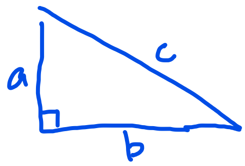
(345 △) $3^2+4^2=5^2$

Chapter 10



1. $a^2+b^2=c^2$

10.1 Introduction

(5, 12, 13)
$5^2+12^2=13^2$

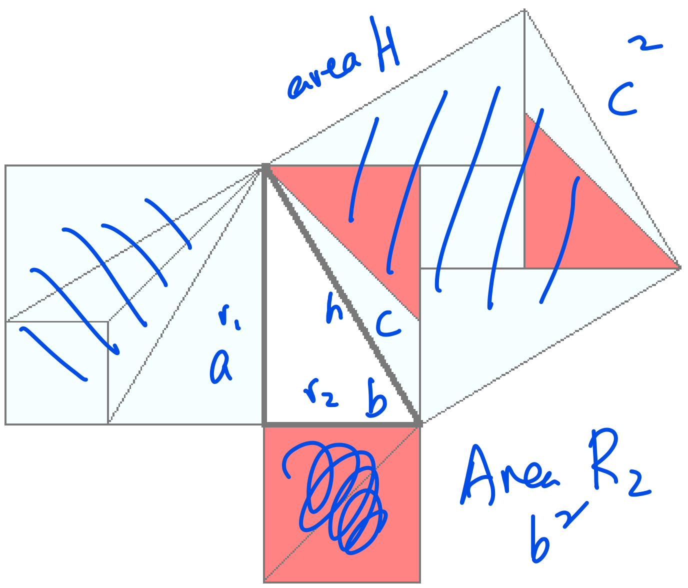
The Pythagorean Theorem was one of the earliest theorems known to ancient civilizations. This famous theorem is named for the Greek mathematician and philosopher, **Pythagoras. Pythagoras** founded the Pythagorean School of Mathematics in Cortona, a Greek seaport in Southern Italy.

The Pythagorean Theorem is Pythagoras' most famous mathematical contribution. According to legend, Pythagoras was so happy when he discovered the theorem that he offered a sacrifice of oxen.

The Pythagorean Theorem is a statement about triangles containing a right angle. The Pythagorean Theorem states that:

"The area of the square built upon the hypotenuse of a right triangle is equal to the sum of the areas of the squares upon the remaining sides."

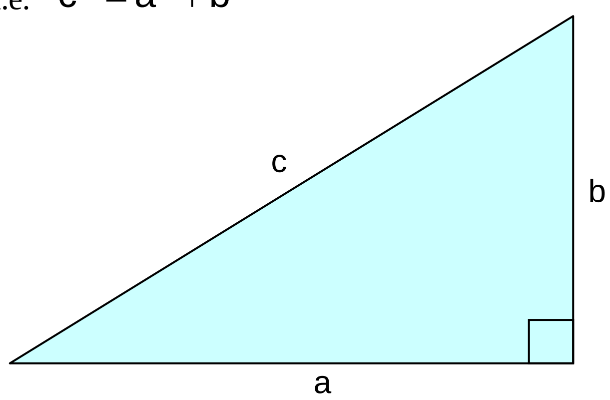
area R_1
a^2



$H = R_1 + R_2$
$c^2 = a^2 + b^2$

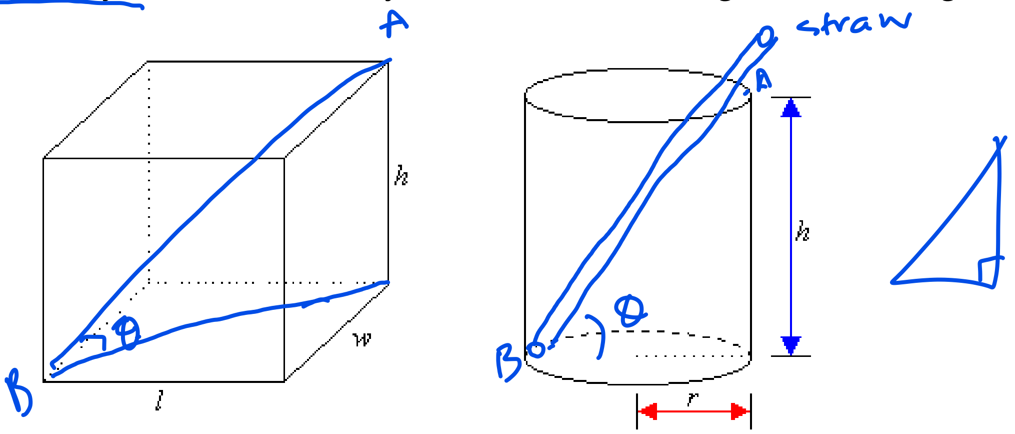
10.2 Pythagoras' Theorem

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides, i.e. $c^2 = a^2 + b^2$



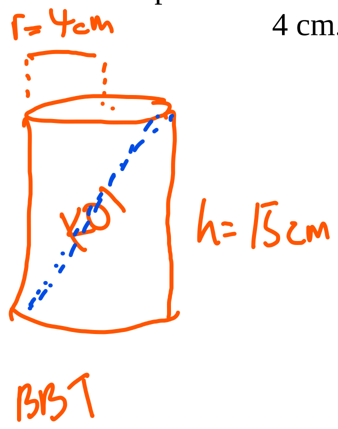
10.3 Pythagoras' Theorem in Three Dimensions

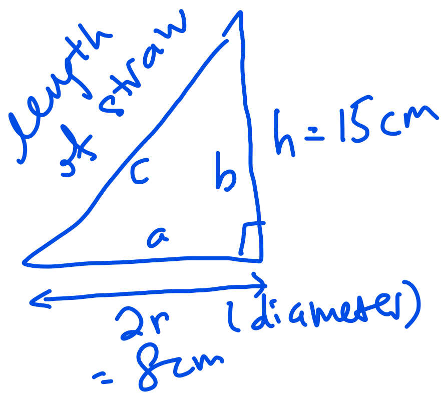
A three-dimensional object can be described by three measurements - length, width and height.



We can use Pythagoras' Theorem to find the length of the longest straw that will fit inside the box or cylinder.

Example 1 A can of drink is in the shape of a cylinder with height 15 cm and radius 4 cm. What is the length of the longest straw that will fit inside the can?





– diameter not known

→ d = ∅

$a^2+b^2=c^2$

$8^2+15^2=c^2$

$64+225=c^2$

$c^2=289 \Rightarrow c=\underline{17\text{ cm}}$

10.4 Pythagorean Triples

A Pythagorean triple is a set of three numbers a, b and c such that $a^2+b^2=c^2$

E.g. 3, 4 and 5 form a Pythagorean triple because $3^2+4^2=5^2$.

3 4 5
6 8 10
9 12 15
15 20 25

Example 2 Determine whether the numbers 6, 8 and 10 form a Pythagorean triple.

3n 4n & 5n

$LHS = (3n)^2 + (4n)^2 = 9n^2 + 16n^2 = 25n^2$

$RHS = (5n)^2 = 25n^2$

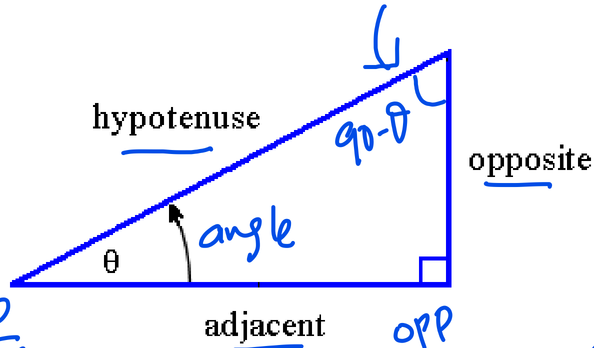
1500 2000 2500

$3\times10^{100}, 4\times10^{100}, 5\times10^{100}$

3,4,5 5,12,13 5n, 12n, 13n

10.5 Trigonometric Ratios

angle – degree
– radian



toa
cah
soh

$\tan\theta = \frac{opp}{adj}$ $\quad \sin\theta = \frac{opp}{hyp}$, $\cos\theta = \frac{adj}{hyp}$

For the angle θ in a right-angled triangle as shown, we name the sides as:

- hypotenuse (the side opposite the right angle)
- adjacent (the side "next to" θ)
- opposite (the side furthest from the angle)

$\tan(90-\theta) = \frac{adj}{opp}$

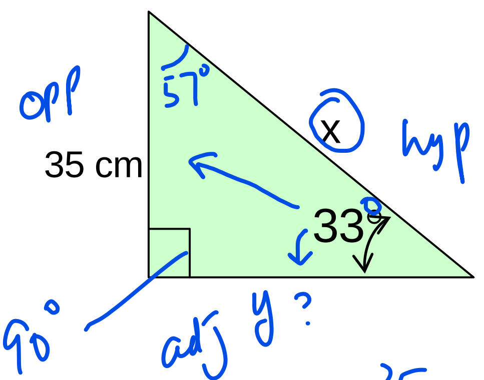
We define the three trigonometrical ratios sine θ, cosine θ, and tangent θ as follows:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

To remember these, many people use TOA, CAH, SOH.

big leg auntie

Example 3 Find the length of the side marked x.



tan $\frac{opp}{adj}$

$\frac{opp}{hyp} = \frac{35}{x} = \sin 33°$

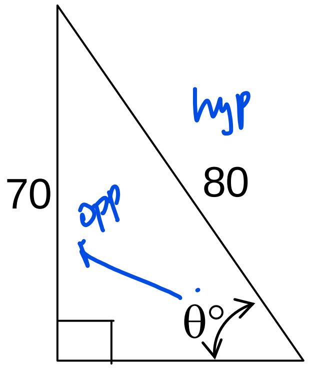
$x = \frac{35}{\sin 33°} =$ ______

in degree mode

$\frac{35}{y} = \tan 33° \Rightarrow y = \frac{35}{\tan 33°}$

$\sin 57 = \frac{y}{x}$, $\cos 57 = \frac{35}{x}$, $\tan 57 = \frac{y}{35}$

Example 4 Find the size of angle marked θ.



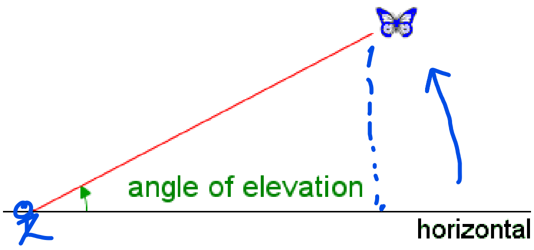
soh

$\sin \theta = \frac{70}{80}$

$\theta = \sin^{-1}\left(\frac{70}{80}\right) =$ ______

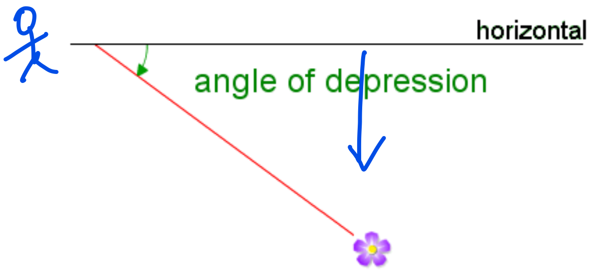
10.6 Angle of elevation:

In surveying, the angle of elevation is the angle from the horizontal looking up to some object:

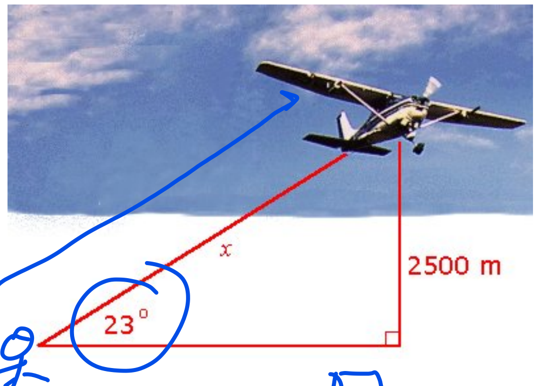


10.7 Angle of depression:

The angle of depression is the angle from the horizontal looking down to some object:



Example 5 The angle of elevation of an aeroplane is 23°. If the aeroplane's altitude is 2500 m, how far away is it?



$\sin 23° = \frac{2500}{x}$

$x = \frac{2500}{\sin 23°} =$ ______

how long is the string

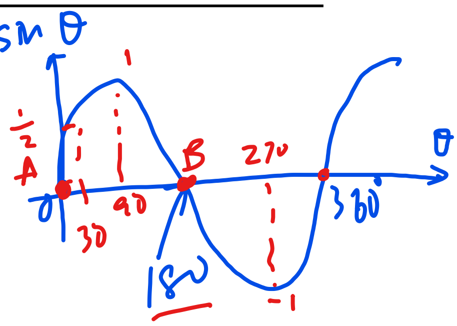
18th floor

50

100 m

Pythagoras theorem.

10.8 Two important identities for supplementary angles are:



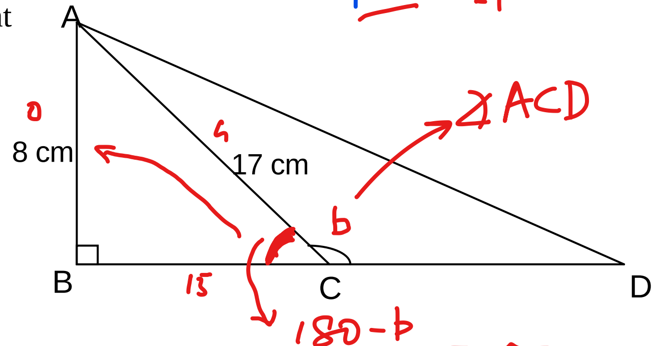
$\sin(180n) = 0$, n = integer

$\sin\theta = \sin(180° - \theta)$ $\qquad$ $\cos\theta = -\cos(180° - \theta)$

$= \sin\theta$ $\qquad$ $= -\cos\theta$

A) $\sin 0 = 0$, $\sin 180 = 0$, $\sin 360 = 0$

Example 6 In the diagram above, BCD is a straight line, AB=8cm, AC=17cm



$\cos(180-\theta) = -\cos\theta$

$\cos\theta = -\cos(180-\theta)$

Calculate (a) BC $\quad$ 15cm

(b) sin∠ACD

(c) cos∠ACD

(a) BC → (Pythagoras theorem)

$AB^2 + BC^2 = AC^2$

$8^2 + BC^2 = 17^2$

$64 + BC^2 = 289$

$BC^2 = 225$

$BC = 15$

$\sin\measuredangle ACD = \sin b = \sin(180-b)$

$= \frac{opp}{hyp} = \frac{AB}{AC} = \frac{8}{17}$

$\cos\measuredangle ACD = \cos b$

$= -\cos(180-b)$

$= -\frac{adj}{hyp} = -\frac{15}{17}$

∡ACD → value

cos ∡ACD → value

10.9 Trigonometric Ratio of Negative Angles

In general, for any negative angle A,

$\sin(-A) = -\sin A$, $\qquad \cos(-A) = \cos A$, $\qquad \tan(-A) = -\tan A$

Eq of circle

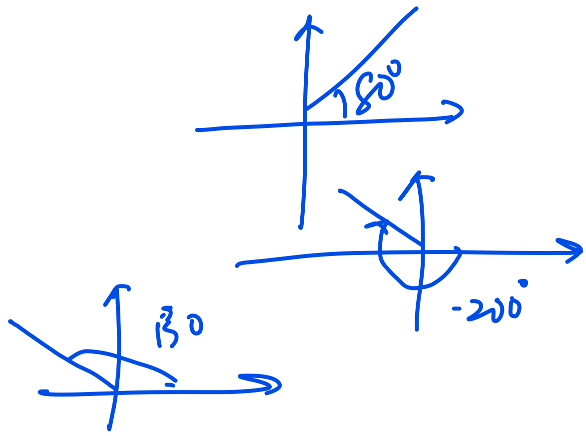
$x^2 + y^2 = r^2$

Example 7 Find the values of $\sin(-80°)$, $\cos(-200°)$ and $\tan(-130°)$.

$\sin(-80°) = -\sin 80°$

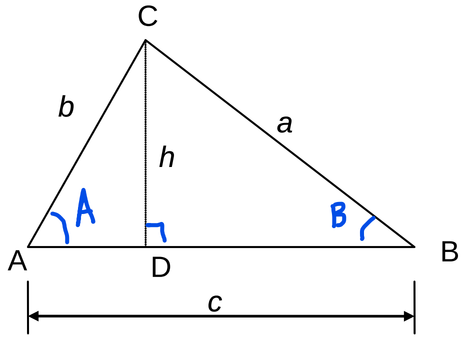
$\cos(-200°) = \cos 200°$

$\tan(-130°) = -\tan 130°$



10.10 The Sine Rule

Consider the triangle ABC, D is the point such that CD is perpendicular to AB. Using the small letters a, b and c for the sides opposite angles A, B and C respectively, the Sine Rule may be derived as follows.



From triangle ADC, From triangle BCD,

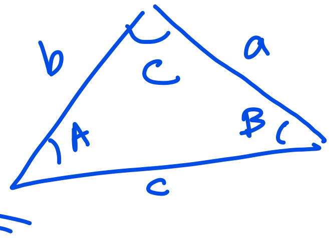
$\sin A = \frac{h}{b}$ or $h = b \sin A$ $\sin B = \frac{h}{a}$ or $h = a \sin B$

Therefore, $a \sin B = b \sin A$

Rearranging, $\frac{a}{\sin A} = \frac{b}{\sin B}$

By drawing a perpendicular from A or B to the opposite side, we can show in a similar manner that

$$\frac{c}{\sin C} = \frac{b}{\sin B} \text{ or } \frac{c}{\sin C} = \frac{a}{\sin A}$$



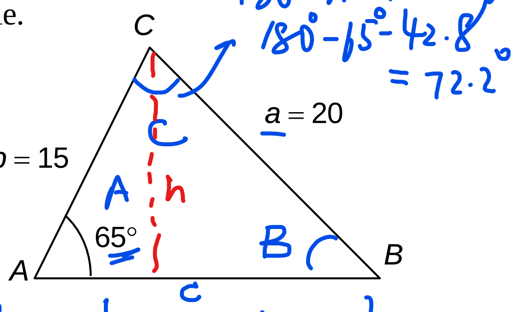
Combining these equations we have the Sine Rule:

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

In order to use the Sine Rule, the following must be known:

a) two sides and an angle opposite one of them, or
b) two angles and a side opposite one of them (knowing two angles and any side is sufficient because two angles, the third is easily found.)

Example 8 If A = 65°, a = 20 cm, and b = 15 cm, solve the triangle.



$180° - A - B$
$180° - 65° - 42.8°$
$= 72.2°$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Solve ∠B, we have a, b and A

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{20}{\sin 65} = \frac{15}{\sin B}$$

$C = 72.2°$ $\{180 - A - B\}$

$$\sin B = \frac{15}{20}\sin 65 = \frac{3}{4}\sin 65$$

$\sin B = 0.67973\ldots$

$B = \sin^{-1}(0.67973)$

$= 42.8°$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

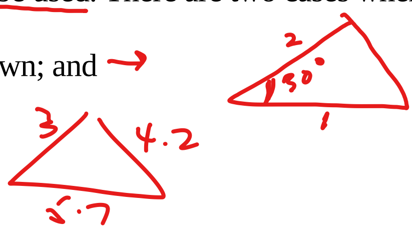
$$c = \frac{a \sin C}{\sin A} = \frac{20 \sin 72.2°}{\sin 65°} =$$

10.11 The Cosine Rule

The Cosine Rule is normally used when the Sine Rule cannot be used. There are two cases when the Sine Rule does not apply in solving triangles:

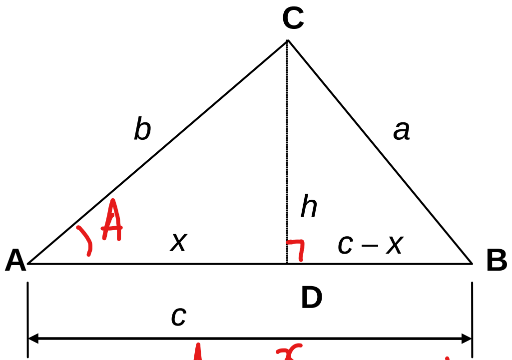
a) when two sides and the included angle are known; and

b) when all three sides are known.



For these cases we use what is called the Cosine Rule.

To derive the Cosine Rule, let's take any general triangle ABC. From C draw CD perpendicular to AB which forms two right triangles as shown at the right.



$\cos A = \frac{x}{b} \Rightarrow x = b\cos A$

By the Pythagorean Theorem, we have: Py

Right From triangle BDC, $a^2 = h^2 + (c - x)^2$

Left From triangle ADC, $b^2 = h^2 + x^2$

Subtracting the two equations, we have $a^2 - b^2 = c^2 - 2cx$.

However, x = b cos A. Therefore, the above equation becomes:

$a^2 = b^2 + c^2 - 2bc\cos A.$

By drawing a perpendicular from A or B to the opposite, we can derive the other two forms of the Cosine Rule in a similar manner.

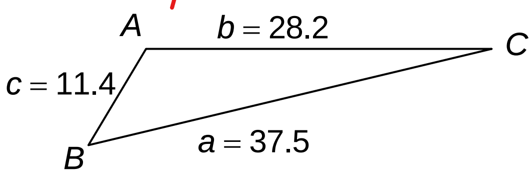
In summary, the Cosine Rule can take any of the following forms:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

general ΔABC
given 3 sides
no ∡
~~sine Rule~~

Cosine Rule

Example 9 If a = 37.5, b = 28.2, and c = 11.4, solve the triangle.



Solve ∡A,

$a^2 = b^2 + c^2 - 2bc\cos A$

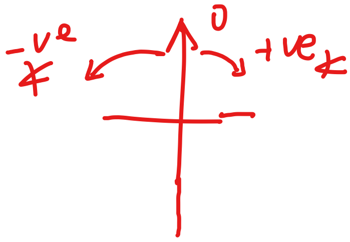
$\measuredangle A = \cos^{-1}\left[\frac{b^2 + c^2 - a^2}{2bc}\right]$

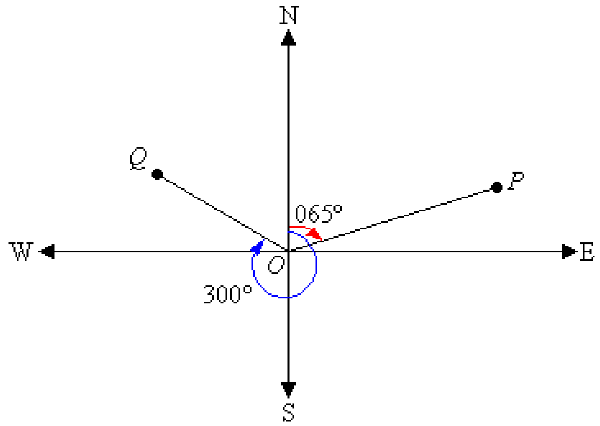
$\measuredangle B = \cos^{-1}\left[\frac{a^2 + c^2 - b^2}{2ac}\right]$

$\measuredangle C = \cos^{-1}\left[\frac{a^2 + b^2 - c^2}{2ab}\right]$

10.12 Bearing

The true bearing to a point is the angle measured in degrees in a clockwise direction from the north line. We will refer to the true bearing simply as the bearing.





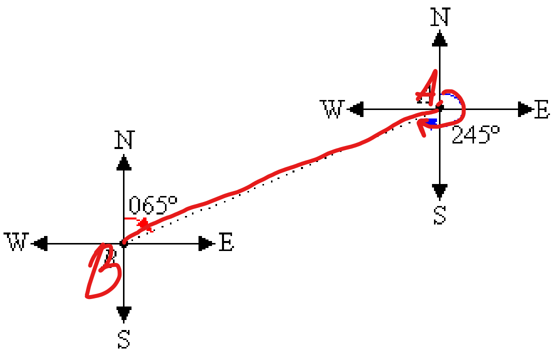
For example, the bearing of point P is 065° which is the number of degrees in the angle measured in a clockwise direction from the north line to the line joining the centre of the compass at O with the point P (i.e. OP).

The bearing of point Q is 300° which is the number of degrees in the angle measured in a clockwise direction from the north line to the line joining the centre of the compass at O with the point Q (i.e. OQ).

<u>Note</u>: The bearing of a point is the number of degrees in the angle measured in a clockwise direction from the north line to the line joining the centre of the compass with the point.

A bearing is used to represent the direction of one point relative to another point.

For example, the bearing of A from B is 065°. The bearing of B from A is 245°.

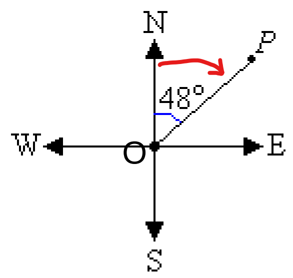


Note:

- Three figures are used to give bearings.
- All bearings are measured in a horizontal plane.

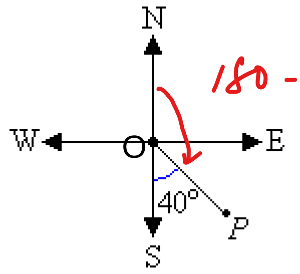
Example 10 State the bearing of the point P in each of the following diagrams:

a.



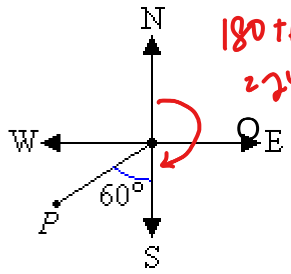
bearing 48°

c.



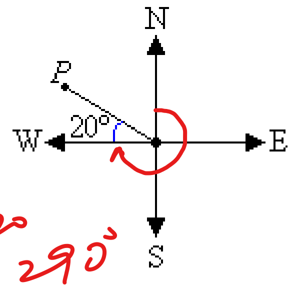
180 - 40 = 140

b.



180 + 60 = 240

d.

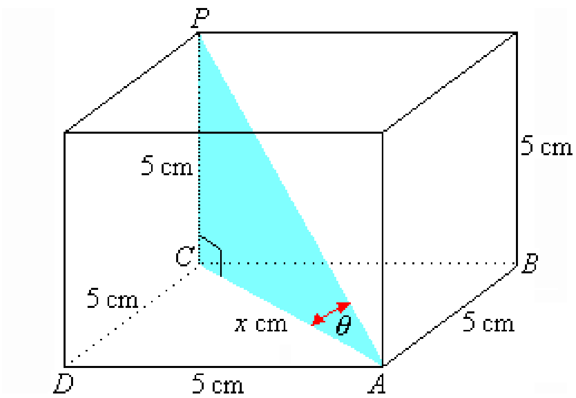


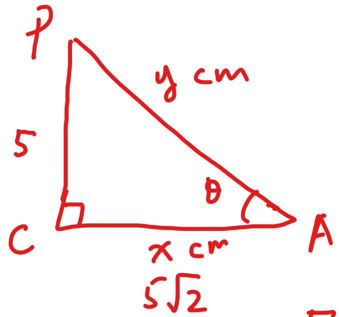
270 + 20 = 290°

10.13 Three-Dimensional Problems

To solve a three-dimensional problem, it is important to be able to visualise right triangles contained in a diagram. Then redraw the right triangles in two dimensions and use an appropriate trigonometric ratio and/or apply Pythagoras' Theorem to obtain the answer.

Example 11 Find the angle between the body diagonal and the base of a cube of side-length 5 cm.



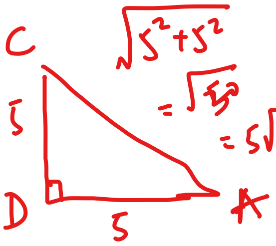


$\tan\theta = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= 35.26°$

$\triangle ACD$



$\sqrt{5^2+5^2}$

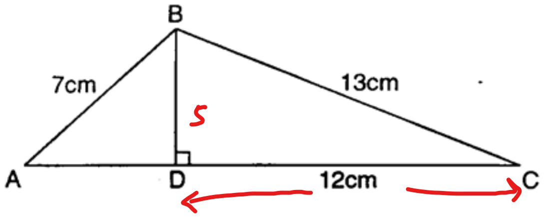
$= \sqrt{50}$

$= 5\sqrt{2} = x$

TUTORIAL 10

1. Refer to the figure below, given that AB=7 cm, BC=13 cm, CD=12 cm. Calculate, correct to 1 decimal place,

   (a) BD, (b) AC.



(a) $BD^2 + 12^2 = 13^2$

$BD = 5.$

(b) $AD^2 + 5^2 = 7^2$

$AD = \sqrt{24}$

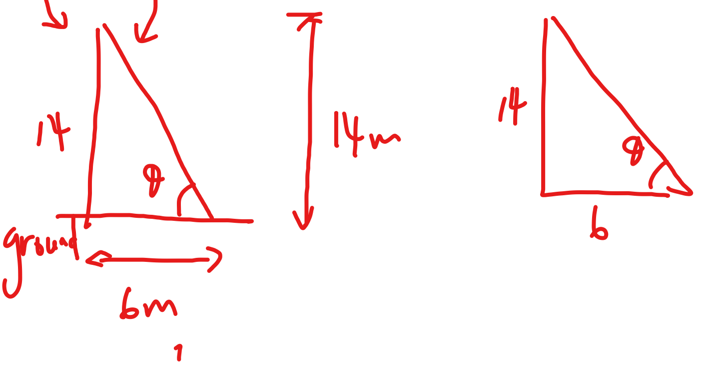
$AC = AD + CD$

$= \sqrt{24} + 12$

$=$ ______ ≈ 17

pytharas' theorem

2 A ladder leans against a building. The foot of the ladder is 6 m from the building. The ladder reaches height of 14 m on the building.
Find to the nearest degree, the angle the ladder makes with the ground.



θ, opp, adj ✓

$\tan\theta = \frac{opp}{adj} = \frac{14}{6} = \frac{7}{3}$

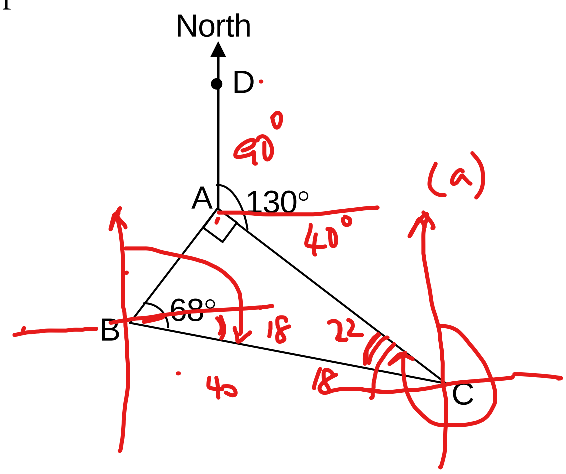
$\theta = \tan^{-1}\left(\frac{7}{3}\right) =$ ______

flat or horizontal plane

3 Point A,B,C and D lie on level ground. The point D is due north of A. ∠CAD=130°, ∠CAB= 90°, and ∠ABC=68°. Find the bearing of

   (a) A from C, (b) B from A,

   (c) C from B.

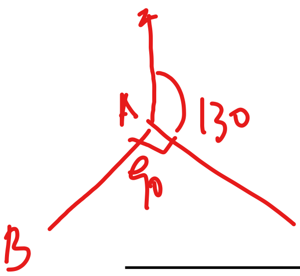


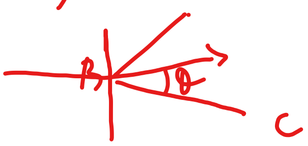
(a) A from C = 270 + 40 = 310°

(b) B from A = 130 + 90 = 220°

(c) C from B = 90 + 18

= 108°

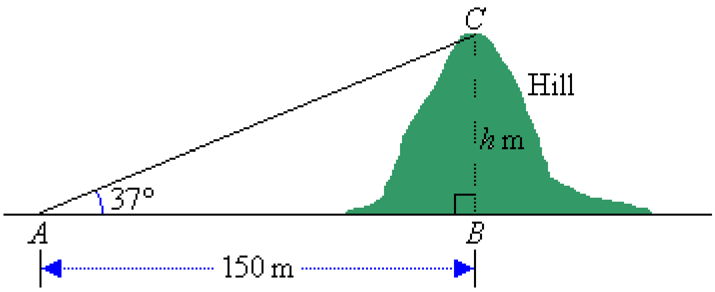




4 A surveyor measures the angle to the peak of a hill from point A, as shown in the diagram. Calculate the height, h, of the hill rounded to 2 decimal places.

$\tan 37^\circ = \frac{h}{150}$

$h = 150 \tan^{-1}(37^\circ)$



5 In the following exercises, find the unknown angles and lengths:

a) A = 75°, B = 34°, a = 102 mm
b) A = 19°, C = 105°, c = 11.1 mm
c) A = 36°, B = 77°, b = 2.5 m
d) a = 23 m, c = 18.2 m, A = 49° 19'
e) a = 9 m, b = 11 m, C = 60°
f) a = 8.16 m, c = 7.14 m, B = 37° 18'
g) a = 5 m, b = 8 m, c = 7 m
h) a = 7.912 m, b = 4.318 m, c = 11.08 m

6 Fig 4 represents part of a roof truss. Calculate the ∠ABC and length of the member BC.

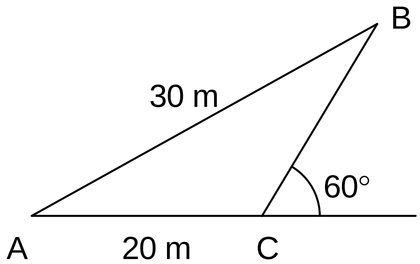
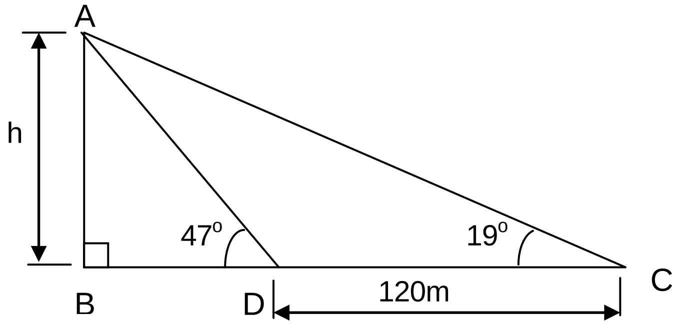


Fig. 4

7 A surveyor measures the angle of elevation of the top of a perpendicular building as 19°. He moves 120m nearer the building and finds the angle of elevation is now 47°. Determine the height of the building.



Challenging Problem

1. ABCD represents the rectangular sloping surface of a desk. ABEF is a rectangle which is horizontal, and CE and DF are vertical lines. AB = DC = FE = 40 cm, BC = AD = 30 cm, ∠CBE = ∠DAF = 35°.
Calculate (a) AC, (b) CE, (c) ∠FAE.