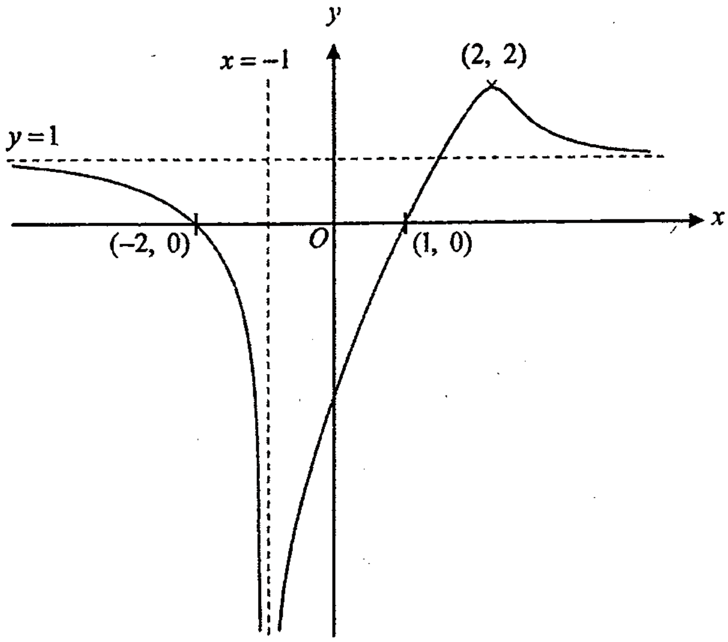
2023 EJC JC1 Promo

1 A function f is defined by $f(x)=x^3+ax^2+bx+c$, where a, b and c are constants. The graph of $y=f(x)$ passes through the points $(-2, 1)$ and $(2, -3)$. The point $(2,1)$ lies on the graph of $y=f(x+1)$.
Find the values of a, b and c. [4]

2 The diagram below shows the graph of $y=f(x)$. The curve passes through the x-axis at $(-2,0)$ and $(1,0)$, and has a maximum point with coordinates $(2,2)$. The lines $x=-1$ and $y=1$ are asymptotes to the graph.



Stating the equations of any asymptotes and the coordinates of any points of intersection with the axes and stationary points, where possible, sketch the graphs of

(a) $y=3f(x+a)$, where a is a positive constant such that $1<a<2$, [3]

(b) $y=f'(x)$. [3]

3 (a) Sketch, on the same diagram, the curves with equations $y=\left|\frac{x+1}{x-2}\right|$ and $y=\ln\left(1-\frac{x}{2}\right)+\frac{1}{2}$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. Label the two curves clearly. [5]

(b) Hence solve the inequality $\left|\frac{x+1}{x-2}\right| \geq \ln\left(1-\frac{x}{2}\right)+\frac{1}{2}$. [1]

4 Referred to the origin O, let P, Q and R be distinct points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively.

(a) Show that $(\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})=\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}$. [2]

(b) Give the geometrical meaning of $\frac{1}{2}|\,\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}\,|$. [2]

(c) Given that $\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}=\mathbf{0}$, $PR=3QR$, and that $PQ>PR$, express $\mathbf{r}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. [3]

5 (a) Verify that $\dfrac{3}{(r+1)!}-\dfrac{2}{r!}-\dfrac{1}{(r-1)!}=\dfrac{-r^2-3r+1}{(r+1)!}$. [1]

(b) Hence find $\displaystyle\sum_{r=1}^{n}\frac{-r^2-3r+1}{(r+1)!}$. [3]

(c) Use your answer to part (b) to find $\displaystyle\sum_{r=3}^{n}\frac{-r^2-r+3}{r!}$. [3]

6 (a) The first n terms of a series are given by $\log_a 3+\log_a 27+\log_a 243+...+\log_a 3^{2n-1}$, where a is a positive constant.

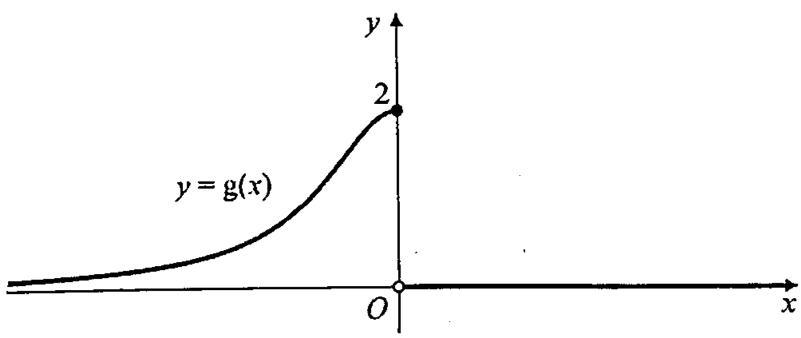
(i) Show that the series is an arithmetic series. [2]

(ii) Given that sum of the first 30 terms of the series is 300, find the value of a. [2]

(b) A geometric series has first term c and common ratio r, where c and r are non-zero. An arithmetic series has first term b and common difference d, where b and d are non-zero. It is given that the 5th, 8th and 10th terms of the arithmetic series are equal to the 2nd, 3rd and 4th of the geometric series respectively. Show that r satisfies the equation $3r^2-5r+2=0$ and hence find the sum to infinity in terms of c. [4]

7 A function g is defined by

$$g(x)=\begin{cases}\dfrac{2}{1+x^2} & \text{if } x\le 0,\\ 0 & \text{if } x>0.\end{cases}$$



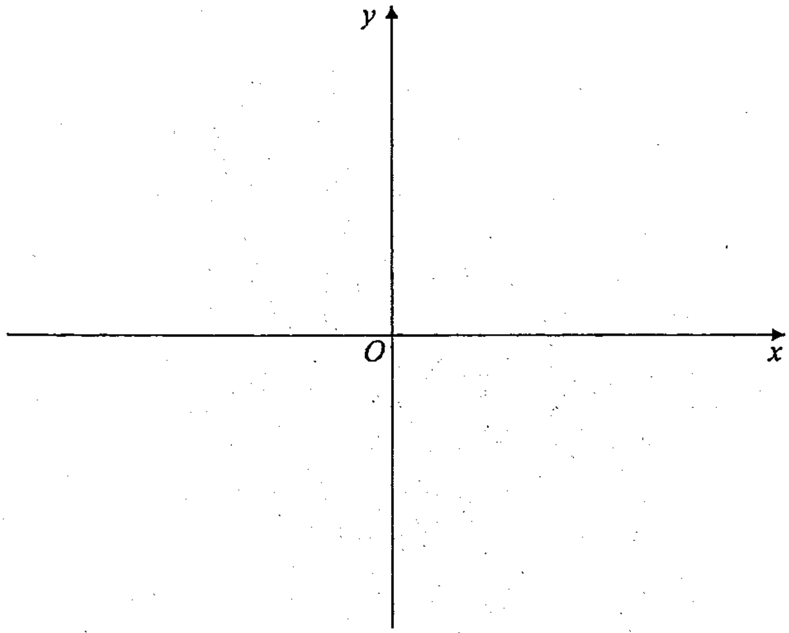
(a) Give a reason why g does not have an inverse. [1]

(b) The function g^{-1} exists if the domain of g is restricted to $x\le k$. State the greatest possible value of k. [1]

In the rest of the question, the domain of g is $x \le k$, where k takes the value determined in part **(b)**.

(c) Find $g^{-1}(x)$ and state the domain of g^{-1}. [3]

(d) Sketch, on the axes given below, the graphs of $y = g(x)$ and $y = g^{-1}(x)$. Label the two graphs clearly. Write down the equation of the line in which the graph of $y = g(x)$ must be reflected in order to obtain the graph of $y = g^{-1}(x)$. [3]



8 The curve C has equation

$$x + y = (x - y)^2.$$

It is given that C has only one turning point.

(a) Show that $1 - \frac{dy}{dx} = \frac{2}{1 + 2x - 2y}$. [4]

(b) Hence, or otherwise, show that $\frac{d^2y}{dx^2} = \left(1 - \frac{dy}{dx}\right)^3$. [3]

(c) Hence state, with a reason, whether the turning point is a minimum or a maximum. [2]

9 It is given that $y = \ln\left(2 - e^{-2x}\right)$.

(a) Show that $\frac{dy}{dx} = 4e^{-y} - 2$. [2]

(b) Hence find the Maclaurin series for y, up to and including the term in x^2. [3]

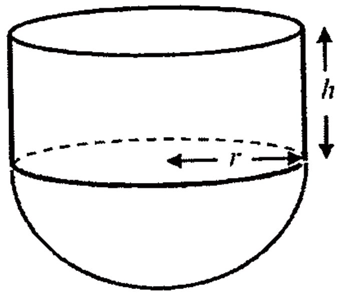
(c) Using standard series from the List of Formulae (MF26), expand $\ln\left(2 - e^{-2x}\right)$ as far as the term in x^2, and use this expansion as a check on the correctness of the series found in part **(b)**. [4]

10 **(a)** Find $\int \sin 3x \cos x \, dx$. [2]

(b) Find $\int \frac{x}{x^2+4x+13} \, dx$. [4]

(c) Use the substitution $x = 3\sin\theta$ to find $\int \sqrt{9-x^2} \, dx$. [4]

11 [A sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



A water fountain is to be constructed in the middle of Bishan East Park. It consists of a hemisphere with radius r m joined to an open cylinder with radius r m and height h m (see diagram).

The thickness of the fountain is negligible. It is given that the fountain, when filled to the brim, can hold a fixed volume k m^3 of water.

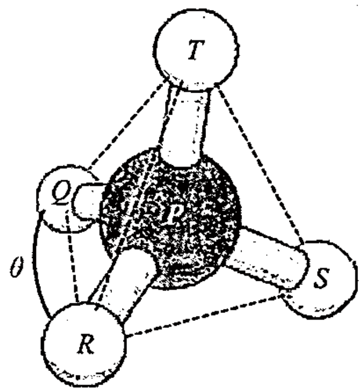
(a) The interior of the fountain is to be painted with a layer of special reflecting paint. The cost of painting is \$3 per m^2 for the hemispherical surface and \$2.50 per m^2 for the cylindrical wall. Show that the total cost of painting, \$$C$, is given by \$$\left(\frac{8}{3}\pi r^2 + \frac{5k}{r}\right)$. [3]

(b) Using differentiation, find the value of r, in terms of k, such that C is a minimum. [4]

Keeping C at a minimum, it is now given that $k = 50$.

(c) Find the numerical values of r and h. [2]

(d) When the fountain is filled to the brim, a leak develops at the joint between the cylinder and the hemisphere. Water leaks at a constant rate of 0.002 m^3 per minute. Assuming that water is neither lost nor added to the fountain in any other way, find the rate at which the level of water is decreasing. [3]



Methane (CH_4) is a chemical compound with a tetrahedral structure. The 4 hydrogen (H) atoms form a regular tetrahedron, and the carbon (C) atom is in the centre.

Let the centre of the C-atom be the point P, and the centres of the 4 H-atoms be the points Q, R, S and T. The coordinates of P, Q, R and S are $(-1,0,2)$, $(-2,-1,1)$, $(-2,1,3)$ and $(0,-1,a)$ respectively.

The angle θ subtended by any two C-H bonds at the C-atom, such as angle QPR, is known as the H-C-H bond angle (see diagram above).

(a) Find the bond angle, correct to 2 decimal places. [3]

(b) By using the fact that $QS = RS$, show that $a = 3$. [2]

(c) Find a cartesian equation for plane π, which contains the points P, Q and R. [3]

(d) F is the point on π that is closest to the point S.

(i) State a vector equation for the line SF. [1]

(ii) Hence, show that the coordinates of F are $(0,0,2)$. [3]

(iii) Given that the point T is the mirror image of the point S in π, find the position vector of T. [2]

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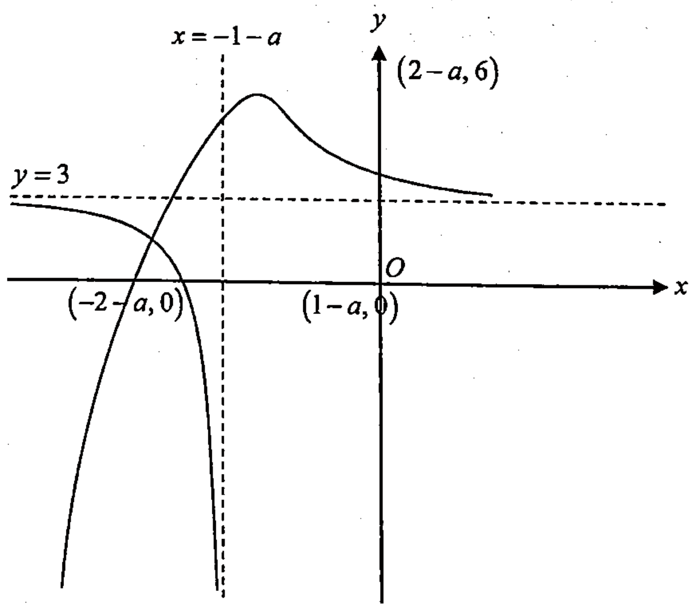
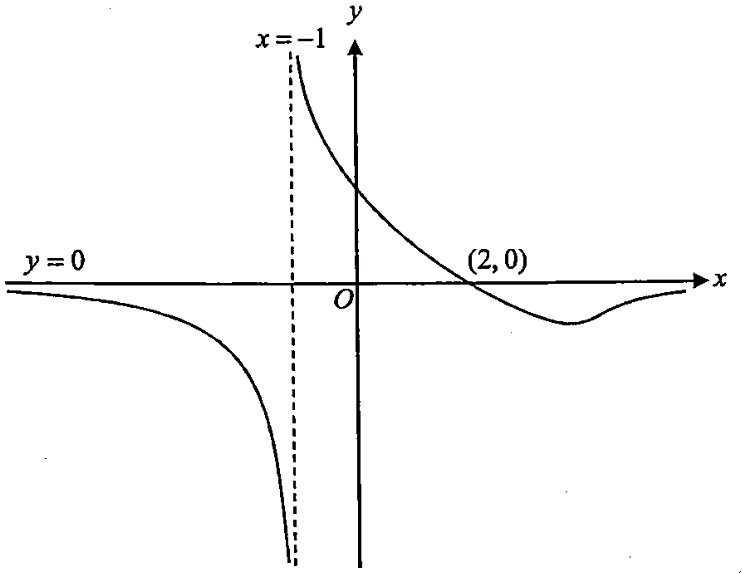
1	Solution
	At $(-2,1)$, $1=(-2)^3+a(-2)^2+b(-2)+c \Rightarrow 4a-2b+c=9$ --- (1) At $(2,-3)$, $-3=(2)^3+a(2)^2+b(2)+c \Rightarrow 4a+2b+c=-11$ --- (2) Since $(2,1)$ lies on $y=\mathrm{f}(x+1)$, <u>Method 1:</u> consider that $(3,1)$ lies on $y=\mathrm{f}(x)$: $1=(3)^3+a(3)^2+b(3)+c \Rightarrow 9a+3b+c=-26$ --- (3) <u>Method 2</u>: use $y=\mathrm{f}(x+1)=(x+1)^3+a(x+1)^2+b(x+1)+c$ $1=(2+1)^3+a(2+1)^2+b(2+1)+c \Rightarrow 9a+3b+c=-26$ --- (3) Solving (1), (2) and (3), $a=-2, b=-5, c=7$ [or $\mathrm{f}(x)=x^3-2x^2-5x+7$]
2	**Solution**
(a)	

(b)

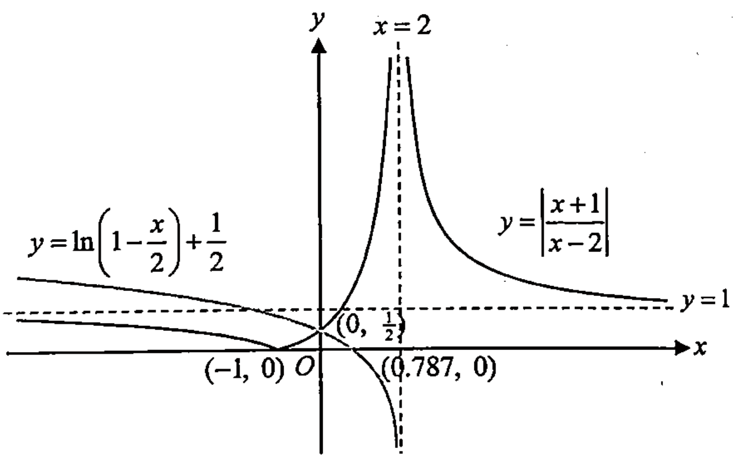


3 **Solution**

(a)

$y=\left|\frac{x+1}{x-2}\right|=\left|1+\frac{3}{x-2}\right|$. HA: $y=1$. VA: $x=2$

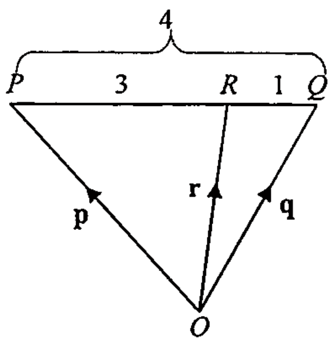
For $\ln\left(1-\frac{x}{2}\right)$: $1-\frac{x}{2}\neq 0 \Rightarrow x\neq 2$, i.e. $x=2$ is a VA.



(b) From the graph, $0 \leq x < 2$

4 **Solution**

(a) [Consider any triplet of distinct points P, Q, R with position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively.]

	LHS $= (\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})$ $= (\mathbf{r}-\mathbf{p})\times\mathbf{r}-(\mathbf{r}-\mathbf{p})\times\mathbf{q}$ $= \mathbf{r}\times\mathbf{r}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{q}+\mathbf{p}\times\mathbf{q}$ $= \mathbf{r}\times\mathbf{p}+\mathbf{q}\times\mathbf{r}+\mathbf{p}\times\mathbf{q}$ $\quad (\because \mathbf{r}\times\mathbf{r}=0$ and $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a})$ $= \mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}$ = RHS $\therefore\ \mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}=(\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})$ (shown)
(b)	$\frac{1}{2}\lvert\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}\rvert=\frac{1}{2}\lvert(\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})\rvert$ $=\frac{1}{2}\left\lvert\left(\overline{OR}-\overline{OP}\right)\times\left(\overline{OR}-\overline{OQ}\right)\right\rvert$ $=\frac{1}{2}\left\lvert\overline{PR}\times\overline{QR}\right\rvert$ $\therefore\ \frac{1}{2}\lvert\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}\rvert$ represents the area of ΔPQR. <u>Alternative</u> $\frac{1}{2}\lvert\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}\rvert$ represents half the area of a parallelogram with PR and QR as adjacent sides.
(c)	Given $\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}=\mathbf{0}$, $\Rightarrow (\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})=\mathbf{0}$ (from result in **(a)**) $\Rightarrow \overline{PR}\times\overline{QR}=\mathbf{0}$ $\because P,Q,R$ are distinct points, $\Rightarrow \overline{PR}\neq\mathbf{0}$, and $\overline{QR}\neq\mathbf{0}$, $\Rightarrow \overline{PR}\parallel\overline{QR}$, i.e. P,Q,R are <u>collinear</u> points. Given also that $PR=3QR$, and $PQ>PR$, $\therefore$ Point R divides PQ internally in the ratio 3:1, i.e. $PR:RQ=3:1$.  By the ratio theorem, position vector $\mathbf{r}=\frac{\mathbf{p}+3\mathbf{q}}{4}$. <u>Alternative (to show collinear)</u>

	Given $\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}=\mathbf{0}$, $\Rightarrow$ Area of $\Delta PQR=\frac{1}{2}\lvert\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}\rvert$ (by part **(b)**) $=\frac{1}{2}\lvert 0\rvert$ $=0$ i.e. ΔPQR is a degenerate triangle i.e. P,Q,R are <u>collinear</u> points.
5	**Solution**
(a)	$\frac{3}{(r+1)!}-\frac{2}{r!}-\frac{1}{(r-1)!}=\frac{3-2(r+1)-1(r)(r+1)}{(r+1)!}$ $=\frac{-r^2-3r+1}{(r+1)!}$ (verified)
(b)	$\sum_{r=1}^{n}\frac{-r^2-3r+1}{(r+1)!}$ $=\sum_{r=1}^{n}\left(\frac{3}{(r+1)!}-\frac{2}{r!}-\frac{1}{(r-1)!}\right)$ $=\frac{3}{2!}-\frac{2}{1!}-\frac{1}{0!}$ $+\frac{3}{3!}-\frac{2}{2!}-\frac{1}{1!}$ $+\frac{3}{4!}-\frac{2}{3!}-\frac{1}{2!}$ $\vdots$ $+\frac{3}{(n-1)!}-\frac{2}{(n-2)!}-\frac{1}{(n-3)!}$ $+\frac{3}{n!}-\frac{2}{(n-1)!}-\frac{1}{(n-2)!}$ $+\frac{3}{(n+1)!}-\frac{2}{n!}-\frac{1}{(n-1)!}$ $=\frac{3}{(n+1)!}+\frac{1}{n!}-4$
(c)	<u>Method 1: change of variable</u>

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$$\sum_{r=3}^{n}\frac{-r^2-r+3}{r!}=\sum_{r+1=3}^{r+1=n}\frac{-(r+1)^2-(r+1)+3}{(r+1)!}$$
$$=\sum_{r=2}^{n-1}\frac{-r^2-3r+1}{(r+1)!}$$
$$=\sum_{r=1}^{n-1}\frac{-r^2-3r+1}{(r+1)!}-\left(-\frac{3}{2}\right)$$
$$=\left[\left(\frac{3}{n!}+\frac{1}{(n-1)!}-4\right)\right]+\frac{3}{2}$$
$$=\left(\frac{3}{n!}+\frac{1}{(n-1)!}\right)-\frac{5}{2}$$

<u>Method 2: Listing</u>

$$\sum_{r=3}^{n}\frac{-r^2-r+3}{r!}=\frac{-3^2-3+3}{3!}+\ldots+\frac{-n^2-n+3}{n!}$$
$$=\sum_{r=2}^{n-1}\frac{-(r+1)^2-(r+1)+3}{(r+1)!}$$
$$=\sum_{r=1}^{n-1}\frac{-r^2-3r+1}{(r+1)!}-\left(-\frac{3}{2}\right)$$
$$=\left(\frac{3}{n!}+\frac{1}{(n-1)!}-4\right)-\left(-\frac{3}{2}\right)$$
$$=\left(\frac{3}{n!}+\frac{1}{(n-1)!}\right)-\frac{5}{2}$$

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<tr><td>6</td><td>Solution</td></tr>
<tr><td>(a)(i)</td><td>

$$d=u_n-u_{n-1}$$
$$=\log_a 3^{2n-1}-\log_a 3^{2(n-1)-1}$$
$$=\log_a\frac{3^{2n-1}}{3^{2n-3}}$$
$$=\log_a 9 \text{ which is a constant independent of } n$$

Therefore, the series is an arithmetic series.

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<tr><td>(a)(ii)</td><td>

<u>Method 1: use $S_n=\frac{n}{2}(a+l)$</u>

$$S_{30}=\frac{30}{2}\left[\log_a 3+\log_a 3^{2(30)-1}\right]=300$$

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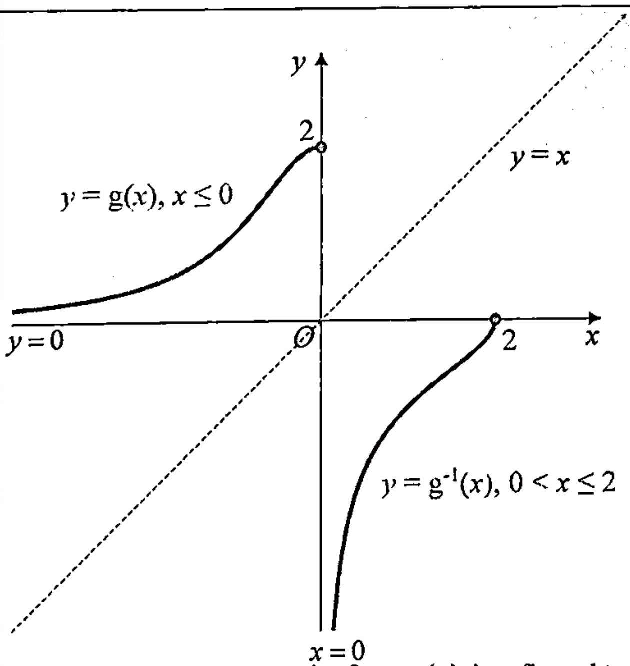
	$15\left(\log_a 3^{60}\right)=300$ $900\log_a 3=300$ $\log_a 3=\frac{1}{3}$ $a^{\frac{1}{3}}=3$ $a=27$ Method 2: use $S_n=\frac{n}{2}\left[2a+(n-1)d\right]$ $S_{30}=\frac{30}{2}\left[2(\log_a 3)+29\left(\log_a 9\right)\right]=300$ $15\left[2\left(\log_a 3\right)+29\left(\log_a 3^2\right)\right]=300$ $900\left(\log_a 3\right)=300$ $\log_a 3=\frac{1}{3}$ $a^{\frac{1}{3}}=3$ $a=27$
(b)	$b+4d=cr$ (1) $b+7d=cr^2$ (2) $b+9d=cr^3$ (3) (Eliminate b): $(2)-(1)$: $cr^2-cr=3d$ $(3)-(2)$: $cr^3-cr^2=2d$ (Eliminate d): $\frac{cr^2-cr}{3}=\frac{cr^3-cr^2}{2}$ Since $c, r\neq 0$, we divide both sides by c and r, and rearrange to get $3r^2-5r+2=0$ (shown) Solving, $r=1$ (rejected $\because d\neq 0$) or $r=\frac{2}{3}$ $S_\infty=\frac{c}{1-\frac{2}{3}}=3c$
7	**Solution**
(a)	Explanation 1 ("Horizontal Line Test")

	g is not a one-to-one function as the horizontal line $y=0$ meets the graph of $y=\mathrm{g}(x)$ more than once. <u>Explanation 2 (State two inputs with same output)</u> g is not a one-to-one function as there are distinct inputs producing the same output under function g, e.g. $\mathrm{g}(1)=\mathrm{g}(2)=0$.
(b)	Greatest $k=0$.
(c)	Let $y=\mathrm{g}(x),\ x\le 0$. Then $x=\mathrm{g}^{-1}(y)$. $y=\mathrm{g}(x)=\dfrac{2}{1+x^2}$, since $x\le 0$. $1+x^2=\dfrac{2}{y}$ $x^2=\dfrac{2}{y}-1,\quad x=\pm\sqrt{\dfrac{2}{y}-1}$ Since $x\le 0$, $x=-\sqrt{\dfrac{2}{y}-1}=\mathrm{g}^{-1}(y)$ $\therefore\ \mathrm{g}^{-1}(x)=-\sqrt{\dfrac{2}{x}-1}$ $\mathrm{D}_{\mathrm{g}^{-1}}=\mathrm{R}_{\mathrm{g}}=(0,2]$.
(d)	 The line in which the graph of $y=\mathrm{g}(x)$ is reflected to obtain the graph of $y=\mathrm{g}^{-1}(x)$ is $y=x$.
8	**Solution**

(a)	Method 1: Find $\frac{dy}{dx}$ then simplify Differentiating implicitly w.r.t. x, **Method A: Consider Chain Rule** $1+\frac{dy}{dx}=2(x-y)\left(1-\frac{dy}{dx}\right)$ **Method B: Consider product rule** $x+y=(x-y)^2=x^2-2xy+y^2$ $1+\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+2y\frac{dy}{dx}$ $\Rightarrow 1+\frac{dy}{dx}=2x-2y-(2x-2y)\frac{dy}{dx}$ $\Rightarrow 1-2x+2y=-(1+2x-2y)\frac{dy}{dx}$ $\Rightarrow -\frac{dy}{dx}=\frac{1-2x+2y}{1+2x-2y}$ Add 1 to both sides, $1-\frac{dy}{dx}=\frac{1-2x+2y+(1+2x-2y)}{1+2x-2y}$ $=\frac{2}{1+2x-2y}$ (shown)
	Method 2: Consider adding $1-\frac{dy}{dx}$ to both sides Differentiating implicitly w.r.t. x, $1+\frac{dy}{dx}=2(x-y)\left(1-\frac{dy}{dx}\right)$ Add $1-\frac{dy}{dx}$ to both sides, $1+\frac{dy}{dx}+1-\frac{dy}{dx}=2(x-y)\left(1-\frac{dy}{dx}\right)+\left(1-\frac{dy}{dx}\right)$ $2=(2x-2y+1)\left(1-\frac{dy}{dx}\right)$ $1-\frac{dy}{dx}=\frac{2}{2x-2y+1}$ (shown)
(b)	Diff implicitly w.r.t. x,

	$-\frac{d^2y}{dx^2}=-2(1+2x-2y)^{-1-1}\left(2-2\frac{dy}{dx}\right)$ $=-\frac{4}{(1+2x-2y)^2}\left(1-\frac{dy}{dx}\right)$ $=-\left(1-\frac{dy}{dx}\right)^2\left(1-\frac{dy}{dx}\right)$ $\Rightarrow\frac{d^2y}{dx^2}=\left(1-\frac{dy}{dx}\right)^3$ (shown)
(c)	$\frac{dy}{dx}=0\Rightarrow\frac{d^2y}{dx^2}=1>0$ $\therefore$ minimum point
9	**Solution**
(a)	$y=\ln\left(2-e^{-2x}\right)\Rightarrow\underbrace{e^y=2-e^{-2x}}_{\text{Eqn 1}}\Rightarrow\underbrace{e^{-2x}=2-e^y}_{\text{Eqn 2}}$ <u>Method 1: implicit differentiation</u> Differentiating Eqn 1 implicitly w.r.t. x, $e^y\frac{dy}{dx}=2e^{-2x}$ Then $\frac{dy}{dx}=2e^{-2x}e^{-y}=2\underbrace{\left(2-e^{-y}\right)}_{\text{from Eqn 2}}e^{-y}=4e^{-y}-2$ (shown) <u>Method 2: direct differentiation</u> $\frac{dy}{dx}=\frac{2e^{-2x}}{2-e^{-2x}}=\frac{2\overbrace{\left(2-e^y\right)}^{\text{from Eqn 2}}}{\underbrace{e^y}_{\text{from Eqn 1}}}=4e^{-y}-2$ (shown) <u>Method 3: make x the subject, implicit differentiation</u> From Eqn 2, $e^{-2x}=2-e^y\Rightarrow-2x=\ln\left(2-e^y\right)$ Differentiating implicitly w.r.t. x, $-2=\frac{1}{2-e^y}\left(-e^y\frac{dy}{dx}\right)$ Then $\frac{dy}{dx}=\frac{-2\left(2-e^y\right)}{-e^y}=4e^{-y}-2$ (shown)
(b)	<u>Method 1: further differentiation of result in (a)</u> Differentiating $\frac{dy}{dx}=4e^{-y}-2$ implicitly w.r.t. x, $\frac{d^2y}{dx^2}=4e^{-y}\left(-\frac{dy}{dx}\right)=-4\frac{dy}{dx}e^{-y}$

	When $x=0$, $y=0, \frac{dy}{dx}=2$, $\frac{d^2y}{dx^2}=-4(2)e^{-0}=-8$ $y=(0)+(2)x+\left(\frac{-8}{2!}\right)x^2+\ldots=2x-4x^2+\ldots$ <u>Method 2: direct differentiation of 1st derivative in x</u> $\frac{d^2y}{dx^2}=\frac{-4e^{-2x}\left(2-e^{-2x}\right)-2e^{-2x}\left(2e^{-2x}\right)}{\left(2-e^{-2x}\right)^2}$ $=\frac{-8e^{-2x}}{\left(2-e^{-2x}\right)^2}$ When $x=0$, $y=0, \frac{dy}{dx}=2$, $\frac{d^2y}{dx^2}=\frac{-8e^{-0}}{\left(2-e^{-0}\right)^2}=-8$ $y=(0)+(2)x+\left(\frac{-8}{2!}\right)x^2+\ldots=2x-4x^2+\ldots$
(c)	From MF26, $e^{-2x}=1+(-2x)+\frac{(-2x)^2}{2}+\ldots=1-2x+2x^2+\ldots$ Then $y=\ln\left(2-e^{-2x}\right)$ $=\ln\left[2-\left(1-2x+2x^2+\ldots\right)\right]$ $=\ln\left[1+\left(2x-2x^2+\ldots\right)\right]$ $=\underbrace{\left(2x-2x^2+\ldots\right)-\frac{\left(2x-2x^2+\ldots\right)^2}{2}+\ldots}_{\text{Using the expansion for }\ln[1+f(x)]}$ $=2x-2x^2-\frac{4x^2}{2}+\ldots$ $=2x-4x^2+\ldots$ This is the same expression as found part **(b)** and hence we can conclude that the expansion is correct.
10	**Solution**

(a)	$\int \sin 3x \cos x \, dx = \frac{1}{2}\int 2\sin 3x \cos x \, dx$ $= \frac{1}{2}\int \sin(3x+x) + \sin(3x-x) \, dx$ $= \frac{1}{2}\int \sin 4x + \sin 2x \, dx$ $= \frac{1}{2}\left[\int \sin 4x \, dx + \int \sin 2x \, dx\right]$ $= \frac{1}{2}\left[\frac{1}{4}\int 4\sin 4x \, dx + \frac{1}{2}\int 2\sin 2x \, dx\right]$ $= \frac{1}{2}\left[\frac{1}{4}(-\cos 4x) + \frac{1}{2}(-\cos 2x)\right] + c$ $= -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + c,$ where c is an arbitrary constant.
(b)	[Since $\frac{d}{dx}(x^2+4x+13) = 2x+4$, we re-write x as $x = A(2x+4)+B$ in order to split the numerator into 2 parts. Compare coefficients to get A and B.] $\int \frac{x}{x^2+4x+13} \, dx = \int \frac{\frac{1}{2}(2x+4)-2}{x^2+4x+13} \, dx$ $= \frac{1}{2}\int \frac{2x+4}{x^2+4x+13} dx - 2\int \frac{1}{x^2+4x+13} dx$ $= \frac{1}{2}\ln\left\|x^2+4x+13\right\| - 2\int \frac{1}{(x+2)^2+3^2} dx$ $= \frac{1}{2}\ln(x^2+4x+13) - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c,$ where c is an arbitrary constant
(c)	Let $x = 3\sin\theta$. Then $\frac{dx}{d\theta} = 3\cos\theta$. Substituting,

	$\int\sqrt{9-x^2}\ \underline{dx}=\int\sqrt{9-(3\sin\theta)^2}\cdot3\cos\theta\ d\theta$ $=\int\sqrt{9(1-\sin^2\theta)}\cdot3\cos\theta\ d\theta$ $=\int\sqrt{9\cos^2\theta}\cdot3\cos\theta\ d\theta$ $=\int3\cos\theta\cdot3\cos\theta\ d\theta$ $=\int9\cos^2\theta\ d\theta$ $=\frac{9}{2}\int\cos2\theta+1\ d\theta$ (double angle formula) $=\frac{9}{2}\left(\frac{1}{2}\sin2\theta+\theta\right)+c$ $=\frac{9}{2}\sin\theta\cos\theta+\frac{9}{2}\theta+c$ $\left[x=3\sin\theta\Rightarrow\sin\theta=\frac{x}{3}\Rightarrow\cos\theta=\sqrt{1-\left(\frac{x}{3}\right)^2}=\frac{\sqrt{9-x^2}}{3}\right]$ So $\int\sqrt{9-x^2}\ \underline{dx}=\frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+c$ $=\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+c,$ where c is an arbitrary constant.
11	**Solution**
(a)	$V=\pi r^2h+\frac{2}{3}\pi r^3=k$ $\Rightarrow h=\frac{k-\frac{2}{3}\pi r^3}{\pi r^2}=\frac{k}{\pi r^2}-\frac{2}{3}r$ $C=3(2\pi r^2)+2.5(2\pi rh)$ $=6\pi r^2+5\pi r\left(\frac{k}{\pi r^2}-\frac{2}{3}r\right)$ $=\frac{8}{3}\pi r^2+\frac{5k}{r}$ (shown)
(b)	$\frac{dC}{dr}=\frac{16\pi r}{3}-\frac{5k}{r^2}$ $\frac{dC}{dr}=0\Rightarrow\frac{16\pi r}{3}-\frac{5k}{r^2}=0$

$\Rightarrow r^3 = \dfrac{15k}{16\pi}$

$\Rightarrow r = \sqrt[3]{\dfrac{15k}{16\pi}}$

<u>Check min using second derivative method</u>

$\dfrac{d^2C}{dr^2} = \dfrac{16\pi}{3} + \dfrac{10k}{r^3} > 0 \quad (\because k, r > 0)$

So C is minimum when $r = \sqrt[3]{\dfrac{15k}{16\pi}}$.

<u>Check min using first derivative method</u>

$\dfrac{dC}{dr} = \dfrac{16\pi r}{3} - \dfrac{5k}{r^2} = \dfrac{16\pi r^3 - 15k}{3r^2}$

r	$\sqrt[3]{\dfrac{15k}{16\pi}}^{-}$	$\sqrt[3]{\dfrac{15k}{16\pi}}$	$\sqrt[3]{\dfrac{15k}{16\pi}}^{+}$
Sign of $\dfrac{dC}{dr}$	$16\pi r^3 - 15k < 0$ $3r^2 > 0$ $\therefore \dfrac{16\pi r^3 - 15k}{3r^2} < 0$	0	$16\pi r^3 - 15k > 0$ $3r^2 > 0$ $\therefore \dfrac{16\pi r^3 - 15k}{3r^2} > 0$
Slope	\	-	/

So C is minimum when $r = \sqrt[3]{\dfrac{15k}{16\pi}}$.

(c) When $k = 50$,

$r = \sqrt[3]{\dfrac{15(50)}{16\pi}} = 2.4619 = 2.46 \text{ (3s.f.)}$ and

$h = \dfrac{50}{\pi(2.4619)^2} - \dfrac{2}{3}(2.4619) = 0.985 \text{ (3 s.f.)}$

(d) [Since the leak is at the joint between cylinder and hemisphere, we need to consider only the volume in the cylindrical part.]

Let V_c be volume of water in the cylindrical part and l be level of water in the cylindrical part.

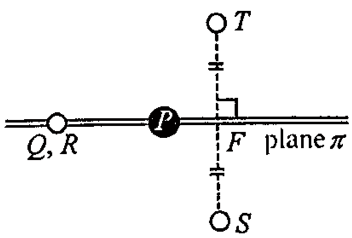
Note that $r = 2.4619$ is a constant, so $V_c = (2.4619)^2 \pi l$

<u>Method 1 – differentiate w.r.t. t</u>

$\dfrac{dV_c}{dt} = (2.4619)^2 \pi \dfrac{dl}{dt}$.

	$\frac{dl}{dt}=\frac{1}{\pi(2.4619)^2}\frac{dV_c}{dt}$ $=\frac{1}{\pi(2.4619)^2}(-0.002)$ $=-1.05\times10^{-4}$ m per minute So the water level decreases at 1.05×10^{-4} m/min. <u>Method 2 – connected rate of change</u> $\frac{dV_c}{dl}=\pi(2.4619)^2$ $\frac{dl}{dt}=\frac{dl}{dV_c}\frac{dV_c}{dt}$ $=\frac{1}{\pi(2.4619)^2}(-0.002)$ $=-1.05\times10^{-4}$ m per minute So the water level decreases at 1.05×10^{-4} m/min. <u>Method 3 – consider proportionality</u> Since surface area is a constant, $\frac{dV_c}{dt}=A\frac{dl}{dt}$ $\frac{dl}{dt}=\frac{1}{\pi(2.4619)^2}(-0.002)$ $=-1.05\times10^{-4}$ m per minute
12	**Solution**
(a)	$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$ $\quad$ $\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}$ $=\begin{pmatrix}-2\\-1\\1\end{pmatrix}-\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$, $\quad$ $=\begin{pmatrix}-2\\1\\3\end{pmatrix}-\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$. Angle $QPR=\cos^{-1}\left(\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\lvert\overrightarrow{PQ}\rvert\lvert\overrightarrow{PR}\rvert}\right)$

	Angle $QPR = \cos^{-1}\left(\dfrac{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}}{\left\|\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\|\left\|\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\|}\right)$ $= \cos^{-1}\left(\dfrac{1+(-1)+(-1)}{\sqrt{3}\sqrt{3}}\right)$ $= \cos^{-1}\left(-\dfrac{1}{3}\right)$ $= 109.47^\circ$ (2 d.p.)
(b)	$QS^2 = RS^2 \quad (\because QS = RS)$ $(0-(-2))^2 + (-1-(-1))^2 + (a-1)^2 = (0-(-2))^2 + (-1-1)^2 + (a-3)^2$ $2^2 + 0^2 + (a-1)^2 = 2^2 + (-2)^2 + (a-3)^2$ $(a^2 - 2a + 1) = 4 + (a^2 - 6a + 9)$ $4a = 12$ $a = 3$ (shown)
(c)	$\overrightarrow{PQ} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$ from **(a)** A vector normal to plane π is $\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\times\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}(-1)(1)-(-1)(1)\\(-1)(-1)-(-1)(1)\\(-1)(1)-(-1)(-1)\end{pmatrix} = \begin{pmatrix}0\\2\\-2\end{pmatrix} = 2\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\therefore \mathbf{n} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\therefore\ \pi : \mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix} = -2$ $\pi : y - z = -2$
(d)(i)	F is the foot of perpendicular from S to plane π, and SF is parallel to a normal vector used for plane π. $\therefore$ Line $SF : \mathbf{r} = \begin{pmatrix}0\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\-1\end{pmatrix},\ \lambda \in \mathbb{R}$.

(d)(ii)	$\because$ F lies on line SF, $\overrightarrow{OF}=\begin{pmatrix}0\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\-1+\lambda\\3-\lambda\end{pmatrix}$, for some $\lambda\in\square$. $\because$ F lies on π, $\overrightarrow{OF}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=-2.$ $\Rightarrow \begin{pmatrix}0\\-1+\lambda\\3-\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=-2,$ $\Rightarrow (-1+\lambda)-(3-\lambda)=-2,$ $-4+2\lambda=-2$ $\lambda=1$ $\therefore \overrightarrow{OF}=\begin{pmatrix}0\\-1+\lambda\\3-\lambda\end{pmatrix}\Bigg\|_{\lambda=1}=\begin{pmatrix}0\\0\\2\end{pmatrix}$ $F(0,\ 0,\ 2)$ (shown) <u>Alternative:</u> $\begin{pmatrix}-1\\0\\2\end{pmatrix}+\mu\begin{pmatrix}-1\\-1\\-1\end{pmatrix}+\alpha\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $-1-\mu-\alpha=0$ $-\mu+\alpha=-1+\lambda$ $2-\mu+\alpha=3-\lambda$ $\mu=-\frac{1}{2},\alpha=-\frac{1}{2},\lambda=1$
(d)(iii)	$\because$ T is the mirror image of S in π, the foot of perpendicular from S to π, i.e. point F, is the midpoint of S and T. By the midpoint theorem (special case of ratio theorem), $\overrightarrow{OF}=\frac{\overrightarrow{OS}+\overrightarrow{OT}}{2}$ $\overrightarrow{OT}=2\overrightarrow{OF}-\overrightarrow{OS}$ $=2\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}0\\-1\\3\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}$  <u>Alternatively,</u> $\because$ T is the mirror image of S in plane π,

	$\overrightarrow{SF} = \overrightarrow{FT}$ $\overrightarrow{OT} = \overrightarrow{OF} + \overrightarrow{FT}$ $= \overrightarrow{OF} + \overrightarrow{SF}$ $= \overrightarrow{OF} + \overrightarrow{OF} - \overrightarrow{OS}$ $= 2\begin{pmatrix}0\\0\\2\end{pmatrix} - \begin{pmatrix}0\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$