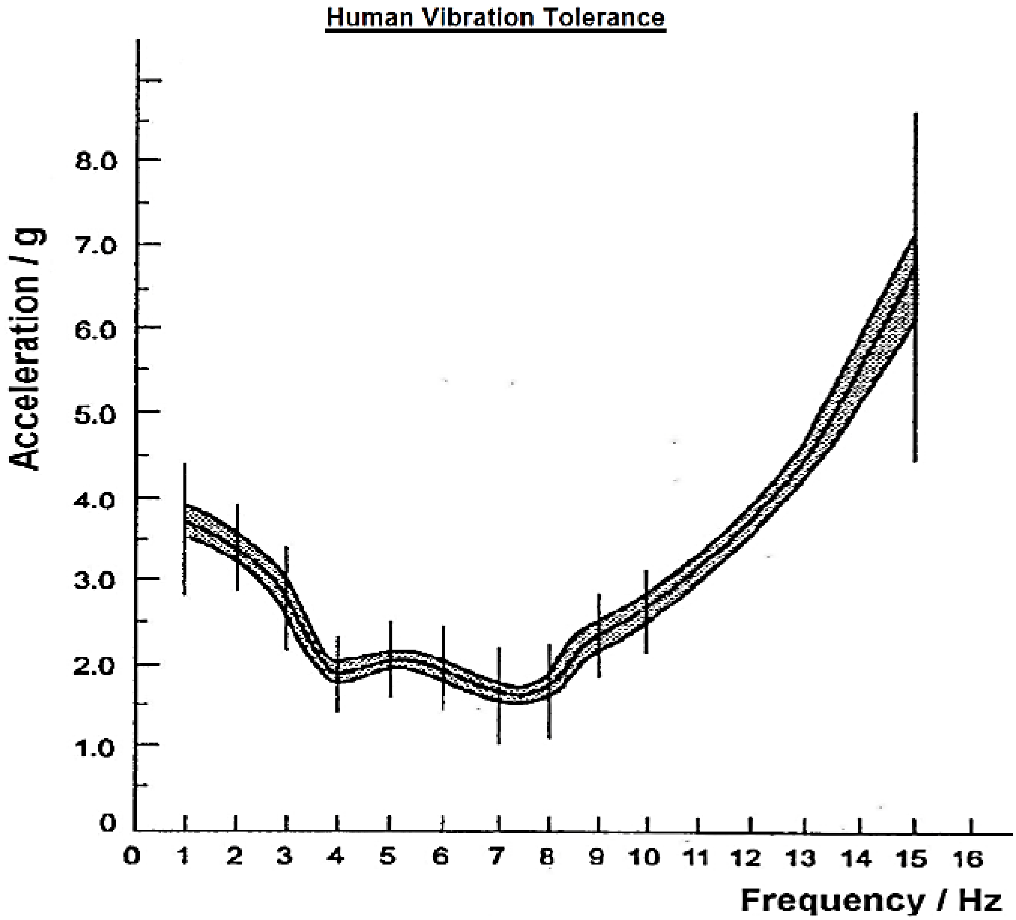


Fig. 6.3

The curves show the value and the range of the limit of the tolerable acceleration at various frequencies. ($g = 9.81\ m\ s^{-2}$)

(a) Estimate from Fig. 6.1, the range of frequencies which would be safe for hand-operated machines. Give a reason for your answer. [3]

(b) Fig. 6.3 shows that humans are very intolerant of acceleration when they are subjected to vibrations between 3 to 9 Hz. Use information from Fig. 6.1 to suggest any two discomforts that might be experienced under these conditions. [2]

(c) Fig. 6.3 shows the maximum tolerable acceleration, during one cycle of an oscillation, plotted against oscillation frequency.

(i) Assuming the oscillation is simple harmonic, write down an appropriate formula that relates the magnitude of the maximum tolerable acceleration to the maximum tolerable amplitude, explain the symbols used. [2]

(ii) Hence calculate the terms given below and complete the following table. [3]

Frequency (Hz)	Angular frequency (rad s^{-1})	Magnitude of the maximum tolerable acceleration (m s^{-2})	Maximum tolerable amplitudes (m)
5.0			

(d) **(i)** Comment on Fig. 6.2 using your knowledge of resonance. [2]

(ii) Hence suggest and explain a medical problem that may arise from the phenomenon depicted in Fig. 6.2. [2]

(e) One important feature in the design of cars is its suspension system that is based on the comfort of the car driver. Assume the formula for the natural frequency of the spring in the suspension is as follows

$$\frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$$

where f = natural frequency of vibration of the spring
m = mass of the car and the driver
k = spring constant (unit is N m^{-1})

(i) From Fig. 6.3, estimate the frequency of vibration that corresponds to the lowest value of tolerable acceleration that a car driver can experience. [1]

(ii) Hence find the spring constant that will resonate at this frequency for a car with a driver that has a total mass of 680 kg. [2]

(iii) Comment on why the spring in the car suspension system is not effective in dissipating energy. Suggest and explain a method which will effectively dissipate the vibrational energy of the spring faster. [3]