RAFFLES INSTITUTION
2022 Year 6 H2 Mathematics Preliminary Examination Paper 2
Questions and Solutions with comments

1 **(i)** Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\cos^2 x}\right)=\dfrac{k\sin x}{\cos^n x}$, where the values of the constants k and n are to be determined. [2]

(ii) Hence use integration by parts to evaluate $\int_0^{\frac{\pi}{4}} \sin^2 x\sec^3 x\,\mathrm{d}x$, leaving your answer in the form $a+b\ln c$, where a, b and c are exact constants. [5]

(i)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\cos^2 x}\right)$ $=\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x)^{-2}$ $=-2(\cos x)^{-3}(-\sin x)$ $=\dfrac{2\sin x}{\cos^3 x}$ (i.e. $k=2$ and $n=3$)	Generally, majority of students were able to do this part well. A very common mistake that a number of students made was the missing of the negative sign after differentiating cosine.
(ii)	$\int_0^{\frac{\pi}{4}} \sin^2 x\sec^3 x\,\mathrm{d}x=\int_0^{\frac{\pi}{4}}\dfrac{\sin^2 x}{\cos^3 x}\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{4}}\dfrac{2\sin x}{\cos^3 x}\cdot\dfrac{\sin x}{2}\,\mathrm{d}x$ $=\left[\dfrac{1}{\cos^2 x}\cdot\dfrac{\sin x}{2}\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\dfrac{1}{\cos^2 x}\cdot\dfrac{\cos x}{2}\,\mathrm{d}x$ $=\dfrac{1}{2}\left[\dfrac{\sin x}{\cos^2 x}\right]_0^{\frac{\pi}{4}}-\dfrac{1}{2}\int_0^{\frac{\pi}{4}}\sec x\,\mathrm{d}x$ $=\dfrac{1}{2}\left[\dfrac{\sin\frac{\pi}{4}}{\cos^2\frac{\pi}{4}}-0\right]-\dfrac{1}{2}\left[\ln(\sec x+\tan x)\right]_0^{\frac{\pi}{4}}$ $=\dfrac{1}{2}\left[\dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{2}}\right]-\dfrac{1}{2}\left[\ln(\sec\dfrac{\pi}{4}+\tan\dfrac{\pi}{4})-\ln(1+0)\right]$ $=\dfrac{1}{\sqrt{2}}-\dfrac{1}{2}\ln(\sqrt{2}+1)$	This part posed some difficulty to some as they could not identify which should be "u" in the by parts process. Students are reminded to pay attention how to apply the "Hence" method for this type of by parts question. Some common errors were as follows: 1. "$\frac{1}{2}$" was missing. 2. Did not recognize $\int_0^{\frac{\pi}{4}}\dfrac{1}{\cos x}\,\mathrm{d}x=\int_0^{\frac{\pi}{4}}\sec x\,\mathrm{d}x$ and proceeded to lengthy and tedious working which often were wrong. 3. $\int_0^{\frac{\pi}{4}}\dfrac{1}{\cos x}\,\mathrm{d}x=\left[\ln\lvert\cos x\rvert\right]_0^{\frac{\pi}{4}}$ 4. $\dfrac{2}{\sqrt{2}}+1=\dfrac{2+\sqrt{2}}{2}$

2 **Do not use a calculator in answering this question.**

(a) Let $\mathrm{f}(z)$ be a polynomial in z of degree 4 with real coefficients. The equation $\mathrm{f}(z)=0$ has four roots, namely α, β, γ and δ such that they satisfy the following two conditions:

$$\alpha\beta\gamma\delta<0 \text{ and } \alpha^2+\beta^2+\gamma^2+\delta^2<0.$$

Based on the two conditions, a student concludes that the equation $\mathrm{f}(z)=0$ has one positive real root, one negative real root and a pair of complex conjugate roots.

State, with reasons, whether the student's claim is true. [3]

(b) It is given that $\mathrm{g}(z)=z^4+z^3-2z^2+4z-24$.

Verify that $z=2\mathrm{i}$ is a root of the equation $\mathrm{g}(z)=0$. Hence find the other roots of the equation. [5]

(a) **[3]**	The student's claim is true. Reasons: 1. From $\alpha^2+\beta^2+\gamma^2+\delta^2<0$, it shows that at least one of the roots is complex. 2. As the coefficients of $\mathrm{f}(z)$ are real, we know that complex roots exist in conjugate pairs, so there is at least one pair of complex conjugate roots. 3. If there are 2 pairs of complex conjugate roots, then $\alpha\beta\gamma\delta>0$. However, given that $\alpha\beta\gamma\delta<0$, then there is only a pair of complex conjugate roots and 2 real roots of opposite signs. So with 1, 2 and 3, we can conclude that the equation $\mathrm{f}(z)=0$ has one positive real root, one negative real root and a pair of complex conjugate roots.	This part posed some difficulty to a number of students as they took the wrong approach of using the claim to check on the conditions instead of the other way as intended by the questions. These students often could not explain why with the claim they began with could satisfy the second condition. Students are also reminded to read the question carefully and answer the question completely as some did not state whether the claim is true or not and just proceeded to give reasons to justify.
(b) **[5]**	Since $\mathrm{f}(2\mathrm{i})=(2\mathrm{i})^4+(2\mathrm{i})^3-2(2\mathrm{i})^2+4(2\mathrm{i})-24$ $=16-8\mathrm{i}+8+8\mathrm{i}-24=0,$ so $z=2\mathrm{i}$ is a root of the equation $\mathrm{f}(z)=0$ (verified). As complex roots occur in conjugate pair, so $z=-2\mathrm{i}$ is the other complex root. Now $(z-2\mathrm{i})(z+2\mathrm{i})=z^2+4$.	Most students were able to handle this part well though they are reminded to improve in their method to use the most efficient way to solve as suggested here and not continue to use their very tedious and long method.

	Hence $z^4+z^3-2z^2+4z-24=\left(z^2+4\right)\left(z^2+az-6\right)$. Comparing the coefficient of z^3: $a=1$. Thus $z^4+z^3-2z^2+4z-24=\left(z^2+4\right)\left(z^2+z-6\right)$ $=\left(z^2+4\right)\left(z+3\right)\left(z-2\right)$ So the other 3 roots of the equation $\mathrm{f}\left(z\right)=0$ are $-2\mathrm{i}$, 2 and -3.	Some students missed out verifying that $z=2\mathrm{i}$ is a root of the equation and a number of students did not know that $z=2\mathrm{i}$ is a root while $z-2\mathrm{i}$ is a factor.

3 The line L_1 has equation

$$1-y=\frac{z-1}{2},\ x=2,$$

and meets the the xy-plane at point P. The point A has position vector $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ with reference to the origin O.

(i) Find a vector equation of the line L_2 which passes through O and P. [3]

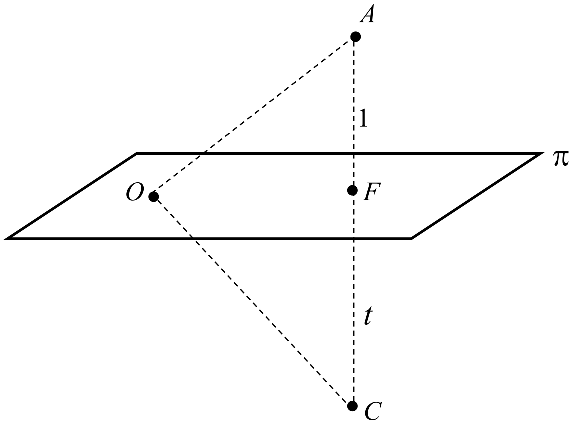
(ii) Find an equation of the plane π containing both L_1 and L_2, in the scalar product form. [2]

(iii) The points A and C are on different sides of π such that AC is perpendicular to π. The distance of C from π is t times the distance of A from π. Find, in terms of t, the position vector of C. [5]

(iv) Find the value of t such that the line OC is parallel to the plane with equation $\mathbf{r}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}=2$. [2]

(i) **[3]**	**Method 1** The z-coordinate of P is 0, since P lies on the xy-plane. Thus, putting $z=0$ into L_1, $1-y=\frac{0-1}{2} \quad\Rightarrow\quad y=\frac{3}{2}$ The position vector of P is $\overrightarrow{OP}=\begin{pmatrix}2\\\frac{3}{2}\\0\end{pmatrix}$. Hence, equation of L_2 is $\mathbf{r}=\lambda\begin{pmatrix}4\\3\\0\end{pmatrix},\ \lambda\in\mathbb{R}$ **Method 2** Vector equation of line L_1: $\mathbf{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}0\\-1\\2\end{pmatrix},\ \mu\in\mathbb{R}$ Equation of xy-plane : $\mathbf{r}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ To find P, consider $\left(\begin{pmatrix}2\\1\\1\end{pmatrix}+\mu\begin{pmatrix}0\\-1\\2\end{pmatrix}\right)\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=0$	This part is well done. Quite a significant had careless mistake, missing negative sign or placed at the wrong component. The vector equation of a line is of the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b},\lambda\in\mathbb{R}$. Missing "$\mathbf{r}=$" (i.e. writing just $\mathbf{a}+\lambda\mathbf{b}$) is considered incomplete. This is analogous to writing the Cartesian equation of a 2D straight line as $mx+c$ (i.e. without $y=$).

	$1+2\mu=0$ $\mu=-\frac{1}{2}$ Hence, $\overrightarrow{OP}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\left(-\frac{1}{2}\right)\begin{pmatrix}0\\-1\\2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4\\3\\0\end{pmatrix}$. Hence, equation of L_2 is $\mathbf{r}=\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}$, $\lambda\in\mathbb{R}$	
(ii) **[2]**	$\begin{pmatrix}4\\3\\0\end{pmatrix}\times\begin{pmatrix}0\\-1\\2\end{pmatrix}=\begin{pmatrix}6\\-8\\-4\end{pmatrix}=-2\begin{pmatrix}-3\\4\\2\end{pmatrix}$ Hence, equation of π is $\mathbf{r}\bullet\begin{pmatrix}-3\\4\\2\end{pmatrix}=0$	This part is also well done.
(iii) **[5]**	**<u>Method 1 (Find the foot of perpendicular first)</u>** Let F be the foot of perpendicular from A to π. Equation of the line AF is $\mathbf{r}=\begin{pmatrix}3\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}-3\\4\\2\end{pmatrix}$. Since F lies on the line AF, then $\overrightarrow{OF}=\begin{pmatrix}3\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}-3\\4\\2\end{pmatrix}$, for some $\lambda\in\mathbb{R}$. Since F is also on π, then $\left[\begin{pmatrix}3\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}-3\\4\\2\end{pmatrix}\right]\bullet\begin{pmatrix}-3\\4\\2\end{pmatrix}=0$ $(-9-4+4)+\lambda(9+16+4)=0$ $\lambda=\frac{9}{29}$ Hence, $\overrightarrow{OF}=\begin{pmatrix}3\\-1\\2\end{pmatrix}+\frac{9}{29}\begin{pmatrix}-3\\4\\2\end{pmatrix}=\frac{1}{29}\begin{pmatrix}60\\7\\76\end{pmatrix}$.	Quite a few wrote that $\overrightarrow{AF}=\begin{pmatrix}3\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}-3\\4\\2\end{pmatrix}$. Recall that "$\mathbf{r}$" in the equation of a line represents the **<u>position vector</u>** of a point on the line (i.e. a vector from the **origin** to a point on the **line**). You need get the concept of the line correct. See Chap 4C page 2.



Method 1.1 (Find C using ratio theorem)

By Ratio Theorem, $\overrightarrow{OF} = \dfrac{t\overrightarrow{OA} + \overrightarrow{OC}}{(t+1)}$

$$\overrightarrow{OC} = (t+1)\overrightarrow{OF} - t\overrightarrow{OA} = \frac{(t+1)}{29}\begin{pmatrix}60\\7\\76\end{pmatrix} - t\begin{pmatrix}3\\-1\\2\end{pmatrix} = \frac{1}{29}\begin{pmatrix}60-27t\\7+36t\\76+18t\end{pmatrix}$$

A few applied the ratio theorem with respect to another point on π (for example, P) instead of O. Although not wrong, but it involves more computation.

Method 1.2 (Find C without using ratio theorem)

$$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \frac{1}{29}\begin{pmatrix}60\\7\\76\end{pmatrix} - \begin{pmatrix}3\\-1\\2\end{pmatrix} = \frac{1}{29}\begin{pmatrix}-27\\36\\18\end{pmatrix}$$

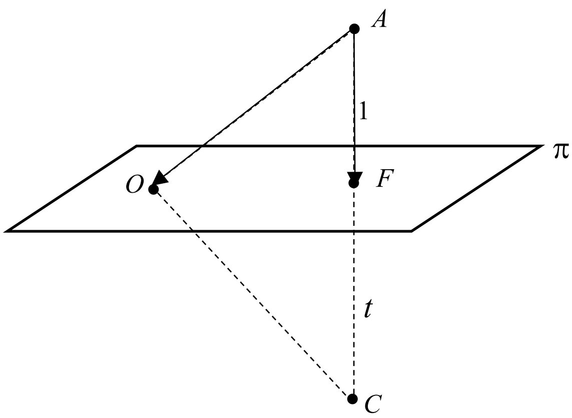
Since $\overrightarrow{FC} = t\overrightarrow{AF}$, then $\overrightarrow{OC} = \overrightarrow{OF} + \overrightarrow{FC} = \overrightarrow{OF} + t\overrightarrow{AF}$

$$= \frac{1}{29}\begin{pmatrix}60\\7\\76\end{pmatrix} + \frac{t}{29}\begin{pmatrix}-27\\36\\18\end{pmatrix}$$

Although $\left|\overrightarrow{CF}\right| = t\left|\overrightarrow{AF}\right|$, but $\overrightarrow{CF} \neq t\overrightarrow{AF}$ as $\overrightarrow{CF}$ and $\overrightarrow{AF}$ are in opposite direction. In fact, $\overrightarrow{CF} = -t\overrightarrow{AF}$.

Quite a few wrote $\overrightarrow{AC} = t\overrightarrow{AF}$, when it should be $\overrightarrow{AC} = (t+1)\overrightarrow{AF}$

<u>**Method 2 (Use projection vector)**</u>



$$\overrightarrow{AF} = \left(\frac{\overrightarrow{AO} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}}{\left| \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|} \right) \frac{\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}}{\left| \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|} = \frac{1}{29} \left(\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right) \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = \frac{9}{29} \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$$

Since $\overrightarrow{AC} = (t+1)\overrightarrow{AF}$, then

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + (t+1)\overrightarrow{AF}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \frac{9(t+1)}{29} \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = \frac{1}{29} \begin{pmatrix} 60 \\ 7 \\ 76 \end{pmatrix} + \frac{t}{29} \begin{pmatrix} -27 \\ 36 \\ 18 \end{pmatrix}$$

Method 3 (Use length of projection)

Let $\overrightarrow{OC} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\overrightarrow{AC} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$$\overrightarrow{AC} \perp \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}: \quad \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right) \bullet \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = 0$$

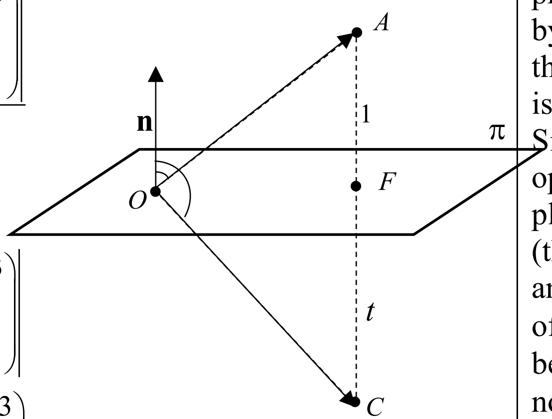
$$-y + 2z = 5 \quad ----(1)$$

$$\overrightarrow{AC} \perp \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}: \quad \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right) \bullet \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = 0$$

$$4x + 3y = 9 \quad ----(2)$$

$$\left|\overrightarrow{CF}\right| = t\left|\overrightarrow{AF}\right|$$

$$\frac{\left| \overrightarrow{OC} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|} = t \frac{\left| \overrightarrow{OA} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|}$$



$$\left| \overrightarrow{OC} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right| = t \left| \overrightarrow{OA} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|$$

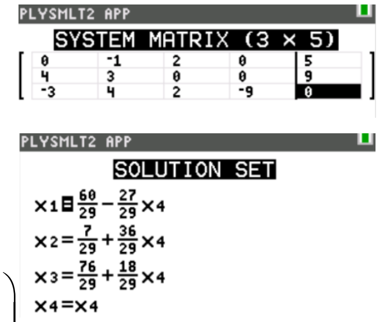
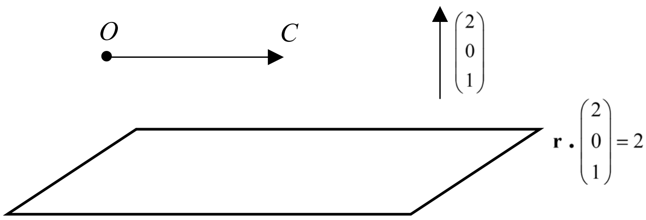
$$\overrightarrow{OC} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = -t\overrightarrow{OA} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \Rightarrow -3x + 4y + 2z = -(-9t) = 9t \quad ----(3)$$

Those who wrote $\left| \overrightarrow{OA} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|$ in

$$\overrightarrow{AF} = \frac{\left| \overrightarrow{OA} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|} \frac{\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}}{\left| \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \right|}$$

will lose marks as the projection vector needs to take into consideration of the direction of $\overrightarrow{OA}$.

The sign of the dot product is determined by the angle between the 2 vectors, whether it is acute or obtuse. Since A and C are on opposite sides of the plane, one of the angles (the angle between OA and the normal vector of π, and the angle between OC and the normal vector of π) will be acute and the other angle will be obtuse. Thus, there is a

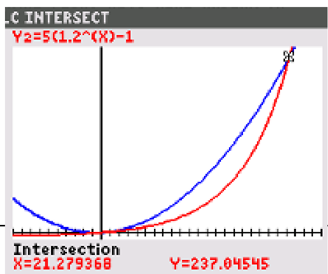
	Solving (1), (2), (3) using GC, we get $x=\frac{60}{29}-\frac{27}{29}t$ $y=\frac{7}{29}+\frac{36}{29}t$ $z=\frac{76}{29}+\frac{18}{29}t$  Thus, $\overrightarrow{OC}=\frac{1}{29}\begin{pmatrix}60\\7\\76\end{pmatrix}+\frac{t}{29}\begin{pmatrix}-27\\36\\18\end{pmatrix}$	need to put the negative sign when removing the "modulus".
(iv) **[2]**	 *OC* must be perpendicular to the normal vector of the plane. $\overrightarrow{OC}\bullet\begin{pmatrix}2\\0\\1\end{pmatrix}=0$ $\frac{1}{29}\begin{pmatrix}60-27t\\7+36t\\76+18t\end{pmatrix}\bullet\begin{pmatrix}2\\0\\1\end{pmatrix}=0$ $2(60-27t)+(76+18t)=0$ $-36t+196=0$ $t=\frac{49}{9}$	By drawing a simple diagram (on the left), one can see that *OC* will be perpendicular to the normal vector of the plane. Quite a few wrote $\overrightarrow{OC}\bullet\begin{pmatrix}2\\0\\1\end{pmatrix}=2$. This actually means that *C* is a point on the plane $\mathbf{r}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}=2$, which is NOT what the question meant. There are also a few who wrote $t\begin{pmatrix}-27\\36\\18\end{pmatrix}\bullet\begin{pmatrix}2\\0\\1\end{pmatrix}=0$

4 **(a)** The first 3 terms of a geometric progression are a, b, 2 and the first 3 terms of an arithmetic progression are 2, a, b, with non-zero common difference. Find the values of a and b. [4]

(b) $u_1,\ u_2,\ u_3,\ u_4, \ldots,\ u_{2n}$ is a sequence of $2n$ positive terms, with $n > 1$. The odd-numbered terms form an arithmetic progression with common difference p and the even-numbered terms form a geometric progression with common ratio $\frac{6}{5}$. Given that $u_1 = u_2 = p$ and the sum of the odd-numbered terms is less than the sum of the even-numbered terms, find the least value of n. [4]

(c) A geometric progression of n terms has first term q and common ratio r, where q is non-zero and $r \neq 1$. For $k \leq n$, find the difference between the sum of the last k terms and the sum of the first k terms, simplifying your answer. [5]

(a) **[4]**	Given a, b, 2 is GP: $\frac{b}{a} = \frac{2}{b} \quad \Rightarrow \quad b^2 = 2a \quad$ - (1) Given 2, a, b is AP: $a - 2 = b - a \quad \Rightarrow \quad 2a - b = 2 \quad$ - (2) Solving the 2 equations: $b^2 - b - 2 = 0$ $(b-2)(b+1) = 0$ $b = 2$ or $b = -1$ If $b = 2$, then $a = 2$, $\Rightarrow$ common difference, $b - a = 0$ (reject) If $b = -1$, then $a = \frac{1}{2}$, $\Rightarrow$ common difference, $b - a \neq 0$ $\therefore a = \frac{1}{2},\ b = -1$	This is a simple question making use of common ratio in GP and common difference in AP. Quite a handful of students tend to overthink and complicate the working, in the end making the working really long and often getting final answers wrong.
(b) **[4]**	$p,\ p,\ p+d,\ pr,\ p+2d,\ pr^2 \ldots,\ p+(2n-1)d,\ pr^{2n-1}$ Sum of odd terms $= \frac{n}{2}\left[2p + (n-1)p\right] = \frac{pn}{2}(n+1)$ Sum of even terms $= \frac{p\left(1.2^n - 1\right)}{1.2 - 1} = 5p\left(1.2^n - 1\right)$ Given $\frac{pn}{2}(n+1) < 5p\left(1.2^n - 1\right) \Rightarrow \frac{n}{2}(n+1) < 5\left(1.2^n - 1\right)$ **(since $p > 0$)** Using GC, $n > 21.3$ Least value of n is 22.	An **important** presentation issue: $\frac{6^n}{5} \neq \left(\frac{6}{5}\right)^n$. If this **WRONG** presentation were taken into consideration, even more students will have fewer marks! Also, fortunately $p > 0$ (inferred because question states that



	Alternatively, From G.C., $n=21,\ \frac{n}{2}(n+1)-5(1.2^n-1)=5.9744>0$ $n=22,\ \frac{n}{2}(n+1)-5(1.2^n-1)=-18.03<0$ Hence, the least value of n is 22.	sequence consists of positive terms) and will not make a difference in the division over the inequality. Otherwise, even more students would have lost marks over this. Please note that there are **TWO** points of intersection when comparing the 2 curves $\Rightarrow$ there will be TWO regions, $0<n<1$ and $n>21.27$. However, it is given that $n>1$.
(c) **[4]**	Sum of 1st k terms $=\frac{q(r^k-1)}{r-1}$ Sum of last k terms $=\frac{q(r^n-1)}{r-1}-\frac{q(r^{n-k}-1)}{r-1}$ $=\frac{q}{r-1}(r^n-1-r^{n-k}+1)$ $=\frac{q}{r-1}(r^n-r^{n-k})=\frac{q}{r-1}r^{n-k}(r^k-1)$ **Alternatively**: By writing the sequence backwards $qr^{n-1},qr^{n-2},qr^{n-1},......,qr^2,qr,q$ (first term = qr^{n-1} / common ratio = $\frac{1}{r}$) Sum of last k terms $=\frac{qr^{n-1}\left(\left(\frac{1}{r}\right)^k-1\right)}{\frac{1}{r}-1}=\frac{qr^n}{1-r}\left(\frac{1-r^k}{r^k}\right)$ $=\frac{qr^{n-k}(1-r^k)}{1-r}=\frac{q}{r-1}r^{n-k}(r^k-1)$	Very few got full marks (despite doing everything right) because of a small detail: the question asks for **DIFFERENCE**, which means that the **modulus** will be necessary as there is no mention of which sum is a larger value. Quite a handful of students assumed that S_{n-k} refers to the sum of last k terms, which is **WRONG**!

	Difference required $=\left\lvert\frac{q\left(r^k-1\right)}{r-1}-\frac{q}{r-1}r^{n-k}\left(r^k-1\right)\right\rvert$ $=\left\lvert\frac{q}{r-1}\left[\left(r^k-1\right)-r^{n-k}\left(r^k-1\right)\right]\right\rvert$ $=\left\lvert\frac{q}{r-1}\left(r^k-1\right)\left(1-r^{n-k}\right)\right\rvert$	

5 A club in a school has 5 members from Class P, 4 members from Class Q and 3 members from Class R. Five members are to be chosen for an upcoming competition.

(i) Find the number of ways the team of five can be chosen so that it has exactly two members from each of Class P and Class Q. [1]

(ii) Find the number of ways the team of five can be chosen so that it has at least two members from Class R. [2]

(iii) All the 12 members of the club go to a cinema. Find the number of ways they can sit in a row so that no more than 2 members from Class P are next to each other. [4]

(i) **[1]**	2 from each of P and Q, 1 from R: ${}^5C_2\times{}^4C_2\times{}^3C_1=180$	Generally well done
(ii) **[2]**	2 from Class R & 3 from P or Q: ${}^3C_2\times{}^9C_3=252$ 3 from Class R & 2 from P or Q: ${}^3C_3\times{}^9C_2=36$ Total number of ways is 288 **Alternatively,** Consider complement cases of none from R or one from R: ${}^{12}C_5-{}^9C_5-{}^3C_1\times{}^9C_4=288$	Generally well done.
(iii) **[4]**	Class P students are all separated : $7!\times{}^8C_5\times5!=33868800$ separated in groups of 2, 1, 1, 1 : $7!\times{}^8C_1\times{}^7C_3\times5!=169344000$ (or $7!\times{}^8C_4\times4\times5!$) separated in groups of 2, 2, 1 : $7!\times{}^8C_2\times{}^6C_1\times5!=101606400$ (or $7!\times{}^8C_3\times3\times5!$) Total number of ways is 304 819 200 **Alternatively,** consider the complement method: separated in groups of 3, 1, 1 : $7!\times{}^8C_3\times3\times5!=101\ 606\ 400$ separated in groups of 3, 2 : $7!\times{}^8C_2\times2!\times5!=33\ 868\ 800$ separated in groups of 4, 1 : $7!\times{}^8C_2\times2!\times5!=33\ 868\ 800$ 5 members of P are seated in a group : $8!\times5!=4\ 838\ 400$	Important to consider cases carefully. 1) Permute the non-P student 2) Not more than two members means that we can have group(s) of 2 or less. 3) Beware of double counting 4) Consider permuting the other 7 students 5) Consider slotting in the groups of P

	Total number of ways = $12!-101606400-33868800-33868800-4838400=304819200$	6) Consider the arrangement of these groups 7) Consider the arrangement of P members in these groups

6 A car insurance company collected the following data about the percentage occurrence of accident-involved vehicles, p% for vehicles of different weight, w tons.

w (tons)	2.2	1.9	1.7	1.5	1.3	1.1	1.0	0.9
p (%)	2.6	3.2	3.8	4.3	5.4	5.3	7.4	8.6

(i) Calculate the value of the product moment correlation coefficient between w and p, and explain whether your answer suggests that a linear model is appropriate. [2]

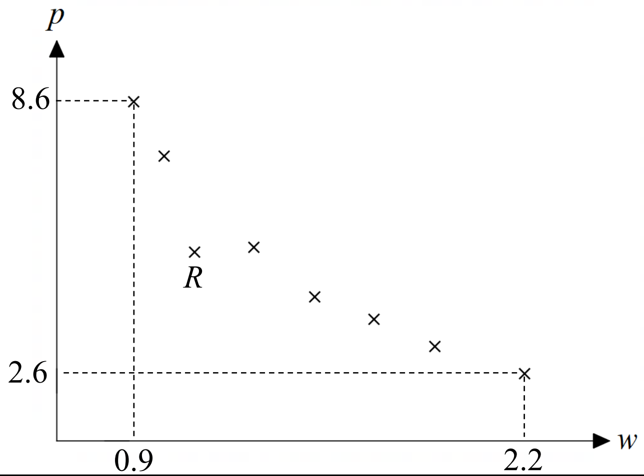
(ii) Draw a scatter diagram of the data. [1]

One of the values of p appears to be incorrect.

(iii) Indicate the corresponding point on your diagram by labelling it R, and explain why the scatter diagram for the remaining points may be consistent with a model of the form $\ln p = a + bw$. [2]

(iv) Omitting R, calculate least squares estimates of a and b for the model $\ln p = a + bw$. [2]

(v) Assume that the value of w at R is correct. Estimate the value of p for this value of w. [1]

(i) **[2]**	Since $r=-0.92821=-0.928$ (3.s.f) is close to -1, it suggests a strong negative linear correlation, a linear model seems appropriate.	Focus on commenting on just the pmcc value.
(ii) **[1]**		The number of points must be drawn correctly. The range of p and w should be given.

(iii) **[2]**	With the point R removed, the values of p decreases as w increases, but by decreasing amounts. Hence it is consistent with a model of the form $\ln p = a + bw$.	Focus on commenting on the scatter diagram, and there is no need to compute the new pmcc value.
(iv) **[2]**	From GC, $\ln p = 2.910569 - 0.916387w$ … (1) $a = 2.91$ (3 s.f.) $b = -0.916$ (3 s.f.)`	To conclude answers in 3.s.f
(v)	Substitute $w = 1.1$ into (1) : $\ln p = 2.910569 - 0.916387(1.1)$ $p = 6.70$ (to 3 s.f.)	A reminder to use at least 5.s.f answers of (iv) for accuracy

7 On average, 11% of the students in school A have been infected before by a contagious disease. Every class has 20 students. The number of students in a class who has been infected by the disease before is denoted by X.

(i) State, in the context of this question, two assumptions needed for X to be well modelled by a binomial distribution. Explain why your assumptions may not be met. [4]

Assume now that the context above is well-modelled by a binomial distribution.

(ii) A class is chosen at random. Find the probability that at least 1 but fewer than 10 students in the class has been infected by the disease before. [2]

Fatihah is a student reporter tasked to ask students in a randomly chosen class, one by one, if they have been infected by the disease before.

(iii) Find the probability that the 20$^{\text{th}}$ student she asks is the fifth student who has been infected by the disease before. [2]

(iv) Without doing any further calculation, is the probability found in **(iii)** higher or lower than the probability that there are exactly 5 students in the class who have been infected by the disease before? Explain your answer. [2]

(i) **[4]**	<u>Condition 1</u> The event that a student infected by the disease is independent of any other student infected by the disease. This might not be true. Since the disease is contagious, it can easily be passed from one student to another. <u>Condition 2</u> The probability of a student infected by the disease is constant. This might not be true as the probability will depend on each individual's lifestyle or exposure, some will have a higher chance of catching the disease.	Generally well done with many students using the contagious nature of the disease to argue against both assumptions convincingly. It is advisable that students list the 2 assumptions and write out their explanations separately as some students who gave a single explanation was not specific enough.

(ii) **[2]**	$X \sim \mathrm{B}(20, 0.11)$ $\mathrm{P}(1 \le X < 10) = \mathrm{P}(1 \le X \le 9)$ $= \mathrm{P}(X \le 9) - \mathrm{P}(X = 0)$ $= 0.999983 - 0.097230$ $= 0.902753$ $= 0.903$	A number of students incorrectly used the value of $\mathrm{P}(X \le 10)$ instead of $\mathrm{P}(X < 10)$ in their calculations giving a value of 0.902757 which still gives the answer as 0.903. These students were not given the marks.
(iii) **[2]**	Let Y denote the number of students who has once been infected by the disease before out of 19. $Y \sim \mathrm{B}(19.0.11)$ Required probability $= \mathrm{P}(Y = 4) \times 0.11 = 0.0109$ (to 3s.f.) <u>Alternative solution</u> Required probability $= \frac{{}^{19}C_4}{{}^{20}C_5} \mathrm{P}(X = 5) = 0.0109$ (to 3 s.f.)	More than 50% of students were able to figure out the solution but not all of them define the RV Y but just wrote out the values of $\mathrm{P}(Y = 4)$.
(iv) **[2]**	Lower. The event described in part **(iii)** is a subset of having exactly 5 students in the class who have been infected by the disease before. For example, it is possible that the first 5 students approached are the only 5 students in the class who has once been infected. This event would have been considered under "exactly 5 students" in the class have once been infected, but not considered in **(iii).**	Most students gave the answer as lower with some students working out the numerical values to compare although the question stated "without doing any further calculation". Students need to provide an example to show that the event in **(iii)** is indeed a subset as merely stating that it is a subset is a generic response when the answer to **(iii)** is a lower value.

8 A game is played by throwing a fair coin and two fair four-sided dice. The dice are coloured red or blue, and have faces numbered from 1 to 4. If the coin shows a head, then the score is the number shown on the red die. Otherwise, the score is the sum of the numbers shown on the two dice.

(i) Show that the probability that a game results in a score of 4 is $\frac{7}{32}$. [2]

(ii) Find the expectation and variance of the score. [5]

(iii) The game is played 35 times. Estimate the probability that the average score of the 35 games is at least 4, given that the first and second games result in a score of 3 and 4 respectively. [3]

(i) **[2]**	Let X be the score obtained in one game. $\mathrm{P}(X=4)=\mathrm{P}((\mathrm{H},4),(\mathrm{T},1+3),(\mathrm{T},2+2))$ $=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 3=\frac{7}{32}$	Candidates were usually able to show the required probability. Note the importance of listing the cases.
(ii) **[5]**	$\mathrm{P}(X=1)=\mathrm{P}(\mathrm{H},1)=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$ $\mathrm{P}(X=2)=\mathrm{P}((\mathrm{H},2),(\mathrm{T},1+1))=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{5}{32}$ $\mathrm{P}(X=3)=\mathrm{P}((\mathrm{H},3),(\mathrm{T},1+2))=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 2=\frac{3}{16}$ $\mathrm{P}(X=5)=\mathrm{P}((\mathrm{T},1+4),(\mathrm{T},2+3))=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 4=\frac{1}{8}$ $\mathrm{P}(X=6)=\mathrm{P}((\mathrm{T},2+4),(\mathrm{T},3+3))=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 3=\frac{3}{32}$ $\mathrm{P}(X=7)=\mathrm{P}(\mathrm{T},3+4)=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot 2=\frac{1}{16}$ $\mathrm{P}(X=8)=\mathrm{P}(\mathrm{T},4+4)=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{32}$ $\mathrm{E}(X)=\sum_{r=1}^{8} r\,\mathrm{P}(X=r)=\frac{15}{4}$ (or 3.75) $\mathrm{E}(X^2)=\sum_{r=1}^{8} r^2\,\mathrm{P}(X=r)=\frac{35}{2}$ (or 17.5) $\mathrm{Var}\,X=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=\frac{35}{2}-\left(\frac{15}{4}\right)^2=\frac{55}{16}$	Most candidates are able to calculate the probabilities correctly and subsequently the expectation and variance. Others made errors in the number of cases or made slips during the use of the calculator. A handful erroneously worked out the sum of $r\left[\mathrm{P}(X=r)^2\right]$ for $\mathrm{E}(X^2)$.

(iii) **[3]**	$\mathrm{P}\left(\frac{X_1+\cdots+X_{35}}{35}\geq 4 \,\middle\|\, X_1=3 \text{ and } X_2=4\right)$ $=\frac{\mathrm{P}\left(\frac{X_1+\cdots+X_{35}}{35}\geq 4 \cap \left(X_1=3 \text{ and } X_2=4\right)\right)}{\mathrm{P}\left(X_1=3 \text{ and } X_2=4\right)}$ $=\frac{\mathrm{P}\left(\frac{3+4+X_3+\cdots+X_{35}}{35}\geq 4\right)\mathrm{P}\left(X_1=3\right)\mathrm{P}\left(X_2=4\right)}{\mathrm{P}\left(X_1=3\right)\mathrm{P}\left(X_2=4\right)}$ $=\mathrm{P}\left(X_3+\cdots+X_{35}\geq 4\times 35-3-4\right)$ $=\mathrm{P}\left(X_3+\cdots+X_{35}\geq 133\right)$ The sample size 33 is large. By Central Limit Theorem, $X_3+\cdots+X_{35}\sim \mathrm{N}\left(33\times\frac{15}{4}=\frac{495}{4},\ 33\times\frac{55}{16}=\frac{1815}{16}\right)$ approximately. $\mathrm{P}\left(X_3+\cdots+X_{35}\geq 133\right)=0.1925638=0.193\ \text{(3sf)}$	This part proves more challenging for candidates than the earlier parts. It also revealed much misunderstanding about the concept of Central Limit Theorem (CLT). A significant number thought that CLT approximates X as a normal distribution and so $X_3+\cdots+X_{35}$ also follows a normal distribution when it should have been CLT approximates the sum $X_3+\cdots+X_{35}$ as a normal distribution instead because CLT is not to be used for X.

9 A large company claims that its employees work an average of 41 hours a week. After receiving feedback from some employees that they work longer than claimed, a survey involving 200 randomly chosen employees is conducted. The amount of time they spend at work per week, x thousand hours, are summarised below.

$$\sum x = 8.71 \qquad \sum x^2 = 0.505$$

(i) Calculate unbiased estimates of the population mean and variance of the amount of time spent by an employee at work per week. [2]

(ii) Test, at the 5% significance level, whether the average working hours of an employee per week is more than 41. State hypotheses for the test, defining any symbols you use. [5]

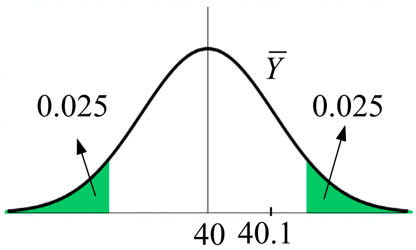
(iii) After the company restructures its operations, it claims that the average working hours a week is now 40. The human resource manager takes a random sample of 12 employees and finds that they spend an average of 40.1 hours per week at work. Suppose the population variance is k hours^2.

(a) Stating a necessary assumption, find the range of possible values of k if the manager concludes that there is insufficient evidence to reject the company's claim that the average working hours is 40, at the 5% significance level. [4]

(b) Explain why the Central Limit Theorem does not apply in this context. [1]

9 (i) [2]	Unbiased estimate of population mean is $\frac{8.71}{200} = 0.04355$ thousand hrs = 43.55 hours Unbiased estimate of population variance is $\frac{1}{199}\left(0.505 - \frac{8.71^2}{200}\right) = 6.31555\times10^{-4}$ (thousand hours)^2 = 632 hours^2 (3sf)	Most students who obtained 2 marks would have gotten zero, if marking demanded the correct units. Proper presentation must be taken note of, the correct name of each item, and not just "mean"/"variance". Many faltered for s^2 as they multiplied by 1000 instead of 1000^2
(ii) [5]	Let μ be the population mean amount of time, in thousand hours, an employee spends at work per week. Null Hypothesis $H_0: \mu = 0.041$ Alternative Hypothesis $H_1: \mu > 0.041$ Perform a 1-tail test at 5% significance level. Under H_0, since sample size 200 is large, $\overline{X} \sim N\left(0.041, \frac{6.31555\times10^{-4}}{200}\right)$ approximately by Central Limit Theorem. From the sample, $\bar{x} = 0.04355$ Using a z-test, p-value $= P\left(\overline{X} \geq 0.04355\right) = 0.0756 > 0.05$	Clear handwriting cannot be over-emphasized. Cambridge markers have stated they cannot mark what they can't read. Many of our students need to improve in this aspect. Many did not define μ properly, which is such a regret since this was done before in • Chap S5 Eg 4 • Tut S5 Qn 9 • Recent CT Central Limit Theorem must be stated.

<table>
<tr><td></td><td>We do not reject H_0. There is insufficient evidence, at the 5% significance level, that the average working hours of an employee per week is more than 41.</td><td>Proper conclusion statement – do follow what has been given in the notes and solutions – students who choose to rephrase do so at their own risk of missing out key terms or putting it in a wrong logical order.</td></tr>
<tr><td>(iii)
(a)
[4]</td><td>Assume that the number of working hours per week of an employee follows a normal distribution.

Let Y be the number working hours per week of an employee after the restructuring.
$H_0: \quad \mu = 40$
$H_1: \quad \mu \neq 40$
Under H_0, $\overline{Y} \sim N\left(40, \frac{k}{12}\right)$
If H_0 is not rejected at the 5% significance level,
$$P\left(\overline{Y} \geq 40.1\right) > 0.025$$


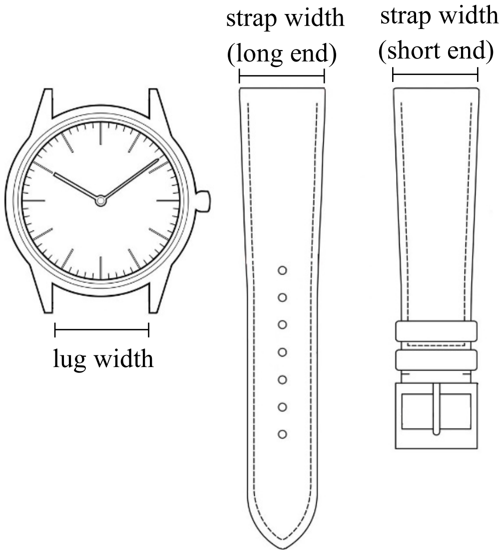
$$P\left(Z \geq \frac{40.1-40}{\sqrt{\frac{k}{12}}}\right) > 0.025$$
$$\frac{0.1}{\sqrt{\frac{k}{12}}} < 1.9600$$
$$\frac{k}{12} > \left(\frac{0.1}{1.9600}\right)^2$$
$$k > 0.0312 \quad \text{(3sf)}$$</td><td>Many students faltered here, they need to know the difference between the working hours and the mean working hours.

Also a 2-tail test reverse method question does not necessarily involve 2 inequalities. In this case the claimed mean and sample mean are both known, the inequality is only 1 directional. Just need to half the probability.

An large number of students worked towards rejecting H_0 instead or only did a 1-tailed test.

A handful number of students should also take note not to convert an irrational answer into an exact fraction.</td></tr>
<tr><td>(b)</td><td>The Central Limit Theorem does not apply here as the sample size 12 is small.</td><td>Most students got the mark here, but they should take note not to write extra irrelevant or inaccurate information, which may not always be ignored.</td></tr>
<tr><td colspan="3">Overall, this question was done below expectations with many students making mistakes that arose from unfamiliarity. With the abundance of suggested solutions, students should not be complacent but spare some time to properly study Hypothesis Testing questions and their solutions.</td></tr>
</table>

10 In this question you should state the parameters of any normal distributions you use.



A leather craftsman company handcrafts customized leather watch straps according to the lug width of their customers' watches. Over a period of time it is found that the lug widths of their customers' watches are normally distributed; 85% of the watches have lug widths less than 21 mm, and 15% of the watches have lug widths less than 19 mm.

(a) Find the mean and the standard deviation of the lug width of their customers' watches. [3]

The widths of the straps made by the company follow the normal distribution with mean 19.6 mm and standard deviation 1.1 mm.

(b) Find the expected number of straps of width more than 20.2 mm in a randomly chosen batch of 40 straps. [3]

The straps are handcrafted in pairs. Each pair consists of a long end strap and a short end strap. The strap widths of both the long end straps and the short end straps follow the same normal distribution, and are independent of each other. In order for the strap to fit into the lug of the watch, the strap width needs to be shorter than the lug width of the watch. If the strap width is less than 0.2 mm shorter than the lug width of the watch, it is considered a good fit. Otherwise, the strap is a bad fit. A pair of strap is only usable for the watch if the long end strap and the short end strap are both good fits.

(c) A customer walks in with a watch with lug width of 20 mm. He randomly chooses a pair of straps. Show that the probability the pair of straps is usable for his watch is 0.00487, correct to 3 significant figures. [3]

(d) Another customer walks in with two watches of lug widths 18.5 mm and 20 mm respectively. He randomly chooses 2 pairs of straps of different designs. Find the probability that at least 1 pair of straps are usable for any of his watches, giving your answer correct to 5 decimal places. [4]

(a) **[3]**	Let X denote the lug width (in mm) of their customers' watches. $X \sim \mathrm{N}(\mathrm{E}(X), \sigma^2)$ $\mathrm{E}(X) = 20$	General reminder for **(a) – (d)** : Do remember to work with higher degree of accuracy in your intermediate working.

	$\mathrm{P}(X < 21) = 0.85$ $\mathrm{P}\left(Z < \dfrac{21-20}{\sigma}\right) = 0.85$ From G.C. $\dfrac{1}{\sigma} = 1.03643$ $\sigma = 0.96484 = 0.965$ (to 3s.f.)	- - - - - - - - In **(a)**, E(X) is most conveniently deduced by symmetry. Standard deviation (σ) obtained is clearly non-exact, and hence you should be leaving your ans in 3 s.f. (and not as a rational number).
(b) **[3]**	Let W denote the strap width (in mm) made by the company. $W \sim \mathrm{N}(19.6, 1.1^2)$ $\mathrm{P}(W > 20.2) = 0.29272$ Let Y denote the number of straps of width more than 20.2mm out of 40. $Y \sim \mathrm{B}(40, 0.29272)$. $\mathrm{E}(Y) = 40 \times 0.29272 = 11.708 = 11.7$ (to 3s.f.)	Common mistakes : 1) finding P($\overline{W}$ >20.2) instead of P(W>20.2). 2) finding <u>mode</u> instead of <u>expected number</u>. - - - - - <u>NOTE</u> : Expected number is a statistical value and you should not round off/down/up to an integer.
(c) **[3]**	$\mathrm{P}(19.8 < W \le 20) = 0.069798$ Required probability = $0.069798^2 = 0.0048717 = 0.00487$ (to 3s.f.)	The value "0.00487" is <u>given</u> in the question. Hence, there is a need to show detailed working, and the value (0.0048717) right before your conclusion.

(d) **[4]**	<u>**General comment**</u> : This part proves to be difficult for most students. To facilitate discussion, let's set some background. Let's refer to the watches with lug widths 18.5 mm and 20 mm as the 18.5mm-watch and 20mm-watch respectively. Let α be the probability that a pair of straps is usable for the 18.5mm-watch, and β be the probability that a pair of straps is usable for the 20mm-watch. Clearly, $\alpha = \left[\mathrm{P}(18.3 < W \le 18.5)\right]^2 = 0.0016013$ (to 7 d.p.). From part **(c)**, $\beta = \left[\mathrm{P}(19.8 < W \le 20)\right]^2 = 0.0048717$ (to 7d.p.) <u>**IMPORTANT NOTE**</u> : When a pair of strap is <u>not</u> usable for the 18.5mm-watch, it may still be usable for the 20mm-watch. Hence, the probability $1 - \alpha$ only suggests the chance that a pair of straps is not usable for the 18.5mm-watch, but it doesn't rule out that the pair is usable for the 20mm-watch. <u>**Method 1**</u> : Probability that a pair of straps is not usable for both watches $= 1 - \alpha - \beta$ Hence, the required probability $= 1 -$ P(both pairs of straps are not usable for both watches) $= 1 - (1-\alpha-\beta)^2$ … (*) $= 0.0129041904 = 0.01290$ (5 d.p.) *Note that the expression (*) when expanded is* $2\alpha + 2\beta - 2\alpha\beta - \alpha^2 - \beta^2$. ... (**) *This expression can provide a quick check for any other approaches to this question you may have in mind.* <u>**Method 2**</u> : Let Q be the number of pairs of straps (out of 2) that are usable (for either the 18.5mm-watch or the 20mm-watch. $Q \sim \mathrm{B}(2,\ \alpha+\beta)$ Hence, the required probability $= \mathrm{P}(Q \ge 1) = 1 - \mathrm{P}(Q = 0)$ $= 1 - (1-(\alpha+\beta))^2 = 1 - (1-\alpha-\beta)^2$ (same as (*)) $= 0.0129041904 = 0.01290$ (5 d.p.)	

	Method 3 : Case 1 : One pair of straps is usable for 18.5mm-watch, but the other pair is not usable for the 20mm-watch. Probability = $\alpha(1-\beta)\times 2! \; -\alpha^2$ $= 2\alpha - 2\alpha\beta - \alpha^2$ *[Many students using this approach forgot to multiple by 2, and/or forgot to subtract double counted cases.]* Case 2 : One pair of straps is usable for 20mm-watch, but the other pair is not usable for the 18.5mm-watch. Probability = $\beta(1-\alpha)\times 2! \; -\beta^2$ $= 2\beta - 2\alpha\beta - \beta^2$ Case 3 : One pair of straps is usable for the 18.5mm-watch and the other pair is usable for the 20mm-watch. Probability = $2\alpha\beta$ Hence, the required probability $= \left(2\alpha - 2\alpha\beta - \alpha^2\right) + \left(2\beta - 2\alpha\beta - \beta^2\right) + 2\alpha\beta$ $= 2\alpha + 2\beta - 2\alpha\beta - \alpha^2 - \beta^2$ (same as (**)) = 0.0129041904 = 0.01290 (5 d.p.) **Method 4** : Case 1 : There is a usable pair of straps (out of the 2) for the 18.5mm-watch. Probability = 1 – P(both pairs of straps are not usable for the 18.5mm-watch) $= 1 - (1-\alpha)^2 = 2\alpha - \alpha^2$ OR Probability = $\alpha^2 + 2\alpha(1-\alpha) = 2\alpha - \alpha^2$ Case 2 : There is a usable pair of straps (out of the 2) for the 20mm-watch. Probability = 1 – P(both pairs of straps are not usable for the 20mm-watch) $= 1 - (1-\beta)^2 = 2\beta - \beta^2$ OR Probability = $\beta^2 + 2\beta(1-\beta) = 2\beta - \beta^2$ Case 3 : One pair of straps is usable for the 18.5mm-watch and the other pair is usable for the 20mm-watch. Probability = $2\alpha\beta$ *[Note that case 3 is repeated in case 1 and case 2.]*	

Hence, the required probability

$$= \left(2\alpha - \alpha^2\right) + \left(2\beta - \beta^2\right) - 2\alpha\beta$$

$$= 2\alpha + 2\beta - 2\alpha\beta - \alpha^2 - \beta^2 \quad \text{(same as (**))}$$

$$= 0.0129041904$$

$$= 0.01290 \text{ (5 d.p.)}$$

This approach may be *re-packaged* as the following :

Let A be the event that there is a usable pair of straps for the 18.5mm-watch.

Let B be the event that there is a usable pair of straps for the 20mm-watch.

Required probability

$$= \mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$$= [\alpha^2 + 2\alpha(1-\alpha)] + [\beta^2 + 2\beta(1-\beta)] - 2\alpha\beta$$

$$= 2\alpha + 2\beta - 2\alpha\beta - \alpha^2 - \beta^2 \quad \text{(same as (**))}$$

$$= 0.0129041904 = 0.01290 \text{ (5 d.p.)}$$

NOTE : $\mathrm{P}(A) \neq \alpha^2$; $\mathrm{P}(B) \neq \beta^2$; $\mathrm{P}(A \cap B) \neq \alpha\beta$.

Method 5 :

Case 1 : First pair of straps chosen is usable.

Probability = $(\alpha + \beta)(1)$

Case 2 : Second pair of straps chosen is usable.

Probability = $(1)(\alpha + \beta)$

Case 3 : One pair of straps is usable for the 18.5mm-watch and the other pair is usable for the 20mm-watch.

Probability = $2\alpha\beta$

[Note that case 3 is repeated in case 1 and case 2.]

Case 4 : 2 pairs of straps are usable for the same watch.

Probability = $\alpha^2 + \beta^2$

[Note that case 4 is also repeated in case 1 and case 2.]

NOTE : Case 3 and Case 4 may be more conveniently combined as "First and second pair of straps are both usable".

Probability = $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

Hence, the required probability

$$= 2(\alpha + \beta) - 2\alpha\beta - \alpha^2 - \beta^2$$

$$= 2\alpha + 2\beta - 2\alpha\beta - \alpha^2 - \beta^2 \quad \text{(same as (**))}$$

$$= 0.0129041904 = 0.01290 \text{ (5 d.p.)}$$

Method 6 :

Let $m = \mathrm{P}(W \le 18.3) + \mathrm{P}(18.5 < W \le 19.8) + \mathrm{P}(W > 20)$

$= 0.8901858519$

$n = \mathrm{P}(18.3 < W \le 18.5) = 0.0400162748$,

$s = \mathrm{P}(19.8 < W \le 20) = 0.0697978733$.

P(a pair of straps is not usable)

$= m^2 + 2mn + 2ms + 2ns$ (or just $1 - nn - ss$)

Hence, the required probability

$= 1 - \left(m^2 + 2mn + 2ms + 2ns\right)^2$

$= 0.0129041904$

$= 0.01290$ (5 d.p.)