RAFFLES INSTITUTION
2020 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 20

MATHEMATICS 9758/01

PAPER 1

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

FOR EXAMINER'S USE						
Q1	Q2	Q3	Q4	Q5	Q6	Q7
/4	/4	/5	/6	/7	/9	/10
Q8	Q9	Q10	Q11	Q12	Total	
/10	/10	/11	/12	/12	/100	

This document consists of **28** printed pages and **0** blank page.

1 A particle is moving along the curve with equation $16x^2+9y^2=144$.
$\dfrac{\mathrm{d}y}{\mathrm{d}t}=2\text{ cm s}^{-1}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ is positive when $x=\sqrt{5}$.
Find the rate of increase of x at this instant. [4]

2 Mr. Li invested a total of \$30000 and divided this sum into three accounts, which paid 2%, 3% and 5% annual interest respectively.

At the end of the first year, Mr. Li withdrew all the money out from the 2% and 5% accounts and gave the interest earned to his son. The amount in the 3% account, including interest, was re-invested in the same account for another year.

At the end of the second year, Mr. Li withdrew all the money out of the 3% account and gave the interest earned to his son.

The total interest received by the son from the three accounts was \$1423.50.

Given that the amount invested in the 2% account was \$1000 more than the amount invested in the 5% account, find the amounts invested in each of the three accounts. [4]

3 **(i)** Prove that for $x>0$, the substitution $y=ux$ reduces the differential equation $(y-x)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}-\dfrac{y}{x}\right)=y^2+2x^2$ to

$$\left(\frac{u}{u^2+2}-\frac{1}{u^2+2}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)=1.$$ [2]

(ii) Hence find the general solution of the differential equation

$$(y-x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{y}{x}\right)=y^2+2x^2 \text{ for } x>0.$$ [3]

4 **(i)** Sketch the curve with equation $x^2+y^2-6x=7$. [2]

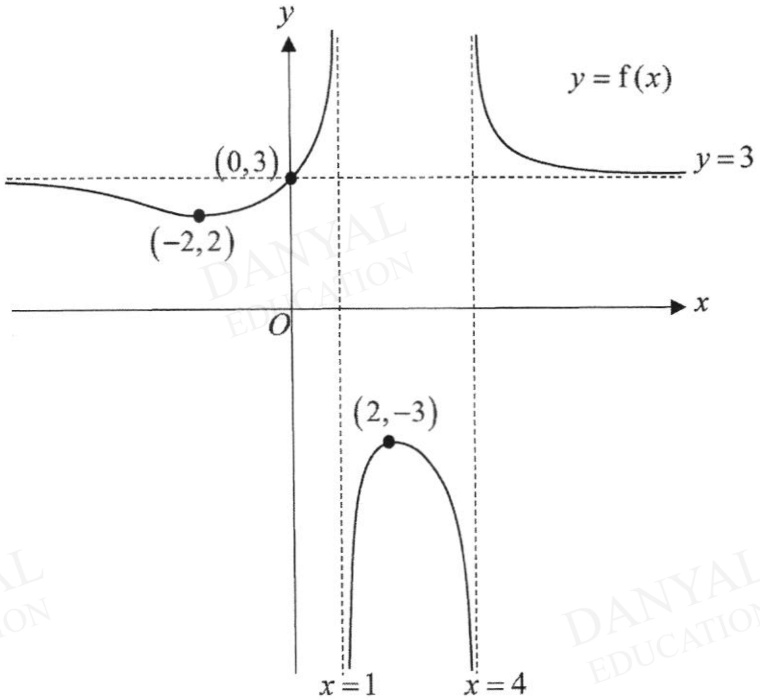
(ii) The region R is bounded by the curve $x^2+y^2-6x=7$, for $x\geq 3$, and the line $x=3$.
Given that $\int_0^a \sqrt{a^2-x^2}\,\mathrm{d}x=\dfrac{\pi a^2}{4}$, find the exact volume of the solid of revolution formed when R is rotated completely about the y-axis. [4]

5 **(a)** Find $\int x\cos x^2 \, dx$. [1]

(b) Use integration by parts to find $\int x\cos 2x \, dx$. [3]

Hence or otherwise find $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos^2 x \, dx$. [3]

6 The diagram below shows the curve of $y=f(x)$. The curve has a minimum point at $(-2, 2)$, a maximum point at $(2,-3)$ and cuts the y-axis at $(0,3)$. The lines $x=1$, $x=4$ and $y=3$ are the asymptotes to the curve.



On separate diagrams, draw sketches of the following graphs, stating the exact coordinates of any turning points and/or points of intersection with the axes, and the equations of any asymptotes, where possible.

(a) $y=f(1-x)$. [3]

(b) $y=\dfrac{1}{f(x)}$. [3]

(c) $y=f'(x)$. [3]

7 It is given that $\mathrm{f}(r)=\dfrac{2^r}{r-2}$. Show that $\mathrm{f}(r+2)-\mathrm{f}(r)=\dfrac{(3r-8)2^r}{r(r-2)}$. [2]

(i) Show that $\displaystyle\sum_{r=3}^{n}\frac{(3r-8)2^r}{r(r-2)}=\frac{(3n-2)2^{n+1}}{n(n-1)}-16$. [4]

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{(3r-2)2^r}{r(r+2)}$ in the form $\dfrac{(3n+A)2^{n+1}}{(n+B)(n+1)}+C$ where A, B and C are integers to be determined. [4]

8 **Do not use a calculator in answering this question.**

(a) **(i)** Solve the equation $z^2=4\mathrm{i}-3$. [3]

(ii) Solve the equation $z^4+6z^2+25=0$. [3]

(b) Find the modulus and argument of the complex number $w=\dfrac{8-2\mathrm{i}}{5+3\mathrm{i}}$. Hence find the possible values of the positive integer n for which w^n is real. [4]

9 A curve C has parametric equations

$$x=t\ln t,\qquad y=\frac{4}{\mathrm{e}^t}+\mathrm{e}^t,$$

for $t\geq\dfrac{1}{2}$.

(i) C meets the y-axis at point P and line L is the normal to C at P. Show that the equation of L is

$$y=\frac{\mathrm{e}}{4-\mathrm{e}^2}x+\frac{4+\mathrm{e}^2}{\mathrm{e}}.$$ [4]

(ii) Sketch the curve C, stating the coordinates of any turning points and points of intersection with the axes. [3]

(iii) The finite region bounded by C, L and the line $x=\dfrac{1}{2}\ln\left(\dfrac{1}{2}\right)$ is denoted by R. Find the area of R. [3]

10 The plane π_1 contains the point $A(1,2,-1)$ and the line l with equation $\dfrac{x-1}{2}=\dfrac{2-z}{3}$, $y=-1$. The plane π_2 contains the point $B(-5.5,3,2)$ and meets π_1 in the line l.

(i) Find the equation of π_1 in scalar product form. [3]

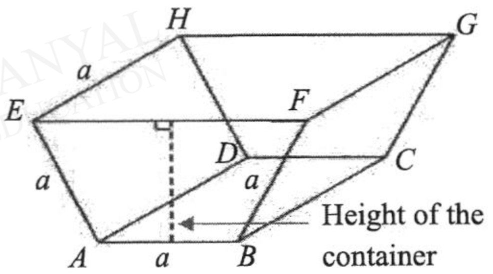
(ii) Show that the vector $\overrightarrow{BF}$ is $\begin{pmatrix}4.5\\-4\\3\end{pmatrix}$, where F is the foot of perpendicular from B to l. [3]

(iii) Find the exact value of the shortest distance from B to π_1. [2]

(iv) Hence or otherwise find the acute angle between π_1 and π_2, giving your answer to the nearest $0.1°$. [3]

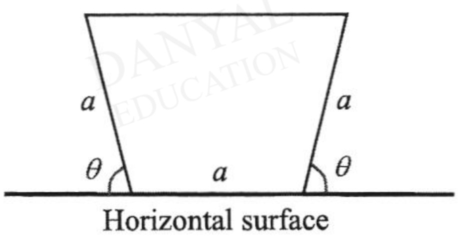
11 Figure 1 shows an open container in the form of a trapezoidal prism *ABCDEFGH* with square base *ABCD* and $AB = AE = BF = EH = a$ cm, where a is a constant. The container is made of plastic of negligible thickness and is placed on a horizontal surface. The faces *BCGF* and *ADHE* are inclined at an angle θ radians, $0<\theta<\dfrac{\pi}{2}$, to the horizontal surface, and faces *ABFE* and *DCGH* are perpendicular to base *ABCD*. Figure 2 shows its cross-sectional view.

Figure 1



Trapezoidal Prism

Figure 2



Cross-Sectional View

(i) Show that the volume V cm^3 of the container is given by $V = a^3\sin\theta(1+\cos\theta)$. [2]

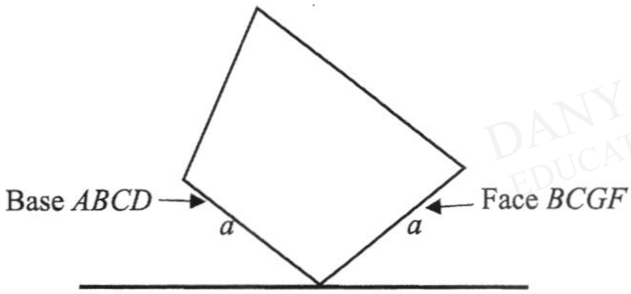
(ii) Use differentiation to find, in terms of a, the maximum value of V in exact form, proving that it is a maximum. [5]

(iii) A particular container is constructed with $\theta = \dfrac{\pi}{3}$ and it is filled with water to half its height. Find, in terms of a, the exact volume of water in this container. [3]

The container is then tilted in the direction of the face $BCGF$ until face $BCGF$ and base $ABCD$ makes the same angle with the horizontal surface.

Figure 3 shows its cross-sectional view.

Figure 3



Cross-Sectional View

Explain if it is possible to tilt the container to this position without any water flowing out from the container. [2]

12 The von Bertalanffy growth model, introduced in 1938, is widely used in fisheries studies. It is used to predict the length, L mm of a fish over a period of time, t years. If L_∞ is the maximum length for a species, then the model assumes that the rate of growth in length of a fish is proportional to $L_\infty - L$.

(i) By setting up and solving a differential equation, show that the general solution of this differential equation is given by $L = L_\infty - Ae^{-kt}$, where k is the constant of proportionality and A is a positive constant. [5]

For the species of fish known as the Atlantic croaker, it has been determined that $L_\infty = 419$ mm and at one year of age, its length is 219 mm and the rate of growth in length is 55 mm per year. Using the above model, obtain an expression for L in terms of t. [3]

(ii) Find its age when the Atlantic croaker grows to a length of 300 mm. [2]

(iii) Sketch a graph of L against t. [2]

RAFFLES INSTITUTION
2020 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 20

MATHEMATICS 9758/02

PAPER 2

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

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Q6	Q7	Q8	Q9	Q10	Q11	Sub-Total	
/7	/8	/9	/12	/12	/12	/60	/100

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RAFFLES INSTITUTION
Mathematics Department

Section A: Pure Mathematics [40 marks]

1 An arithmetic series has first term $\frac{3}{2}$ and fourth term u_4.

A geometric series also has first term $\frac{3}{2}$ and fourth term u_4.

Given that the common ratio of the geometric series is non-negative and the sum of its first 3 terms is $\frac{21}{2}$, find the sum of the first 10 odd-numbered terms of the arithmetic series. [5]

2 In a geometric series, the ratio of the sum of the first 8 terms to the sum of the first 4 terms is 17:16. Find the 2 possible values of the common ratio, r_1, r_2 where $r_1 > r_2$. The sum to infinity of the geometric series with first term a and common ratio r_1 is denoted by S_1 and the sum to infinity of the geometric series with first term b and common ratio r_2 is denoted by S_2. Find $S_1 : S_2$ in terms of a and b. [5]

3 The functions f and g are defined by

$$\mathrm{f}(x) = 1 + 3\mathrm{e}^{-x}, \qquad x \in \mathbb{R}, x > 0,$$
$$\mathrm{g}(x) = |x-1|(x-3), \quad x \in \mathbb{R}, x < c \text{ where } c \text{ is a real constant.}$$

(i) Given that $c = 4$, determine if the composite functions fg and gf exist, justifying your answers. Find the range of the composite function that exists. [4]

(ii) Given that g^{-1} exists, state the largest possible value of c. Using this value of c, find $\mathrm{g}^{-1}(x)$. [4]

4 Given that $y = e^{\tan^{-1}\left(\frac{x}{2}\right)}$, show that $\left(4+x^2\right)\frac{dy}{dx} = 2y.$ [2]

(i) By repeated differentiation of the above result, find the Maclaurin series for $e^{\tan^{-1}\left(\frac{x}{2}\right)}$ up to and including the term in x^3. [5]

(ii) Hence find the Maclaurin series for $\frac{e^{\tan^{-1}\left(\frac{x}{2}\right)}}{(1+x)^2}$ up to and including the term in x^2. [3]

5 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. Point C lies on OA, between O and A, such that $OC : CA = 2:1$. Point D lies on OB produced such that $OD : BD = 3:2$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of **a** and **b**. [2]

(ii) Show that the point E where the lines BC and AD meet has position vector $\frac{4}{3}\mathbf{a}-\mathbf{b}$. [4]

(iii) Show that the area of triangle CDE can be written as $k\left|\mathbf{a}\times\mathbf{b}\right|$, where k is a constant to be found. [3]

(iv) It is given that the point F is on BO produced, and OE bisects the angle AOF. Find the ratio $OA : OB$. [3]

Section B: Probability and Statistics [60 marks]

6 For events X and Y, it is given that $\mathrm{P}(X \cap Y') = \dfrac{1}{2}$, $\mathrm{P}(X \cup Y') = \dfrac{3}{4}$ and $\mathrm{P}(X|Y') = \dfrac{50}{63}$. Find

(i) $\mathrm{P}(Y')$, [2]

(ii) $\mathrm{P}(X)$, [2]

(iii) $\mathrm{P}(X \cap Y)$ and state with a reason whether X and Y are independent events. [3]

7 In a factory, machines pack sugar into bags of 1 kg each on average, with variance σ^2 kg^2. The manufacturer is concerned that the machines are putting too much sugar into the bags and decides to carry out a hypothesis test. A random sample of 8 bags are selected and their total mass is 8.4 kg.

(i) Stating a necessary assumption, carry out a test of the manufacturer's concern at the 5% significance level if $\sigma = 0.08$. [5]

(ii) Use an algebraic method to calculate the range of values of σ^2 for which the null hypothesis would not be rejected at the 5% significance level. [3]

8 **In this question you should state clearly the values of the parameters of any normal distribution you use.**

In a supermarket, the masses in grams of apples have the distribution $\mathrm{N}(90, 13^2)$ and the masses in grams of potatoes have the distribution $\mathrm{N}(170, 30^2)$.

(i) Find the probability that the mass of a randomly chosen potato is more than twice the mass of a randomly chosen apple. [3]

A certain salad recipe requires 5 apples and 6 potatoes.

(ii) Find the probability that the total mass of 5 randomly chosen apples and 6 randomly chosen potatoes is between 1.2 and 1.5 kilograms. [3]

The salad recipe requires the apples and potatoes to be prepared by peeling and slicing them. The process reduces the mass of each apple by 15% and the mass of each potato by 25%.

(iii) Find the probability that the total mass, after preparation, of 5 randomly chosen apples and 6 randomly chosen potatoes is not more than 1.2 kilograms. [3]

9 The continuous random variable X has the distribution $N(\mu, \sigma^2)$.
It is known that $P(X < k) = 0.2$ and $P(X < 7) = 0.8$.

(i) Show that $P(k < X < 7) = 0.6$ and write down the value of $P(\mu < X < 7)$. [2]

(ii) Express μ in terms of k. [1]

You may use $\sigma^2 = 12$ for the rest of the question.

(iii) Show that $\mu = 4.0845$ correct to 4 decimal places. [3]

(iv) Find $P(|X| < k)$. [2]

It is given that $2P(X \le r) = 3P(X > r)$ for a certain constant, r.

(v) Ten independent observations of X are randomly selected. Find the probability that there are more observations of X with values greater than r than observations of X with values less than r in the selection. [4]

10 A group of 13 people consists of 6 single men and 5 single women and a married couple. A committee of 7 is to be selected from the group.

(i) Find the number of committees that can be formed if there is no restriction in the selection. [1]

(ii) Show that there are 658 such committees with more women than men. [2]

Given that a committee of 7 people is to be selected from the group such that the committee contains more women than men,

(iii) find the probability of getting a committee that consists of 4 single women and 3 single men. [2]

(iv) find the probability of getting a committee that contains at least one married member. [2]

The 4 women and 3 men who were finally selected for the committee included the married couple. The 7 members sit at random around a table with 7 chairs.

(v) Find the probability that the men are all separated from each other. [2]

(vi) Given that the men are all separated from each other, find the probability that the married couple sit next to each other. [3]

11 In conjunction with the Great Singapore Sale, a certain electronics store is having a lucky draw for their customers. In each round of the lucky draw, the customer draws two balls randomly, one after another, with replacement from a box containing 20 red balls, 30 blue balls and 50 white balls. The colour of each ball drawn is noted and points are awarded accordingly as follows.

Colour of ball	**Point(s)**
Red	5
Blue	4
White	1

The customer's score in each round of the lucky draw is the total number of points awarded for the balls drawn. To illustrate, a customer scores a total of 5 points if he draws a blue ball and a white ball in a round of the lucky draw, regardless of the order of appearance of the balls.
You may assume that the 100 balls are indistinguishable from each other apart from their colour.

Let X denote the total number of points scored by a customer in a round of the lucky draw.

(i) Tabulate the probability distribution of X. [3]

(ii) Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = 6.02$. [2]

Mr Lim participated in 50 rounds of the lucky draw.

(iii) Using a suitable approximation, find the probability that Mr Lim's average score is at least 6. [3]

A customer wins a cash voucher if his total score for one round of the lucky draw is more than 6.

The total number of cash vouchers won by Mr Tan in n rounds of the lucky draw is denoted by Y.

(iv) Find the least value of n such that there is a probability of more than 0.7 that Mr Tan will win more than 3 cash vouchers in total. [4]

2020 RI H2 Mathematics Prelim Paper 1 Solutions

1 A particle is moving along the curve with equation $16x^2+9y^2=144$.

$\frac{dy}{dt}=2\text{ cm s}^{-1}$ and $\frac{dx}{dt}$ is positive when $x=\sqrt{5}$.

Find the rate of increase of x at this instant. [4]

[4]

$16x^2+9y^2=144$

When $x=\sqrt{5}$: $y=\pm\sqrt{\frac{144-16(5)}{9}}=\pm\frac{8}{3}$

Differentiate equation with respect to x:

$$32x+18y\left(\frac{dy}{dx}\right)=0 \Rightarrow \frac{dy}{dx}=\frac{-32x}{18y}=-\frac{16x}{9y}$$

$$\frac{dx}{dt}=\frac{dx}{dy}\times\frac{dy}{dt}=-\frac{9y}{16x}\times 2=-\frac{9y}{8x}$$

For $\frac{dx}{dt}>0$, x and y have different signs, and so the particle increases with respect to x at $(\sqrt{5},-\frac{8}{3})$.

[Alternative 1 for position of particle : Since $\frac{dy}{dt}, x>0$, particle moves in anti-clockwise direction. Hence for $\frac{dx}{dt}>0$, y should be negative.]

[Alternative 2 for position of particle : differentiate w.r.t t and get $32x\frac{dx}{dt}+18y\frac{dy}{dt}=0$. Since $\frac{dx}{dt},\frac{dy}{dt}, x>0$, y should be negative.]

At $(\sqrt{5},-\frac{8}{3})$, $\frac{dx}{dt}=-\frac{9}{8}\times\frac{-8}{3\sqrt{5}}=\frac{3}{\sqrt{5}}$ cms^{-1}

At $(\sqrt{5},-\frac{8}{3})$, its rate of increase is $\frac{3}{\sqrt{5}}$ cms^{-1}

[Alternative for $\frac{dx}{dt}$,

$32x\frac{dx}{dt}+18y\frac{dy}{dt}=0 \Rightarrow 32\sqrt{5}\frac{dx}{dt}+18\left(\frac{-8}{3}\right)(2)=0 \Rightarrow \frac{dx}{dt}=\frac{3}{\sqrt{5}}$]

Comment: As the value of y is **negative** in this case, rejection of any answer would **need to be justified** using the conditions given in the question.

2 Mr. Li invested a total of \$30000 and divided this sum into three accounts, which paid 2%, 3% and 5% annual interest respectively.

At the end of the first year, Mr. Li withdrew all the money out from the 2% and 5% accounts and gave the interest earned to his son. The amount in the 3% account, including interest, was re-invested in the same account for another year.

At the end of the second year, Mr. Li withdrew all the money out of the 3% account and gave the interest earned to his son. The total interest received by the son from the three accounts was \$1423.50.

Given that the amount invested in the 2% account was \$1000 more than the amount invested in the 5% account, find the amounts invested in each of the three accounts. [4]

[4]

Let x, y and z be the amounts he invested into the 2%, 3% and 5% accounts respectively.

$x+y+z=30000$ -------- (1)

$0.02x+(1.03^2-1)y+0.05z=1423.50$ -------- (2)

$x-z=1000$ -------- (3)

From GC, solving the 3 equations, $x=8000$, $y=15000$, $z=7000$

Comment: Student should try to identify this type of question as one that is easily solved with the GC using a system of three equations.

A common mistake is that students computed the interest for the 3% account only for the second year ($0.03\times1.03y=0.0309y$) and leaving out the interest for the first year ($0.03y$). Computing the interest earned over two years as $(1.03^2-1)y=0.0609y$ would be a safer way.

3	(i)	Prove that for $x>0$, the substitution $y=ux$ reduces the differential equation $(y-x)\left(\frac{dy}{dx}-\frac{y}{x}\right)=y^2+2x^2$ to $\left(\frac{u}{u^2+2}-\frac{1}{u^2+2}\right)\left(\frac{du}{dx}\right)=1$. [2]
	(ii)	Hence find the general solution of the differential equation $(y-x)\left(\frac{dy}{dx}-\frac{y}{x}\right)=y^2+2x^2$ for $x>0$. [3]

(i) [2]

$y=ux \Rightarrow \frac{dy}{dx}=x\frac{du}{dx}+u$

$(y-x)\left(\frac{dy}{dx}-\frac{y}{x}\right)=y^2+2x^2$

$\Rightarrow (ux-x)\left(\left(x\frac{du}{dx}+u\right)-u\right)=u^2x^2+2x^2$

$\Rightarrow (ux-x)\left(x\frac{du}{dx}\right)=u^2x^2+2x^2$

$\Rightarrow (u-1)\left(\frac{du}{dx}\right)=u^2+2 \qquad \because x>0$

$\Rightarrow \left(\frac{u-1}{u^2+2}\right)\left(\frac{du}{dx}\right)=1$

$\Rightarrow \left(\frac{u}{u^2+2}-\frac{1}{u^2+2}\right)\left(\frac{du}{dx}\right)=1$

Asked to reduce the DE to one with $\frac{du}{dx}$ so clearly u is a function of x, and cannot be a constant in $y=ux$

(ii) [3]

$\left(\frac{u}{u^2+2}-\frac{1}{u^2+2}\right)\left(\frac{du}{dx}\right)=1$

$\Rightarrow \int \frac{u}{u^2+2}-\frac{1}{u^2+2}\,du=\int 1\,dx$

$\Rightarrow x=\frac{1}{2}\ln(u^2+2)-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$

$x=\frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2+2\right)-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}x}\right)+C$

Remember to replace u by $u=\frac{y}{x}$ for the final expression.

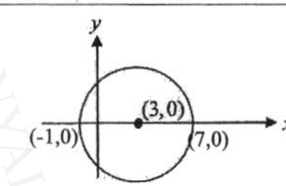
4	(i)	Sketch the curve with equation $x^2+y^2-6x=7$. [2]
	(ii)	The region R is bounded by the curve $x^2+y^2-6x=7$, for $x\ge 3$, and the line $x=3$. Given that $\int_0^a \sqrt{a^2-x^2}\,dx=\frac{\pi a^2}{4}$, find the exact volume of the solid of revolution formed when R is rotated completely about the y-axis. [4]

(i) [2]

$x^2+y^2-6x=7$

$(x-3)^2+y^2-9=7$

$(x-3)^2+y^2=4^2$



Circle with centre (3, 0) and radius 4.

Use compass set to draw a circle. (use equal scale)

Indicate clearly "critical features" like centre and radius.

(ii) [4]

$x=3 \Rightarrow 9+y^2-6(3)=7 \Rightarrow y^2=16 \Rightarrow y=\pm 4$

$x^2+y^2-6x=7 \Rightarrow (x-3)^2+y^2=4^2 \quad \Rightarrow x=3\pm\sqrt{4^2-y^2}$

Since $x\ge 3$, $x=3+\sqrt{4^2-y^2}$.

Volume of solid generated

$=\pi\int_{-4}^{4} x^2\,dy-\pi(3)^2(2(4))$

$=\pi\int_{-4}^{4}\left(3+\sqrt{4^2-y^2}\right)^2 dy-72\pi$

$=\pi\int_{-4}^{4}\left(9+6\sqrt{4^2-y^2}+16-y^2\right) dy-72\pi$

$=\pi\left[25y-\frac{y^3}{3}\right]_{-4}^{4}+(6\pi)(2)\left[\frac{\pi}{4}(4^2)\right]-72\pi$

$=\pi\left(200-\frac{128}{3}\right)+48\pi^2-72\pi$

$=\frac{256}{3}\pi+48\pi^2$

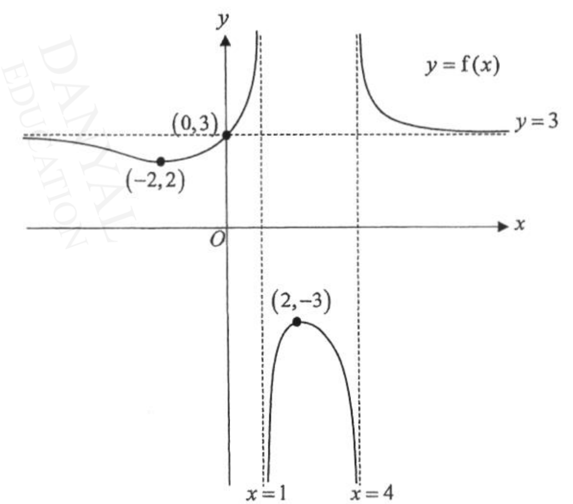
R is rotated completely about the y-axis so the integral involved should be "…dy".

Note that volume of required region is to take "outer volume" subtract "inner volume" which is a cylinder.

5	(a)	Find $\int x\cos x^2\,\mathrm{d}x$.	[1]
	(b)	Use integration by parts to find $\int x\cos 2x\,\mathrm{d}x$.	[3]
		Hence or otherwise find $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos^2 x\,\mathrm{d}x$.	[3]

(i) [1]	$\int x\cos x^2\ \mathrm{d}x = \frac{1}{2}\sin x^2 + c$	More generally, $\int f'(x)\cos(f(x))\ \mathrm{d}x$ $= \sin(f(x)) + c$ Don't forget "+ c"
(ii) [3]	$\int x\cos 2x\ \mathrm{d}x = \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x$ $u = x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$ $\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad v = \frac{1}{2}\sin 2x$ $\int x\cos 2x\ \mathrm{d}x = \frac{x}{2}\sin 2x + \frac{\cos 2x}{4} + c$	
[3]	$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos^2 x\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\left(\frac{\cos 2x + 1}{2}\right)\mathrm{d}x$ $= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos 2x\mathrm{d}x + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\mathrm{d}x$ $= \frac{1}{2}\left[\frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{1}{2}\left[\frac{x^2}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $= \frac{1}{2}\left[\left(0 - \frac{1}{4}\right) - \left(\frac{\pi}{8} + 0\right)\right] + \frac{1}{2}\left[\frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right)\right]$ $= \frac{3\pi^2}{64} - \frac{1}{8} - \frac{\pi}{16}$	Double angle formula for cosine is given in MF26

6 The diagram below shows the curve of $y = \mathrm{f}(x)$. The curve has a minimum point at $(-2, 2)$, a maximum point at $(2, -3)$ and cuts the y-axis at $(0, 3)$. The lines $x = 1$, $x = 4$ and $y = 3$ are the asymptotes to the curve.



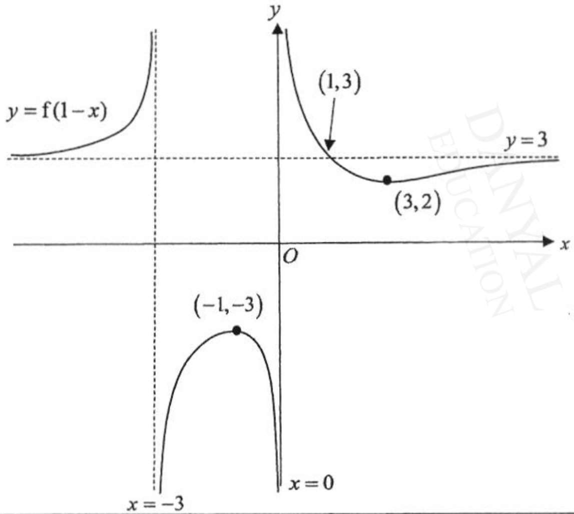
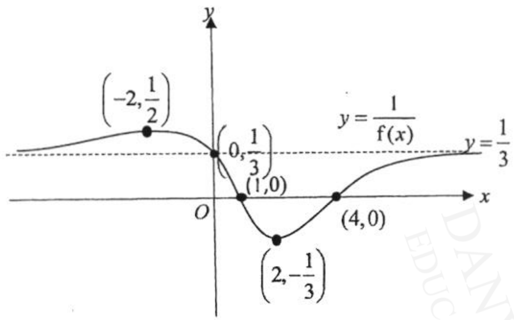
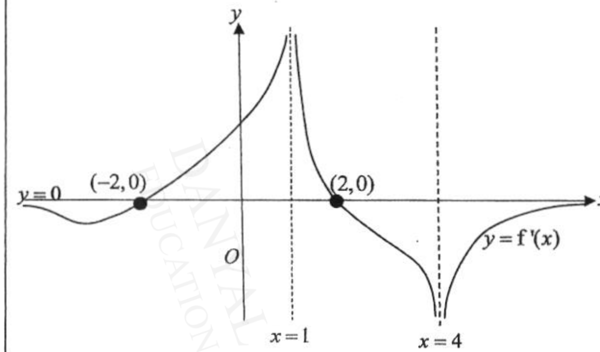
On separate diagrams, draw sketches of the following graphs, stating the exact coordinates of any turning points and/or points of intersection with the axes, and the equations of any asymptotes, where possible.

(a)	$y = \mathrm{f}(1-x)$.		[3]
(b)	$y = \frac{1}{\mathrm{f}(x)}$.		[3]
(c)	$y = \mathrm{f}'(x)$.		[3]

Label axes and origin. Use a ruler to draw any lines.

Write neatly and use a "dark pencil".

Please remove unwanted pencil markings cleanly with an eraser.

(a) **[2]**		
(b) **[3]**		The **only** asymptote is the line $y=\frac{1}{3}$. As such, there **should not** be other lines (such as $x=1$ or $x=4$) seen in the sketch.
(c) **[3]**		

7 It is given that $f(r)=\dfrac{2^r}{r-2}$. Show that $f(r+2)-f(r)=\dfrac{(3r-8)2^r}{r(r-2)}$. [2]

(i) Show that $\sum_{r=3}^{n}\dfrac{(3r-8)2^r}{r(r-2)}=\dfrac{(3n-2)2^{n+1}}{n(n-1)}-16$. [4]

(ii) Hence find $\sum_{r=1}^{n}\dfrac{(3r-2)2^r}{r(r+2)}$ in the form $\dfrac{(3n+A)2^{n+1}}{(n+B)(n+1)}+C$ where A, B and C are integers to be determined. [4]

[2]	$f(r+2)-f(r)=\dfrac{2^{r+2}}{r}-\dfrac{2^r}{r-2}$ $=\dfrac{2^{r+2}(r-2)-2^r r}{r(r-2)}$ $=\dfrac{4r.2^r-8.2^r-r.2^r}{r(r-2)}$ $=\dfrac{(3r-8)2^r}{r(r-2)}$ (shown)	When question says "show that …", clear step-by-step working is expected.
(i) [4]	$\sum_{r=3}^{n}\dfrac{(3r-8)2^r}{r(r-2)}=\sum_{r=3}^{n}\left(f(r+2)-f(r)\right)$ $=\left[\begin{array}{l} f(5)-f(3) \\ +f(6)-f(4) \\ +f(7)-f(5) \\ +f(8)-f(6) \\ \ldots \\ +f(n)-f(n-2) \\ +f(n+1)-f(n-1) \\ +f(n+2)-f(n) \end{array}\right]$ $=\left[f(n+1)+f(n+2)-f(3)-f(4)\right]$ $=\left[\dfrac{2^{n+1}}{n-1}+\dfrac{2^{n+2}}{n}-8-\dfrac{16}{2}\right]$ $=\dfrac{2n.2^n+4n.2^n-4.2^n}{n(n-1)}-16$ $=\dfrac{(2n+4n-4)2^n}{n(n-1)}-16$ $=\dfrac{(3n-2)2^{n+1}}{n(n-1)}-16$	You should not omit the pair of brackets enclosing $f(r+2)-f(r)$. Final answer is given so show 2 "full" cancellations in front and 1 "full" cancellation at the back.
(ii) [4]	$\sum_{r=1}^{n}\dfrac{(3r-2)2^r}{r(r+2)}=\sum_{r=3}^{n+2}\dfrac{(3(r-2)-2)2^{r-2}}{(r-2)r}$ $=\sum_{r=3}^{n+2}\dfrac{(3r-8)2^{r-2}}{r(r-2)}$ $=\dfrac{1}{4}\sum_{r=3}^{n+2}\dfrac{(3r-8)2^r}{r(r-2)}$ $=\dfrac{1}{4}\left[\dfrac{(3(n+2)-2)2^{(n+2)+1}}{(n+2)(n+1)}-16\right]$ $=\dfrac{(3n+4)2^{n+1}}{(n+2)(n+1)}-4$ $\therefore A=4, B=2, C=-4$	Use the denominator $\dfrac{**}{r(r+2)}$ to suggest a replacement of r with "r - 2" which appears in result in (i) Adjust lower and upper limit accordingly.

8 **Do not use a calculator in answering this question.**

(a) (i) Solve the equation $z^2 = 4\text{i} - 3$. [3]

(ii) Solve the equation $z^4 + 6z^2 + 25 = 0$. [3]

(b) Find the modulus and argument of the complex number $w = \frac{8-2\text{i}}{5+3\text{i}}$. Hence find the possible values of the positive integer n for which w^n is real. [4]

(a)(i) [3]

Let $z = x + \text{i}y$, $x, y \in \mathbb{R}$. Then

$z^2 = 4\text{i} - 3 \quad \Rightarrow (x+\text{i}y)^2 = (x^2 - y^2) + 2\text{i}xy = 4\text{i} - 3$

$\Rightarrow \begin{cases} x^2 - y^2 = -3 \\ 2xy = 4 \end{cases}$

$\Rightarrow x^2 - \frac{4}{x^2} = -3$

$\Rightarrow x^4 + 3x^2 - 4 = (x^2+4)(x^2-1) = 0$

$\Rightarrow x = \pm 1$

When $x = 1, y = 2$. When $x = -1, y = -2$

Thus the roots are $1 + 2\text{i}$ and $-1 - 2\text{i}$.

Comment: A calculator is "not allowed" for this question so show **detailed working** (you can check answers using calculator though).

(a)(ii) [3]

$z^4 + 6z^2 + 25 = 0 \quad ---(1)$

$(z^2+3)^2 + 16 = 0$

$z^2 + 3 = \pm 4\text{i}$

$z^2 = 4\text{i} - 3$ or $-4\text{i} - 3$

or

$z^4 + 6z^2 + 25 = 0 \quad ---(1)$

$z^2 = \frac{-6 \pm \sqrt{36 - 4(25)}}{2}$

$= -3 \pm \frac{8\text{i}}{2} = -3 \pm 4\text{i}$

For $z^2 = 4\text{i} - 3$, $z = 1 + 2\text{i}, -1 - 2\text{i}$

Since (1) is an equation with real coefficients, the roots occur in conjugate pairs. Thus the roots of the equation $z^4 + 6z^2 + 25 = 0$ are $z = 1 \pm 2\text{i}, -1 \pm 2\text{i}$.

(b) [4]

$w = \frac{8-2\text{i}}{5+3\text{i}} = \frac{8-2\text{i}}{5+3\text{i}} \times \frac{5-3\text{i}}{5-3\text{i}}$

$= \frac{40 - 10\text{i} - 24\text{i} + 6\text{i}^2}{5^2 + 3^2}$

$= \frac{34 - 34\text{i}}{34} = 1 - \text{i}$

Thus $|w| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\arg w = -\frac{\pi}{4}$

For w^n to be real, $w^n = \left(\sqrt{2}\right)^n \left[\cos\left(-\frac{n\pi}{4}\right) + \text{i}\sin\left(-\frac{n\pi}{4}\right)\right]$ is real.

Hence $\sin\left(-\frac{n\pi}{4}\right) = 0 \Rightarrow \frac{n\pi}{4} = k\pi, k \in$, and so

$n = 4k,\ k \in \mathbb{Z}^+$ (since $n > 0$)

Comment: The set of "positive integer" does not include 0.

Also, a general statement that n is a multiple of 4 is incorrect as this includes negative numbers.

9 A curve C has parametric equations

$$x = t\ln t, \qquad y = \frac{4}{e^t} + e^t,$$

for $t \geq \frac{1}{2}$.

(i) C meets the y-axis at point P and line L is the normal to C at P. Show that the equation of L is

$$y = \frac{e}{4-e^2}x + \frac{4+e^2}{e}.$$ [4]

(ii) Sketch the curve C, stating the coordinates of any turning points and points intersection with the axes. [3]

(iii) The finite region bounded by C, L and the line $x = \frac{1}{2}\ln\left(\frac{1}{2}\right)$ is denoted by R. Find the area of R. [3]

(i) [4]

$x = t\ln t \Rightarrow \frac{dx}{dt} = t\left(\frac{1}{t}\right) + \ln t = 1 + \ln t,$

$y = \frac{4}{e^t} + e^t \Rightarrow \frac{dy}{dt} = -4e^{-t} + e^t = \frac{e^{2t} - 4}{e^t},$

$\therefore \frac{dy}{dx} = \frac{e^{2t} - 4}{e^t(1 + \ln t)}$

Now, $x = 0 \Rightarrow t\ln t = 0 \Rightarrow t = 1\ (\because t > 0)$

$\Rightarrow y = \frac{4}{e} + e = \frac{4+e^2}{e}$ and $\frac{dy}{dx} = \frac{e^2 - 4}{e}$

Equation of normal at $P\ (0, \frac{4+e^2}{e})$:

$y - \frac{4+e^2}{e} = -\frac{e}{e^2-4}x \Rightarrow y = \frac{e}{4-e^2}x + \frac{4+e^2}{e}$

Comment: As this is a **"show … "** question, detailed working should be made clear.

(ii) [3]

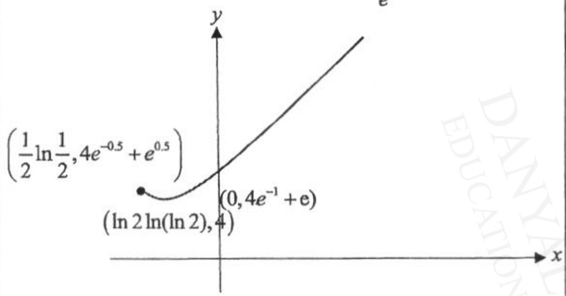
$\frac{dy}{dx} = \frac{e^{2t} - 4}{e^t(1+\ln t)} = 0$

$\Rightarrow e^{2t} - 4 = 0$

$\Rightarrow t = \ln 2$

Comment: For parametric curves, it is important to note the range of t to sketch.

Use the ZOOM function in the GC to check the exact shape/feature of the curve.

	Min occurs at $x=\ln 2(\ln(\ln 2))$, $y=\frac{4}{e^{\ln 2}}+e^{\ln 2}=4$  	Graphs should be drawn **sufficiently large** so that the key features (turning points and relative position) can be seen **clearly**. Where exact values is not stated, students may indicate the coordinates in 3 s.f. . the y values of the turning point, end-point and y-intercept are 4, 4.07 and 4.19, respectively, so the relative position should be indicated clearly in the diagram.
(iii) **[4]**	Area $=\int_{0.5\ln 0.5}^{0}\left(\frac{\mathrm{e}}{4-\mathrm{e}^2}x+\frac{4+\mathrm{e}^2}{\mathrm{e}}\right)-y\,\mathrm{d}x$ $=\int_{0.5\ln 0.5}^{0}\left(\frac{\mathrm{e}}{4-\mathrm{e}^2}x+\frac{4+\mathrm{e}^2}{\mathrm{e}}\right)\mathrm{d}x-\int_{\frac{1}{2}}^{1}\left(\frac{4}{\mathrm{e}^t}+\mathrm{e}^t\right)(1+\ln t)\,\mathrm{d}t$ $=0.0943$ (3s.f.)	For questions involving Area, draw the respective curves and identify the required region, and use the idea of 'Area under Upper curve – Lower Curve' ($\int_a^b y_1-y_2\,\mathrm{d}x$ in this case) Be familiar with how to find area of parametric curves, by using "substitution".

10	The plane π_1 contains the point $A(1,2,-1)$ and the line l with equation $\frac{x-1}{2}=\frac{2-z}{3}$, $y=-1$. The plane π_2 contains the point $B(-5.5,3,2)$ and meets π_1 in the line l.	
	(i)	Find the equation of π_1 in scalar product form. [3]
	(ii)	Show that the vector $\overrightarrow{BF}$ is $\begin{pmatrix}4.5\\-4\\3\end{pmatrix}$, where F is the foot of perpendicular from B to l. [3]
	(iii)	Find the exact value of the shortest distance from B to π_1. [2]
	(iv)	Hence, or otherwise, find the acute angle between π_1 and π_2, giving your answer to the nearest $0.1°$. [3]

(i) **[3]**	$l:\mathbf{r}=\begin{pmatrix}1\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-3\end{pmatrix}$, $\lambda\in\mathbb{R}$ Let C be a point on l such that $\overrightarrow{OC}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$. $\overrightarrow{AC}=\begin{pmatrix}1\\-1\\2\end{pmatrix}-\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}0\\-3\\3\end{pmatrix}$ $\mathbf{n}_1=\begin{pmatrix}0\\-1\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\-3\end{pmatrix}=\begin{pmatrix}3\\2\\2\end{pmatrix}$ $\pi_1:\mathbf{r}.\begin{pmatrix}3\\2\\2\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}.\begin{pmatrix}3\\2\\2\end{pmatrix}=3-2+4=5 \quad\Rightarrow\mathbf{r}.\begin{pmatrix}3\\2\\2\end{pmatrix}=5$	**Use proper vector notation:** Put the tilde beneath the **r**. Write "… $\lambda\in\mathbb{R}$" in the equation of a line.
(ii) **[3]**	$OF=\begin{pmatrix}1\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-3\end{pmatrix}$, for some $\lambda\in$ $\overrightarrow{BF}=\overrightarrow{OF}-\overrightarrow{OB}=\begin{pmatrix}6.5+2\lambda\\-4\\-3\lambda\end{pmatrix}$	

$\overrightarrow{BF}.\begin{pmatrix}2\\0\\-3\end{pmatrix}=0 \Rightarrow 13+4\lambda+9\lambda=0$

$\lambda=-1$

$\therefore \overrightarrow{BF}=\begin{pmatrix}6.5+2(-1)\\-4\\-3(-1)\end{pmatrix}=\begin{pmatrix}4.5\\-4\\3\end{pmatrix}$ (shown)

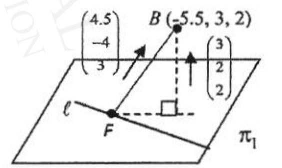
As this is a "show …" question, show the substitution of $\lambda=-1$

(iii) [2]

Shortest distance from B to π_1

= length of projection of $\overrightarrow{BF}$ onto $\mathbf{n}_1$

$=\left|\overrightarrow{BF}\cdot\hat{\mathbf{n}}_1\right|=\dfrac{\begin{pmatrix}4.5\\-4\\3\end{pmatrix}\cdot\begin{pmatrix}3\\2\\2\end{pmatrix}}{\sqrt{17}}=\dfrac{23}{2\sqrt{17}}$ units



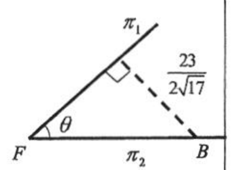
(iv) [3]

Let θ be the acute angle between π_1 and π_2

$\sin\theta=\dfrac{\frac{23}{2\sqrt{17}}}{BF}$

$=\dfrac{23}{2\sqrt{17}}\div\left|\overrightarrow{BF}\right|$

$\theta=\sin^{-1}\dfrac{23}{2\sqrt{17}\sqrt{4.5^2+4^2+3^2}}=24.5°$ (1d.p)



[Alternatively, $\mathbf{n}_2=BF\times\begin{pmatrix}2\\0\\-3\end{pmatrix}=\begin{pmatrix}4.5\\-4\\3\end{pmatrix}\times\begin{pmatrix}2\\0\\-3\end{pmatrix}=\begin{pmatrix}12\\19.5\\8\end{pmatrix}$

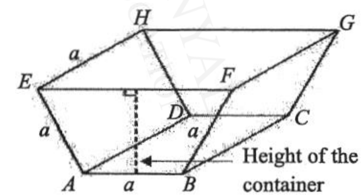
Let θ be the acute angle between π_1 and $\frac{23}{2\sqrt{17}}$

$\theta=\cos^{-1}\dfrac{\begin{pmatrix}3\\2\\2\end{pmatrix}\cdot\begin{pmatrix}12\\19.5\\8\end{pmatrix}}{\sqrt{17}\sqrt{588.25}}=\cos^{-1}\dfrac{91}{\sqrt{17}\sqrt{588.25}}=24.5°$ (1d.p)]

[Another alternative method was to find the acute angle between $\overrightarrow{BF}$ and $\mathbf{n}_1$ using the dot product formula, then subtract it from 90°.]

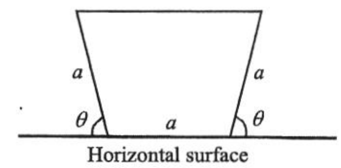
11 Figure 1 shows an open container in the form of a trapezoidal prism $ABCDEFGH$ with square base $ABCD$ and $AB=AE=BF=EH=a$ cm, where a is a constant. The container is made of plastic of negligible thickness and is placed on a horizontal surface. The faces $BCGF$ and $ADHE$ are inclined at an angle θ radians, $0<\theta<\frac{\pi}{2}$, to the horizontal, and faces $ABFE$ and $DCGH$ are perpendicular to base $ABCD$. Figure 2 shows its cross-sectional view.

Figure 1



Trapezoidal Prism

Figure 2



Cross-Sectional View

(i) Show that the volume V cm³ of the container is given by $V=a^3\sin\theta(1+\cos\theta)$. [2]

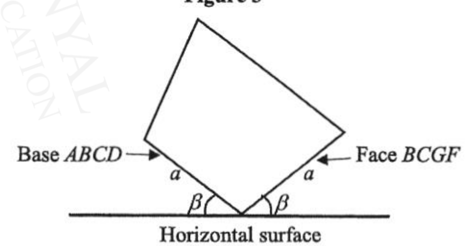
(ii) Use differentiation to find, in terms of a, the maximum value of V in exact form, proving that it is a maximum. [5]

(iii) A container is constructed with $\theta=\frac{\pi}{3}$ and it is filled with water to half its height.

Find, in terms of a, the exact volume of water in the container. [3]

The container is then tilted in the direction of the face $BCGF$ until face $BCGF$ and base $ABCD$ makes the same angle β radians, where β is a constant, with the horizontal. Figure 3 shows its cross-sectional view.

Figure 3

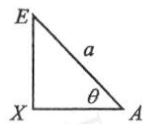


Cross-Sectional View

Explain if it is possible to tilt the container to this position without any water flowing out from the container. [2]

(i) [2]

By trigo ratio, $EX = a\sin\theta$, $AX = a\cos\theta$



$V = a\left\{2\left[\frac{1}{2}(a\cos\theta)(a\sin\theta)\right] + a(a\sin\theta)\right\}$

$= a^3 \sin\theta(1+\cos\theta)$

[Alternatively, by area of trapezium

$V = a\left\{\frac{1}{2}(a\sin\theta)\left[a+(a+2a\cos\theta)\right]\right\} \quad = a^3\sin\theta(1+\cos\theta)$]

Comment: A "show" question so there should be clear explanation in working or illustration on the diagram indicating the method used.

(ii) [5]

$\frac{dV}{d\theta} = a^3\left[\cos\theta(1+\cos\theta)+\sin\theta(-\sin\theta)\right]$

$= a^3\left[\cos\theta + \cos 2\theta\right]$ (Double angle formulae)

$= \begin{cases} 2a^3\cos\frac{3\theta}{2}\cos\frac{\theta}{2} & \text{(Factor Formulae)} \\ a^3\left[2\cos^2\theta + \cos\theta - 1\right] & \text{(Double angle formulae)} \\ = a^3(2\cos\theta-1)(\cos\theta+1) & \end{cases}$

$\frac{dV}{d\theta} = 0 \Rightarrow \cos\frac{3\theta}{2} = 0$ or $\cos\frac{\theta}{2} = 0$

$\Rightarrow \frac{3\theta}{2}$ or $\frac{\theta}{2} = \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$ or π (NA)

[$\frac{dV}{d\theta} = 0 \Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\Rightarrow \theta = \frac{\pi}{3}$ or π (NA)]

$\frac{dV}{d\theta} = a^3\left[\cos\theta + \cos 2\theta\right]$

At $\theta = \frac{\pi}{3}$, $\frac{d^2V}{d\theta^2} = a^3\left[-\sin\theta - 2\sin 2\theta\right]$

$= a^3\left[-\frac{\sqrt{3}}{2} - \sqrt{3}\right] = -\frac{3\sqrt{3}}{2}a^3 < 0$

Hence $\theta = \frac{\pi}{3}$ gives maximum value of V and

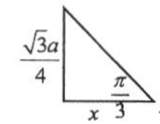
$\max V = a^3\left(\frac{\sqrt{3}}{2}\right)\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}a^3 \text{ cm}^3$

Comment: For students who use first derivative test, **clearly state** what are the signs of the 2 factors of $\frac{dV}{d\theta}$ when $\theta = \frac{\pi}{3}^-, \frac{\pi}{3}^+$ and thus the final sign, to conclude if the gradient is positive or negative.

Note that the conclusion is for the maximum value of V, and not of θ

(iii) [3+2]

Half its height $= \frac{1}{2}\left(a\sin\frac{\pi}{3}\right) = \frac{\sqrt{3}a}{4}$.

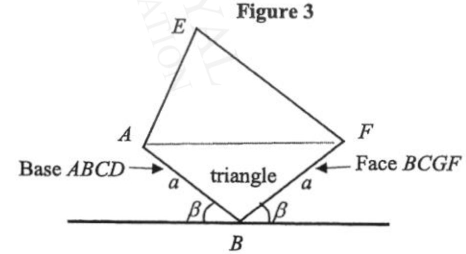


$\therefore \tan\frac{\pi}{3} = \frac{\frac{\sqrt{3}a}{4}}{x} \Rightarrow \sqrt{3} = \frac{\sqrt{3}a}{4x} \Rightarrow x = \frac{a}{4}$

V(half its height)

$= a\left[\left(\frac{\sqrt{3}a}{4}\right)a + \left(\frac{a}{4}\right)\left(\frac{\sqrt{3}a}{4}\right)\right] = \left(\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{16}\right)a^3 = \frac{5\sqrt{3}}{16}a^3 \text{ cm}^3$

Figure 3



Cross-Sectional View

Note that we can consider face $ABFE$ as a possible cross-sectional view in Figure 3. Then $\angle ABF = \frac{2\pi}{3}$ and $AB = BF = a$, and so

Area of triangle $ABF = \frac{1}{2}a^2\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4}a^2$.

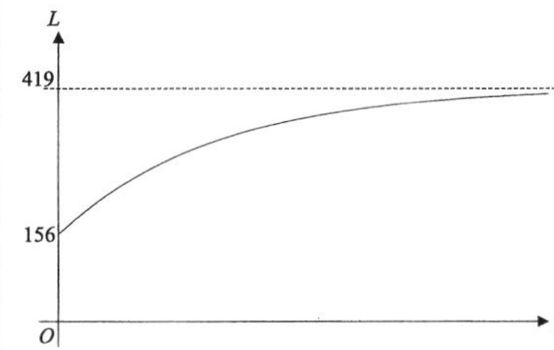
Hence the volume of water that the container can hold at this position is at most $\frac{\sqrt{3}}{4}a^3 < V(\text{half the height})$, and so water will definitely flow out of the container before it reaches this position. So no, it is not possible.

[Alternative explanation (for the case where θ may not be fixed): Note that area of triangle increases with θ and θ is acute. Hence max volume $< \frac{a^3}{2}\sin\frac{\pi}{2} = \frac{a^3}{2} < \frac{5\sqrt{3}}{16}a^3$.]

Comment: Students should explain their working clearly, such as using similar triangles etc.

Please remove all unwanted or replaced working and not leave multiple methods.

12	The von Bertalanffy growth model, introduced by von Bertalanffy in 1938, is a widely used growth curve and is especially important in fisheries studies. It is used to predict the length, L mm of a fish over a period of time, t years. If L_∞ is the maximum length for a species, then the model assumes that the rate of growth in length of a fish is proportional to $L_\infty - L$.	
	(i)	By setting up and solving a differential equation, show that the general solution of this differential equation is given by $L = L_\infty - Ae^{-kt}$, where k is the constant of proportionality and A is a positive constant. [5]
	For the species of fish known as the Atlantic croaker, it has been determined that $L_\infty = 419$ mm and at one year of age, its length is 219 mm and the rate of growth in length is 55 mm per year. Using the above model, obtain an expression for L in terms of t. [3]	
	(ii)	Find its age when the Atlantic croaker grows to a length of 300 mm. [2]
	(iii)	Sketch a graph of L against t. [2]

(i) **[5]**	$\frac{dL}{dt} = k(L_\infty - L)$, where k is the constant of proportionality. $\frac{dL}{dt} = k(L_\infty - L)$ $\frac{1}{L_\infty - L}\frac{dL}{dt} = k$ $\int \frac{1}{L_\infty - L}\, dL = \int k\, dt$ $-\ln(L_\infty - L) = kt + c \qquad \because L_\infty - L > 0$ $L_\infty - L = e^{-(kt+c)}$ $L_\infty - L = Ae^{-kt}$, A is a positive constant $L = L_\infty - Ae^{-kt}$, A is a positive constant Note that $L \neq L_\infty$ in this context. Also, $A > 0$.	A "show …" question so all steps must be clearly explained including why "A" must be positive
[3]	Since $L_\infty = 419$ mm, $L = 419 - Ae^{-kt}$ When $t = 1$, $L = 219$ and thus $219 = 419 - Ae^{-k} \Rightarrow Ae^{-k} = 200$ --- (1) Also, $t = 1$, $\frac{dL}{dt} = 55 \Rightarrow Ake^{-k} = 55$ --- (2) Sub (1) into (2), $k = \frac{55}{200} = \frac{11}{40}$ (or 0.275) (alternatively, using $\frac{dL}{dt} = k(L_\infty - L)$, $55 = k(419 - 219)$) Thus $A = 200(e^{\frac{11}{40}}) = 263.31 = 263$ $L = 419 - 263e^{-\frac{11}{40}t}$	
(ii) **[2]**	When $L = 300$, $300 = 419 - 263.31e^{-\frac{11}{40}t} \Rightarrow t = 2.89$ years	
(iii) **[2]**		**Draw horizontal asymptote.**

2020 RI H2 Mathematics Prelim Paper 2 Solutions

Section A: Pure Mathematics [40 marks]

1	An arithmetic series has first term $\frac{3}{2}$ and fourth term u_4. A geometric series also has first term $\frac{3}{2}$ and fourth term u_4. Given that the common ratio of the geometric series is non-negative and the sum of its first 3 terms is $\frac{21}{2}$, find the sum of the first 10 odd-numbered terms of the arithmetic series. [5]	
1 **[5]**	Given that $u_1 = \frac{3}{2}$, let the common difference of the arithmetic series be d and the common ratio of the geometric series be r. and $\frac{3}{2}+\frac{3}{2}r+\frac{3}{2}r^2=\frac{21}{2}$ --------(1) $r^2+r-6=0$ $(r+3)(r-2)=0$ $\therefore r=2$, since r is non negative. $\frac{3}{2}+3d=\frac{3}{2}r^3$ ----------------(2) Substitute $r=2$ into (2), $d=\frac{7}{2}$. Sum of first 10 odd numbered terms of AP $S=\frac{10}{2}\left[2\left(\frac{3}{2}\right)+9(7)\right]=330$	First ten odd-numbered terms refer to the 1st, 3rd, 5th, … , 19th terms not "odd numbers". The 1st term is $\frac{3}{2}$ and common difference $2d=7$

2	In a geometric series, the ratio of the sum of the first 8 terms to the sum of the first 4 terms is 17:16. Find the 2 possible values of the common ratio, r_1, r_2 where $r_1 > r_2$. The sum to infinity of the geometric series with first term a and common ratio r_1 is denoted by S_1 and the sum to infinity of the geometric series with first term b and common ratio r_2 is denoted by S_2. Find $S_1 : S_2$ in terms of a and b. [5]	
2 **[5]**	$S_4=\frac{a(r^4-1)}{r-1}$ --------(1) $S_8=\frac{a(r^8-1)}{r-1}$ --------(2) $\frac{(2)}{(1)} \Rightarrow \frac{r^8-1}{r^4-1}=\frac{17}{16}$ $\frac{(r^4-1)(r^4+1)}{r^4-1}=\frac{17}{16}$ $\therefore r^4=\frac{1}{16} \Rightarrow r_1=\frac{1}{2}, r_2=-\frac{1}{2}$ When $r_1=\frac{1}{2}$, $S_1=\frac{a}{1-\frac{1}{2}}=2a$ When $r_2=-\frac{1}{2}$, $S_2=\frac{b}{1+\frac{1}{2}}=\frac{2}{3}b$ Ratio is $a:\frac{1}{3}b$ or $3a:b$	Note sum to n terms for a GP formula.

3 The functions f and g are defined by

$$f(x)=1+3e^{-x}, \qquad x\in\mathbb{R}, x>0,$$
$$g(x)=|x-1|(x-3), \quad x\in\mathbb{R}, x<c \text{ where } c \text{ is a real constant.}$$

(i) Given that $c=4$, determine if the composite functions fg and gf exist, justifying your answers. Find the range of the composite function that exists. [4]

(ii) Given that g^{-1} exists, state the largest possible value of c. Using this value of c, find $g^{-1}(x)$. [4]

	Solution	Comments
3(i) [4]	$R_g=(-\infty, 3)$ $D_f=(0,\infty)$ Since $R_g \not\subset D_f$, fg does not exist. $R_f=(1,4)$ $D_g=(-\infty,4)$ Since $R_f\subseteq D_g$, gf exists. $R_{gf}=[g(2), g(4))$ $=[-1,3)$	State the domain and range for each function clearly. Use proper notation. Note that g attains a minimum at $x=2$.
(ii) [4]	Largest possible value of c is 1. For $x<1$, $\|x-1\|=1-x$ $g(x)=(1-x)(x-3)$ $(1-x)(x-3)=y$ $1-(x-2)^2=y$ $x=2-\sqrt{1-y} \quad x<1$ $g^{-1}(x)=2-\sqrt{1-x}, x\in \ , x<0.$	

4 Given that $y=e^{\tan^{-1}\left(\frac{x}{2}\right)}$, show that $(4+x^2)\frac{dy}{dx}=2y$. [2]

(i) By repeated differentiation of the above result, find the Maclaurin series for $e^{\tan^{-1}\left(\frac{x}{2}\right)}$ up to and including the term in x^3. [5]

(ii) Hence find the Maclaurin series for $\frac{e^{\tan^{-1}\left(\frac{x}{2}\right)}}{(1+x)^2}$ up to and including the term in x^2. [3]

[2]

$$y=e^{\tan^{-1}\left(\frac{x}{2}\right)}$$
$$\frac{dy}{dx}=e^{\tan^{-1}\left(\frac{x}{2}\right)}\frac{1}{1+\left(\frac{x}{2}\right)^2}\times\frac{1}{2}$$
$$=\frac{2e^{\tan^{-1}\left(\frac{x}{2}\right)}}{4+x^2}$$
$$(4+x^2)\frac{dy}{dx}=2e^{\tan^{-1}\left(\frac{x}{2}\right)}$$
$$(4+x^2)\frac{dy}{dx}=2y \quad \text{(Shown)}$$

(i) [5]

$(4+x^2)\frac{dy}{dx}=2y$ ---------- (1)

$(4+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=2\frac{dy}{dx}$ ---------- (2)

$(4+x^2)\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+2x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=2\frac{d^2y}{dx^2}$

$(4+x^2)\frac{d^3y}{dx^3}+4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=2\frac{d^2y}{dx^2}$ ---------- (3)

When $x=0, y=1$.

From equations (1), (2), (3) we get

$\frac{dy}{dx}=\frac{1}{2}, \frac{d^2y}{dx^2}=\frac{1}{4}, \frac{d^3y}{dx^3}=-\frac{1}{8}$.

By Maclaurin's Theorem,

	$e^{\tan^{-1}\left(\frac{x}{2}\right)} = 1+\frac{1}{2}x+\frac{1}{4}\left(\frac{x^2}{2!}\right)-\frac{1}{8}\left(\frac{x^3}{3!}\right)$ $= 1+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{48}x^3+\ldots$	
(ii) **[3]**	$e^{\tan^{-1}\left(\frac{x}{2}\right)} = 1+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{48}x^3+\ldots$ $\frac{e^{\tan^{-1}\left(\frac{x}{2}\right)}}{(1+x)^2} = \left(1+\frac{1}{2}x+\frac{1}{8}x^2-\ldots\right)(1+x)^{-2}$ $= \left(1+\frac{1}{2}x+\frac{1}{8}x^2-\ldots\right)\left(1-2x+3x^2-\ldots\right)$ $= 1+\left(\frac{1}{2}-2\right)x+\left(\frac{1}{8}-1+3\right)x^2+\ldots$ $\frac{e^{\tan^{-1}\left(\frac{x}{2}\right)}}{(1+x)^2} = 1-\frac{3}{2}x+\frac{17}{8}x^2+\ldots$	

5	Referred to the origin O, points A and B have position vectors **a** and **b** respectively, where **a** and **b** are non-zero and non-parallel vectors. Point C lies on OA, between O and A, such that $OC : CA = 2:1$. Point D lies on OB produced such that $OD : BD = 3:2$.	
	(i)	Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of **a** and **b**. [2]
	(ii)	Show that the point E where the lines BC and AD meet has position vector $\frac{4}{3}\mathbf{a}-\mathbf{b}$. [4]
	(iii)	Show that the area of triangle CDE can be written as $k\lvert\mathbf{a}\times\mathbf{b}\rvert$, where k is a constant to be found. [3]
	(iv)	It is given that the point F is on BO produced, and OE bisects the angle AOF. Find the ratio $OA : OB$. [3]

(i) **[2]**	$\overline{OC}=\frac{2}{3}\mathbf{a}$, $\overline{OD}=3\mathbf{b}$	
(ii) **[4]**	BC: $\mathbf{r}=\mathbf{b}+s\left(\frac{2}{3}\mathbf{a}-\mathbf{b}\right)$ AD: $\mathbf{r}=\mathbf{a}+t(3\mathbf{b}-\mathbf{a})$ At E, $\mathbf{b}+s\left(\frac{2}{3}\mathbf{a}-\mathbf{b}\right)=\mathbf{a}+t(3\mathbf{b}-\mathbf{a})$ Since **a** and **b** are non-zero and non-parallel, $\frac{2}{3}s=1-t$ (1) $1-s=3t$ (2) Put (1) into (2): $1-s=3\left(1-\frac{2}{3}s\right)=3-2s$ $s=2$ $OE=\mathbf{b}+2\left(\frac{2}{3}\mathbf{a}-\mathbf{b}\right)=\frac{4}{3}\mathbf{a}-\mathbf{b}$	Use proper notation. Do not write $\ell_{BC}=\mathbf{b}+\lambda\ldots$
(iii) **[3]**	Area of CDE is	$\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$

	$\frac{1}{2}\left|\overrightarrow{DC}\times\overrightarrow{CE}\right|=\frac{1}{2}\left|\left(\frac{2}{3}\mathbf{a}-3\mathbf{b}\right)\times\left(\frac{2}{3}\mathbf{a}-\mathbf{b}\right)\right|$ $=\frac{1}{2}\left|-\frac{2}{3}\mathbf{a}\times\mathbf{b}+\frac{2}{3}\mathbf{a}\times3\mathbf{b}\right|$ since $\mathbf{a}\times\mathbf{a}=\mathbf{b}\times\mathbf{b}=\mathbf{0}$ $=\frac{1}{2}\left|\frac{4}{3}\mathbf{a}\times\mathbf{b}\right|=\frac{2}{3}\left|\mathbf{a}\times\mathbf{b}\right|$	
(iv) **[3]**	$\overrightarrow{OE}=\frac{4}{3}\mathbf{a}-\mathbf{b}$ is the diagonal of the rhombus with sides $\frac{4}{3}\mathbf{a}$ and $-\mathbf{b}$. So $\left|\frac{4}{3}\mathbf{a}\right|=\left|-\mathbf{b}\right| \Rightarrow \frac{\left|\mathbf{a}\right|}{\left|\mathbf{b}\right|}=\frac{3}{4}$, i.e. $OA:OB=3:4$ Method 2: $\cos\angle AOE=\cos\angle FOE$ $\frac{\mathbf{a}\cdot\left(\frac{4}{3}\mathbf{a}-\mathbf{b}\right)}{\left|\mathbf{a}\right|\times OE}=\frac{-\mathbf{b}\cdot\left(\frac{4}{3}\mathbf{a}-\mathbf{b}\right)}{\left|\mathbf{b}\right|\times OE}$ $\frac{\frac{4}{3}\left|\mathbf{a}\right|^2-\left|\mathbf{a}\right|\left|\mathbf{b}\right|\cos\theta}{\left|\mathbf{a}\right|}=\frac{-\frac{4}{3}\left|\mathbf{a}\right|\left|\mathbf{b}\right|\cos\theta+\left|\mathbf{b}\right|^2}{\left|\mathbf{b}\right|}$ where $\theta=\angle AOB$ $\frac{4}{3}\left|\mathbf{a}\right|-\left|\mathbf{b}\right|\cos\theta=-\frac{4}{3}\left|\mathbf{a}\right|\cos\theta+\left|\mathbf{b}\right|$ $\frac{4}{3}\left|\mathbf{a}\right|(1+\cos\theta)=\left|\mathbf{b}\right|(1+\cos\theta)$, $1+\cos\theta\neq0$ otherwise $\mathbf{a}\parallel\mathbf{b}$ $\frac{\left|\mathbf{a}\right|}{\left|\mathbf{b}\right|}=\frac{3}{4}$ i.e. $OA:OB=3:4$ Method 3: $\sin\angle AOE=\sin\angle FOE$ $\frac{\left|\mathbf{a}\times\left(\frac{4}{3}\mathbf{a}-\mathbf{b}\right)\right|}{\left|\mathbf{a}\right|\times OE}=\frac{\left|-\mathbf{b}\times\left(\frac{4}{3}\mathbf{a}-\mathbf{b}\right)\right|}{\left|\mathbf{b}\right|\times OE}$ $\frac{\left|-\mathbf{a}\times\mathbf{b}\right|}{\left|\mathbf{a}\right|}=\frac{\left|-\frac{4}{3}\mathbf{b}\times\mathbf{a}\right|}{\left|\mathbf{b}\right|}$ $\frac{\left|\mathbf{a}\right|}{\left|\mathbf{b}\right|}=\frac{\left|-\mathbf{a}\times\mathbf{b}\right|}{\frac{4}{3}\left|-\mathbf{b}\times\mathbf{a}\right|}=\frac{3}{4}$, so $OA:OB=3:4$	**Note alternative solution methods.**

Section B: Probability and Statistics [60 marks]

6	For events X and Y, it is given that $P(X\cap Y')=\frac{1}{2}$, $P(X\cup Y')=\frac{3}{4}$ and $P(X\mid Y')=\frac{50}{63}$. Find		
	(i)	$P(Y')$,	[2]
	(ii)	$P(X)$,	[2]
	(iii)	$P(X\cap Y)$ and state with a reason whether X and Y are independent events.	[3]

(i) **[2]**	$P(X\mid Y')=\frac{P(X\cap Y')}{P(Y')}$ $P(Y')=\frac{P(X\cap Y')}{P(X\mid Y')}$ $=\frac{\left(\frac{1}{2}\right)}{\left(\frac{50}{63}\right)}$ $=\frac{63}{100}$	
(ii) **[2]**	$P(X\cup Y')=P(X)+P(Y')-P(X\cap Y')$ $P(X)=P(X\cup Y')+P(X\cap Y')-P(Y')$ $=\frac{3}{4}+\frac{1}{2}-\frac{63}{100}$ $P(X)=\frac{31}{50}$	
(iii) **[3]**	$P(X)=P(X\cap Y)+P(X\cap Y')$ $P(X\cap Y)=P(X)-P(X\cap Y')$ $=\frac{31}{50}-\frac{1}{2}$ $=\frac{3}{25}$ Since $P(X\cap Y)=\frac{3}{25}\neq\frac{31}{50}\times\frac{37}{100}=P(X)\times P(Y)$, X and Y are not independent events OR Since $P(X)=\frac{31}{50}\neq\frac{50}{63}=P(X\mid Y')$, X and Y' are not independent events So X and Y are not independent events	

7	In a factory, machines pack sugar into bags of 1 kg each on average, with variance σ^2 kg^2. The manufacturer is concerned that the machines are putting too much sugar into the bags and decides to carry out a hypothesis test. A random sample of 8 bags are selected and their total mass is 8.4 kg.	
	(i)	Stating a necessary assumption, carry out a test of the manufacturer's concern at the 5% significance level if $\sigma = 0.08$. [5]
	(ii)	Use an algebraic method to calculate the range of values of σ^2 for which the null hypothesis would not be rejected at the 5% significance level. [3]

7(i) [5]	Let X be the mass of a bag of sugar in kg. The necessary assumption is X follows a normal distribution. $H_0: \mu = 1$ $H_1: \mu > 1$ Perform a 1-tail test at 5% significance level. Under H_0, $\bar{X} \sim N\left(1, \frac{0.08^2}{8}\right)$ $\bar{x} = \frac{8.4}{8} = 1.05$, Using a *z*-test, *p*-value $= P\left(\bar{X} > 1.05\right) = 0.0385 < 0.05$ We reject H_0. There is sufficient evidence, at the 5% significance level, to support the manufacturer's concern.	Use proper notation. Present solution argument clearly.
(ii) [3]	Under H_0, $\bar{X} \sim N\left(1, \frac{\sigma^2}{8}\right)$ *p*-value > 0.05 $P\left(\bar{X} \geq 1.05\right) > 0.05$ $P\left(Z \geq \frac{1.05-1}{\sqrt{\sigma^2/8}}\right) > 0.05$ $0.05\sqrt{\frac{8}{\sigma^2}} < 1.64485$ $\sigma^2 > 8\left(\frac{0.05}{1.64485}\right)^2$ $\sigma^2 > 0.007392$ *OR* $\sigma^2 \in (0.00739, \infty)$	

8	In this question you should state clearly the values of the parameters of any normal distribution you use. In a supermarket, the masses in grams of apples have the distribution $N(90, 13^2)$ and the masses in grams of potatoes have the distribution $N(170, 30^2)$.	
	(i)	Find the probability that the mass of a randomly chosen potato is more than twice the mass of a randomly chosen apple. [3]
	A certain salad recipe requires 5 apples and 6 potatoes.	
	(ii)	Find the probability that the total mass of 5 randomly chosen apples and 6 randomly chosen potatoes is between 1.2 and 1.5 kilograms. [3]
	The salad recipe requires the apples and potatoes to be prepared by peeling and slicing them. The process reduces the mass of each apple by 15% and the mass of each potato by 25%.	
	(iii)	Find the probability that the total mass, after preparation, of 5 randomly chosen apples and 6 randomly chosen potatoes is not more than 1.2 kilograms. [3]

(i) [3]	Let X and Y be the masses in grams of a randomly chosen apple and a randomly chosen potato in grams respectively. i.e. $X \sim N(90, 13^2)$, $Y \sim N(170, 30^2)$. $E(Y-2X) = 170 - 2(90) = -10$ $Var(Y-2X) = 30^2 + 2^2(13^2) = 1576$ $Y - 2X \quad N(-10, 1576)$ Required probability $= P(Y > 2X)$ $= P(Y - 2X > 0)$ $= 0.401$ (3 s.f.)	**Define random variables clearly.**
(ii) [3]	Let $T = X_1 + X_2 + ... + X_5 + Y_1 + Y_2 + ... + Y_6$. $E(T) = 5(90) + 6(170) = 1470$ $Var(T) = 5(13^2) + 6(30^2) = 6245$ $T \quad N(1470, 6245)$ Required probability $= P(1200 < T < 1500)$ $= 0.648$ (3 s.f.)	Note that $X_1 + ... + X_5 + Y_1 + ... + Y_6 \neq 5X + 6Y$ State the distribution of T
(iii) [3]	Let $W = 0.85(X_1 + X_2 + ... + X_5) + 0.75(Y_1 + Y_2 + ... + Y_6)$	

$\mathrm{E}(W) = (0.85)(5)(90) + (0.75)(6)(170) = 1147.5$

$\mathrm{Var}(W) = (0.85^2)(5)(13^2) + (0.75^2)(6)(30^2) = 3648.0125$

$W \sim \mathrm{N}(1147.5,\ 3648.0125)$

Required probability $= \mathrm{P}(W \le 1200)$
$= 0.808$ (3 s.f.)

9 The continuous random variable X has the distribution $\mathrm{N}(\mu, \sigma^2)$.
It is known that $\mathrm{P}(X < k) = 0.2$ and $\mathrm{P}(X < 7) = 0.8$.

(i) Show that $\mathrm{P}(k < X < 7) = 0.6$ and write down the value of $\mathrm{P}(\mu < X < 7)$. [2]

(ii) Express μ in terms of k. [1]

You may use $\sigma^2 = 12$ for the rest of the question.

(iii) Show that $\mu = 4.0845$ correct to 4 decimal places. [3]

(iv) Find $\mathrm{P}(|X| < k)$. [2]

It is given that $2\mathrm{P}(X \le r) = 3\mathrm{P}(X > r)$ for a certain constant, r.

(v) Ten independent observations of X are randomly selected. Find the probability that there are more observations of X with values greater than r than observations of X with values less than r in the selection. [4]

(i) [2]
$\mathrm{P}(k < X < 7) = \mathrm{P}(X < 7) - \mathrm{P}(X < k) = 0.8 - 0.2 = 0.6$ (shown)

$\mathrm{P}(\mu < X < 7) = 0.3$

(ii) [1]
Since $\mathrm{P}(k < X < \mu) = \mathrm{P}(\mu < X < 7) = 0.3$,

by symmetry $\mu = \dfrac{k+7}{2}$.

Draw a diagram!

(iii) [2]
$\mathrm{P}(X < 7) = 0.8 \Rightarrow \mathrm{P}(Z < \frac{7-\mu}{\sqrt{12}}) = 0.8$

Therefore $\dfrac{7-\mu}{\sqrt{12}} = 0.8416$ (4 d.p)

$\dfrac{7-\mu}{\sqrt{12}} = 0.8416 \Rightarrow \mu = 4.0845$

(iv) [3]
$k = 2\mu - 7 = 1.169$

$\mathrm{P}(|X| < k) = 0.135$ (3 s.f)

(v) [4]
$2\mathrm{P}(X \le r) = 3\mathrm{P}(X > r)$

$2(1 - \mathrm{P}(X > r)) = 3\mathrm{P}(X > r)$

$5\mathrm{P}(X > r) = 2$

$\Rightarrow \mathrm{P}(X > r) = 0.4$

Let Y be the number of observations out of 10 with values greater than r.

$Y \sim \mathrm{B}(10, 0.4)$

$\mathrm{P}(Y \ge 6) = 1 - \mathrm{P}(Y \le 5) = 0.166$ (3s.f)

10	A group of 13 people consists of 6 single men and 5 single women and a married couple. A committee of 7 is to be selected from the group.	
	(i) Find the number of committees that can be formed if there is no restriction in the selection. [1]	
	(ii) Show that there are 658 such committees with more women than men. [2]	
	Given that a committee of 7 people is to be selected from the group such that the committee contains more women than men,	
	(iii) find the probability of getting a committee that consists of 4 single women and 3 single men. [2]	
	(iv) find the probability of getting a committee that contains at least one married member. [2]	
	The 4 women and 3 men who were finally selected for the committee included the married couple. The 7 members sit at random around a table with 7 chairs.	
	(v) Find the probability that the men are all separated from each other. [2]	
	(vi) Given that the men are all separated from each other, find the probability that the married couple sit next to each other. [3]	

(i) **[1]**	number of committees if there is no restrictions in the selection $= {}^{13}C_7 = 1716$	
(i) **[2]**	There are 3 cases: Case 1: 4 women 3 men Number of committees = ${}^6C_4 \times {}^7C_3 = 525$ Case 2: 5 women 2 men Number of committees = ${}^6C_5 \times {}^7C_2 = 126$ Case 3: 6 women 1 man Number of committees = ${}^6C_6 \times {}^7C_1 = 7$ Total number of committees = 658 (Shown)	
(iii) **[1]**	P(the committee will consist of 4 single women and 3 single men) $= \frac{{}^5C_4 \times {}^6C_3}{658} = \frac{100}{658} = \frac{50}{329}$	
(iv) **[2]**	Number of committees with no married member $= {}^5C_4 \times {}^6C_3 + {}^5C_5 \times {}^6C_2$ $= 100 + 15$ $= 115$	
	Number of committees with at least one married member $= 658 - 115$ $= 543$ P(the committee will contain at least one married member) $= \frac{543}{658}$	
	Alternative Solution We consider 3 cases: Case 1: Husband in, wife out Number of committees $= {}^5C_4 \times {}^6C_2 + {}^5C_5 \times {}^6C_1 = 75 + 6 = 81$ Case 2: Wife in, husband out Number of committees $= {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2 + {}^5C_5 \times {}^6C_1 = 200 + 75 + 6 = 281$ Case 3: Both husband and wife are in Number of committees $= {}^5C_3 \times {}^6C_2 + {}^5C_4 \times {}^6C_1 + {}^5C_5 = 150 + 30 + 1 = 181$ Total number of such committees = 543 P(the committee will contain at least one married member) $= \frac{543}{658}$	
(v) **[2]**	Number of ways to sit 4 women around the table = 3! Number of ways to slot the 3 men = $4 \times 3 \times 2 = 24$ Number of sitting arrangements where the all the men are separated $= 3! \times 24$ $= 144$ Required probability $= \frac{144}{6!}$ $= \frac{1}{5}$	
(vi) **[3]**	Number of ways to sit 4 women around the table = 3! Number of ways to sit the man next to his wife = 2 Number of ways to sit the other 2 men = ${}^3P_2$ Number of ways to have all the men separated from each other and the married couple sit next to each other $= 3! \times 2 \times 3 \times 2 = 72$ Required probability $= \frac{72}{144}$ $= \frac{1}{2}$	

11 In conjunction with the Great Singapore Sale, a certain electronics store is having a lucky draw for their customers. In each round of the lucky draw, the customer draws two balls randomly, one after another, with replacement from a box containing 20 red balls, 30 blue balls and 50 white balls. The colour of each ball drawn is noted and points are awarded accordingly as follows.

Colour of ball	Point(s)
Red	5
Blue	4
White	1

The customer's score in each round of the lucky draw is the total number of points awarded for the balls drawn. To illustrate, a customer scores a total of 5 points if he draws a blue ball and a white ball in a round of the lucky draw, regardless of the order of appearance of the balls.
You may assume that the 100 balls are indistinguishable from each other apart from their colour.

Let X denote the total number of points scored by a customer in a round of the lucky draw.

(i) Tabulate the probability distribution of X. [3]

(ii) Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = 6.02$. [2]

Mr Lim participated in 50 rounds of the lucky draw.

(iii) Using a suitable approximation, find the probability that Mr Lim's average score is at least 6. [3]

A customer wins a cash voucher if his total score for one round of the lucky draw is more than 6.

The total number of cash vouchers won by Mr Tan in n rounds of the lucky draw is denoted by Y.

(iv) Find the least value of n such that there is a probability of more than 0.7 that Mr Tan will win more than 3 cash vouchers in total. [4]

(i) [2]

x	2	5	6	8	9	10
$\mathrm{P}(X=x)$	$\frac{1}{4}$ =0.25	$\frac{3}{10}$ =0.3	$\frac{1}{5}$ =0.2	$\frac{9}{100}$ =0.09	$\frac{3}{25}$ =0.12	$\frac{1}{25}$ =0.04

"tabulate"

(ii) [3]

Using GC, $\mathrm{E}(X) = 5.4$

$\mathrm{E}(X^2) = 35.18$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 35.18 - (5.4)^2 = 6.02$ (shown)

GC is to be used here.

(iii) [3]

Since $n = 50$ is large, by Central Limit Theorem,

$\overline{X} \sim \mathrm{N}\left(5.4, \frac{6.02}{50}\right)$ approximately.

Required probability $= \mathrm{P}(\overline{X} \geq 6) = 0.0419$ (3 s.f.)

State distribution of $\overline{X}$

(iv) [4]

$\mathrm{P}(\text{winning a cash voucher})$

$= \mathrm{P}(X > 6)$

$= \mathrm{P}(X = 8, 9 \text{ or } 10)$

$= \frac{9}{100} + \frac{3}{25} + \frac{1}{25} = \frac{1}{4}$

$\therefore Y \quad \mathrm{B}\left(n, \frac{1}{4}\right)$

We must have

$\mathrm{P}(Y > 3) > 0.7$

$\Rightarrow 1 - \mathrm{P}(Y \leq 3) > 0.7$

$\Rightarrow \mathrm{P}(Y \leq 3) < 0.3$

Using GC,

n	$\mathrm{P}(Y \leq 3)$
18	0.30569
19	0.26309

$\therefore$ Least $n = 19$

Show sufficient working to justify answer.