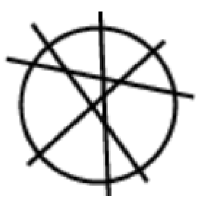
2024 JC2 DHS-EJC-RVHS H3 Math Prelim Suggested Solutions

Qn	Solution
1a	$\dfrac{dx}{dy} = g\left(\dfrac{x}{y}\right)$ $x = vy \Rightarrow \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$ Substituting into the DE: $v + y\dfrac{dv}{dy} = g(v)$ $y\dfrac{dv}{dy} = g(v) - v$ $\displaystyle\int \frac{1}{g(v)-v}\, dv = \int \frac{1}{y}\, dy$
b	$\dfrac{dx}{dy} = \dfrac{y^2 + (y^2+x^2)e^{\left(\frac{x}{y}\right)^2}}{xye^{\left(\frac{x}{y}\right)^2}}$ $= \dfrac{y}{xe^{\left(\frac{x}{y}\right)^2}} + \dfrac{y}{x} + \dfrac{x}{y}$ $= \left(\dfrac{x}{y}\right)^{-1} e^{-\left(\frac{x}{y}\right)^2} + \left(\dfrac{x}{y}\right)^{-1} + \dfrac{x}{y}$
c	Let $g\left(\dfrac{x}{y}\right) = \left(\dfrac{x}{y}\right)^{-1} e^{-\left(\frac{x}{y}\right)^2} + \left(\dfrac{x}{y}\right)^{-1} + \dfrac{x}{y}$ $\displaystyle\int \frac{1}{g(v)-v}\, dv = \int \frac{1}{y}\, dy$ $\displaystyle\int \frac{1}{v^{-1}e^{-v^2} + v^{-1} + v - v}\, dv = \int \frac{1}{y}\, dy$ $\displaystyle\int \frac{v}{e^{-v^2}+1}\, dv = \int \frac{1}{y}\, dy$ $\displaystyle\int \frac{ve^{v^2}}{1+e^{v^2}}\, dv = \int \frac{1}{y}\, dy$ $\dfrac{1}{2}\ln\left(1+e^{v^2}\right) = \ln\lvert y\rvert + c$

	$\ln\lvert y\rvert = \ln\sqrt{1+\mathrm{e}^{v^2}} - c$ $\lvert y\rvert = \mathrm{e}^{\ln\left(1+\mathrm{e}^{v^2}\right)^{\frac{1}{2}}-c} = \sqrt{1+\mathrm{e}^{v^2}}\,\mathrm{e}^{-c}$ $y = \pm\mathrm{e}^{-c}\sqrt{1+\mathrm{e}^{v^2}} = A\sqrt{1+\mathrm{e}^{v^2}}$ $y = A\sqrt{1+\mathrm{e}^{\left(\frac{x}{y}\right)^2}}$ When $x=\sqrt{\ln 9}, y=-\sqrt{2}$ $-\sqrt{2} = A\sqrt{1+\mathrm{e}^{\frac{\ln 9}{2}}} \Rightarrow -\sqrt{2} = A\sqrt{1+(9)^{\frac{1}{2}}} \Rightarrow A = -\frac{\sqrt{2}}{2}$ $y = -\frac{\sqrt{2}}{2}\sqrt{1+\mathrm{e}^{\left(\frac{x}{y}\right)^2}} = -\frac{\sqrt{2+2\mathrm{e}^{\left(\frac{x}{y}\right)^2}}}{2}$.

Qn	Solution
2a	$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(n-1-k+1)!(k-1)!}}{\frac{n!}{(n-k)!k!}}$ $= \frac{(n-1)!}{(n-k)!(k-1)!}\frac{(n-k)!k!}{n!}$ $= \frac{k}{n}$ (shown)
b	Since $s \le n$, and $1 \le i_1 < i_2 < \cdots < i_s \le n$, we need to choose s integers from $\{1, 2, ..., n\}$ for each term $a_{i_1}a_{i_2}\cdots a_{i_s}$ in the sum. Thus, there will be $\binom{n}{s}$ terms in the sum.
c	Similarly, the number of terms will be $\binom{n-1}{s-1}$.
d	$(1+a_1)(1+a_2)\cdots(1+a_n)$ $=1+\sum a_i + \sum_{1\le i<j\le n} a_i a_j + \sum_{1\le i<j<k\le n} a_i a_j a_k$ $+\ldots+\sum_{1\le i_1<i_2<\cdots<i_s\le n} a_{i_1}a_{i_2}\cdots a_{i_s} + \ldots + a_1 a_2 \ldots a_n$

	$\geq 1+\binom{n}{1}(a_1a_2\cdots a_n)^{\frac{1}{n}}+\binom{n}{2}\left(a_1^{\binom{n-1}{1}}a_2^{\binom{n-1}{1}}\cdots a_n^{\binom{n-1}{1}}\right)^{\frac{1}{\binom{n}{2}}}$ $+\dots+\binom{n}{s}\left(a_1^{\binom{n-1}{s-1}}a_2^{\binom{n-1}{s-1}}\cdots a_n^{\binom{n-1}{s-1}}\right)^{\frac{1}{\binom{n}{s}}}+\dots+\left[(a_1a_2\dots a_n)^{\frac{1}{n}}\right]^n$ (by AM-GM and (b), (c)) $=1+\binom{n}{1}g+\binom{n}{2}g^{n\frac{\binom{n-1}{1}}{\binom{n}{2}}}+\dots+\binom{n}{s}g^{n\frac{\binom{n-1}{s-1}}{\binom{n}{s}}}+\dots+g^n$ $=1+\binom{n}{1}g+\binom{n}{2}g^{n\left(\frac{2}{n}\right)}+\dots+\binom{n}{s}g^{n\left(\frac{s}{n}\right)}+\dots+g^n$ (by (a)) $=(1+g)^n$

Qn	**Solution**
3(a)	 Observe that the first time the *n*th line enters the circle, it will create a new region. In addition, each time the *n*th line intersects with each of the previous $n-1$ lines, a new region is created. Additionally, as there are only $n-1$ lines previously, and lines cuts other lines at most once, there can only be at most $n-1$ intersections between the new line and the previous $n-1$ lines, giving rise to at most *n* new regions. Since the total possible maximum can only arise when the maximum number of new regions is created with the addition of each line, we have $m_n - m_{n-1} = 1+(n-1) = n$.

	$m_n = m_1 + \sum_{r=2}^{n}(m_r - m_{r-1})$ $= 2 + \sum_{r=2}^{n} r$ $= 2 + \frac{n-1}{2}\ \ 2+n$ $= \frac{1}{2}(n^2 + n + 2)$
(b)	Note that for each white and black sector on the smaller circle, it will match with exactly $n+1$ sectors and *n* sectors respectively on the bigger circle. Hence there will be in total $n \times\ \ n+1\ \ +\ \ n+1\ \ \times n = 2n\ \ n+1$ such matches across all possible rotations. Since there are $2n+1$ possible rotations of the smaller circle, there will be at least $\left\lceil \frac{2n(n+1)}{2n+1} \right\rceil = \left\lceil n + \frac{n}{2n+1} \right\rceil = n+1$ matches for one of the rotations by Pigeonhole Principle.

Qn	Solution
4(a)	Now, $F_{2n+1} = F_{2n} + F_{2n-1} \Rightarrow F_{2n} = F_{2n+1} - F_{2n-1}$ for $n \geq 1$. $\sum_{i=1}^{n} F_{2i} = \sum_{i=1}^{n} F_{2i+1} - F_{2i-1}$ $= F_3 - F_1$ $+F_5 - F_3$ $+...$ $+F_{2n-1} - F_{2n-3}$ $+F_{2n+1} - F_{2n-1}$ $= F_{2n+1} - F_1 \quad$ (Shown) Similarly, $\sum_{i=1}^{n} F_{2i+1} = F_{2n+2} - F_2$.
(b)	$55 = F_{10} \in S$ $191 = F_7 + F_9 + F_{12} \in S$

Note: We can see this from;

$$\begin{aligned}55 &= 1+54\\ &= 1+2+5+13+34\\ &= F_1+\left(F_3+F_5+F_7+F_9\right)\\ &= F_1+\left(F_{10}-F_2\right)=F_{10}\end{aligned}$$

$$\begin{aligned}191 &= 190+1\\ &= 1+1+3+8+34+144\\ &= F_1+\left(F_2+F_4+F_6\right)+F_9+F_{12}\\ &= F_1+\left(F_7-F_1\right)+F_9+F_{12}\\ &= F_7+F_9+F_{12}.\end{aligned}$$

(ci)

$$\begin{aligned}k+1 &= 1+F_2+F_{u_2}+F_{u_3}+\ldots+F_{u_s}\\ &= F_3+F_{u_2}+F_{u_3}+\ldots+F_{u_s} && \left(\text{if } u_2>4 \text{ are are done}\right)\\ &= F_5+F_{u_3}+F_{u_4}+\ldots+F_{u_s} && \left(\text{else if } u_3>6 \text{ are are done}\right)\\ &= F_7+F_{u_4}+F_{u_5}+\ldots+F_{u_s} && \left(\text{else if } u_4>8 \text{ are are done}\right)\\ &= \ldots\\ &= F_{2s+1} && \left(\text{else if } u_i=2i,\ \text{for all } i\right)\end{aligned}$$

Alternatively

If $u_i=2i$ for all *i*.

Then, $k=F_2+F_4+F_6+\ldots+F_{2s}$ and

$k+1=1+\left(F_{2s+1}-1\right)=F_{2s+1}\in S$.

If $u_i\neq 2i$ for some *i*, consider the smallest *i* where this happens, call this *m* + 1.

(Or consider the largest *m* such that $u_i=u_{i-1}+2$ for all $i\le m$).

So, $u_{m+1}>2m+2$. This implies,

$$\begin{aligned}k &= F_2+F_{u_2}+F_{u_3}+\ldots+F_{u_s}\\ &= F_2+F_4+\ldots+F_{2m}+F_{u_{m+1}}+\ldots+F_{u_s}\end{aligned}$$

Thus from **(a)**,

$$\begin{aligned}k+1 &= F_1+\left(F_2+F_4+\ldots+F_{2m}\right)+F_{u_{m+1}}+\ldots+F_{u_s}\\ &= F_1+\left(F_{2m+1}-F_1\right)+F_{u_{m+1}}+\ldots+F_{u_s}\\ &= F_{2m+1}+F_{u_{m+1}}+\ldots+F_{u_s}\end{aligned}$$

	Since $u_{m+1} > 2m+2 = (2m+1)+1$ we have that, $k+1 = F_{2m+1} + F_{u_{m+1}} + ... + F_{u_s} \in S$.
(cii)	Let P(n) be the statement that $n \in S$ for $n \geq 1$. Base Case: $1 = F_2 \in S$ So, P(1) is true. Inductive Step: Assume that P(k) is true for some $k \geq 1$, i.e. $k \in S$, $k = F_{u_1} + F_{u_2} + ... + F_{u_s}$ for some appropriate sequence $u_1, u_2, ..., u_s$. If $u_1 = 2$, from **(ci)** we are done. If $u_1 = 3$, we can use a similar argument from **(ci)**. Hence it is true. [For more details: If $u_i = 2i+1$ for all i. Then, $k = F_3 + F_5 + F_7 + ... + F_{2s+1}$ and $k+1 = 1 + (F_{2s+2} - F_2) = F_{2s+2} \in S$. If $u_i \neq 2i+1$ for some i, then there exist $m \geq 1$ where $u_{m+1} > 2m+3$. This implies, $k = F_3 + F_{u_2} + F_{u_3} + ... + F_{u_s}$ $\quad = F_3 + F_5 + ... + F_{2m+1} + F_{u_{m+1}} + ... + F_{u_s}$ Thus, $k+1 = F_1 + (F_3 + F_5 + ... + F_{2m+1}) + F_{u_{m+1}} + ... + F_{u_s}$ $\qquad = F_1 + (F_{2m+2} - F_2) + F_{u_{m+1}} + ... + F_{u_s}$ $\qquad = F_{2m+2} + F_{u_{m+1}} + ... + F_{u_s}$ Since $u_m > 2m+3 = (2m+2)+1$ we have that, $k+1 = F_{2m+2} + F_{u_{m+1}} + ... + F_{u_s} \in S$.] If $u_1 \geq 4$, $k+1 = F_2 + F_{u_1} + F_{u_2} + ... + F_{u_s} \in S$ as $u_1 \geq 4 > 2+1$. Hence P(k+1) is also true.

	Conclusion: As P(1) is true, as well as P(*k*) is true $\Rightarrow P(k+1)$ is true, by Principle of Mathematical Induction, P(*n*) is true for $n \geq 1$.

Qn	Solution
5(a)	$\begin{pmatrix} \frac{1}{2}(n-k)+k-1 \\ k-1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(n+k-2) \\ k-1 \end{pmatrix}$ Both *n* and *k* must be of the same parity.
(b)	Without restrictions, each kid has *k* flavours to choose from so there are k^n possible orders. Each order can include exactly *r* flavours where $1 \leq r \leq k$. There are $S(n,r)$ ways to divide the *n* kids into *r* disjoint, non-empty subsets. There are k flavours to assign to the first subset, $k-1$ flavours to assign to the second subset and so on. Thus there are $k(k-1)\ldots(k-r+1) = {}^kP_r$ ways to assign *r* flavours to the subsets.
(ci)	Note that the total number of possible orders is $k!S(n,k)$ (a special case from part (b)) Let A_i be the set of orders where flavour *i* is excluded, $1 \leq i \leq k$. Then $\left\|A_i\right\| = \ k-1\ ^n$ and there are $\begin{pmatrix} k \\ 1 \end{pmatrix}$ such A_i's. Similarly, we have $\left\|A_i \cap A_j\right\| = \ k-2\ ^n$ and there are $\begin{pmatrix} k \\ 2 \end{pmatrix}$ such pairs A_i, A_j. In general, we have $\left\|\bigcap_{j=1}^{r} A_{i_j}\right\| = \ k-r\ ^n$ and there are $\begin{pmatrix} k \\ r \end{pmatrix}$ such sets of $A_{i_1}, A_{i_2}, \ldots A_{i_r}$. Thus number of orders of *n* single scoop cones where no flavour is excluded is $k^n - \begin{pmatrix} k \\ 1 \end{pmatrix}(k-1)^n + \ldots + (-1)^r \begin{pmatrix} k \\ r \end{pmatrix}(k-r)^n + \ldots + (-1)^k \begin{pmatrix} k \\ k \end{pmatrix}(k-k)^n$ $= \sum_{r=0}^{k} (-1)^r \begin{pmatrix} k \\ r \end{pmatrix}(k-r)^n$ Thus combining this result with the answer in (b), we have $S(n,k) = \frac{1}{k!}\sum_{r=0}^{k}(-1)^r \begin{pmatrix} k \\ r \end{pmatrix}(k-r)^n$ where $c_r = (-1)^r \begin{pmatrix} k \\ r \end{pmatrix}$

Qn	Solution
6a	$a \in \{2^n, n \in \mathbb{Z}^+\}$ $b \in \mathbb{Z}^+$
Bi **bii** **ci**	$g^2f(x) = g^2(2x)$ $= 2x+2$ $= f(x+1)$ $= fg(x)$ (shown) $f^2g(x) = 4x+4 = fg^2f(x) = g^2fgf(x) = g^4f^2(x)$ From (b)(i), $f^2g(x) = f\left(g^2f\right)(x) = \left(g^2f\right)gf(x) = g^2\left(g^2f\right)f(x)$ The arrangements are f^2g, fg^2f, g^2fgf and g^4f^2. $g^ifg^jfg^k(x) = 4x+4k+2j+i$
cii	As the coefficient of x is 4, the function f must be applied twice. Therefore $g^ifg^jfg^k(x) = 4x+4k+2j+i$ where i, j, k are non-negative integers describes all possible arrangements of f and g which are equal to a function with a coefficient of x of 4. The number of arrangements that satisfy the condition is the number of non-negative integer solutions for i, j, k to the equation $4k+2j+i=4m$ <u>Method 1</u> As all terms are non-negative, $0 \le k \le m$. Then $j = 2(m-k) - \frac{i}{2}$, so $0 \le j \le 2(m-k)$. Hence there are $2m-2k+1$ corresponding choices for j, after which $i = 4m-4k-2j$ is fixed. The number of arrangements of f and g that are equal to $4x+4m$ is $\sum_{k=0}^{m}(2m-2k+1) = (2m+1)(m+1) - 2\sum_{k=1}^{m}k$ $= (2m+1)(m+1) - 2\frac{m(m+1)}{2}$ $= (m+1)^2$ (shown)

Method 2

$4k+2j+i=4m \Rightarrow i \equiv 0(\text{modulo } 2) \Rightarrow i=2i_0$ for $i_0 \in \mathbb{Z} \cup \{0\}$.

$4k+2j+2i_0=4m \Rightarrow 2k+j+i_0=2m \Rightarrow j, i_0$ both odd or even.

If both odd: $j=2j_1+1, i_0=2i_1+1, i_1, j_1 \in \mathbb{Z} \cup \{0\} \Rightarrow k+j_1+i_1=m-1$. As k, j_1, i_1 are non-negative integers, the number of solutions is $\binom{m-1+3-1}{3-1}=\binom{m+1}{2}$.

If both even: $j=2j_2, i_0=2i_2+1, i_2, j_2 \in \mathbb{Z} \cup \{0\} \Rightarrow k+j_2+i_2=m$. As k, j_2, i_2 are non-negative integers, the number of solutions is $\binom{m+3-1}{3-1}=\binom{m+2}{2}$.

Hence, the total number of ways is

$$\begin{aligned}\binom{m+1}{2}+\binom{m+2}{2} &= \frac{(m+1)(m)}{2}+\frac{(m+2)(m+1)}{2} \\ &= \frac{(m+1)}{2}(m+m+2) \\ &= (m+1)^2.\end{aligned}$$

Method 3 (Proof by Induction)

Let P_m be the proposition that the number of arrangements of f and g that are equal to the function $4x+4m$ is $(m+1)^2$.

P_1 : From (b)(ii), there are $(1+1)^2=4$ arrangements equal to $4x+4$. Therefore P_1 is true.

Assume P_n is true for some $m=n$, $n \in \mathbb{Z}^+$.

P_{n+1} :

As the coefficient of x is 4, the function f must be applied twice.

Therefore $\mathrm{g}^i\mathrm{fg}^j\mathrm{fg}^k(x)=4x+4k+2j+i$ where *i*, *j*, *k* are non-negative integers describes all possible arrangements of f and g which are equal to a function with a coefficient of *x* of 4.

The number of arrangements equal to $4x+4(n+1)$ is the number of non-negative integer solutions for *i*, *j*, *k* to the equation $4k+2j+i=4n+4$.

To count number of arrangements where $k \geq 1$:

This is the number of non-negative integer solutions for *i*, *j*, $k-1$ to the equation $4(k-1)+2j+i=4n$. By induction hypothesis there are $(n+1)^2$ such arrangements.

	<u>To count number of arrangements where $k=0$:</u> This is the number of non-negative integer solutions for *i*, *j* to the equation $2j+i=4n+4$. Note $0 \le j \le 2n+2$, following which there will be one solution for *i*, so there are $2n+3$ such arrangements. Hence there are a total of $(n+1)^2+2n+3=n^2+4n+4=(n+2)^2$ arrangements equal to $4x+4(n+1)$, so P_{n+1} is true. As P_1 is true and P_n is true $\Rightarrow$ P_{n+1} is true, P_m is true for all $m \in \mathbb{Z}^+$.
ciii	As all arrangements of f and g that are equal to $4x+4m$ or $4x+4m+1$ take the form $\text{g}^i\text{fg}^j\text{fg}^k$, we will represent them using the corresponding 3-tuples (i,j,k). Consider the mapping between the set of arrangements of f and g that are equal to the function $4x+4m$ to the set of arrangements of f and g that are equal to the function $4x+4m+1$ defined by: $(i,j,k) \mapsto (i+1,j,k)$ <u>This map is injective:</u> As $(\alpha+1,\beta,\gamma)=(i+1,j,k) \Rightarrow (\alpha,\beta,\gamma)=(i,j,k)$. <u>This map is surjective:</u> Let (α,β,γ) represent an arrangement of f and g that is equal to the function $4x+4m+1$. As the constant term $4m+1$ is odd, then α must be odd, and in particular $\alpha>0$. Hence $(\alpha-1,\beta,\gamma)$ represents an arrangement equal to the function $4x+4m$ that maps to (α,β,γ). Therefore the mapping is a bijection, and there are $(m+1)^2$ arrangements of f and g that are equal to the function $4x+4m+1$.

Qn	Solution
7(ai)	Let $b=a+1$. $11^2+a^2=(a+1)^2 \Rightarrow 2a+1=121$ So $a=60 \Rightarrow$ $(11,60,61)$ is a Pythagorean triple.
(aii)	Let $(2k+1)^2+m^2=(m+1)^2$ $(2k+1)^2=2m+1 \Rightarrow m=2k^2+2k$ So $(2k+1, 2k^2+2k, 2k^2+2k+1)$ is a Pythagorean triple for all positive integers *k*.

	This infinite family of triples is primitive since $2k^2+2k$ and $2k^2+2k+1$ are consecutive integers, so no primes divide both of them.
(bi)	$\gcd(2,2,3)=1$ but $\gcd(2\times 2, 2\times 3, 3\times 2)=2$
(bii)	Let (x,y,z) be a primitive Pythagorean triple. Claim: $\gcd(x,y)=1$. Proof: Suppose otherwise for contradiction. Let $p\mid x,y$, *p* prime. Then $p\mid x^2+y^2=z^2$. So by Euclid's Lemma, $p\mid z$ too. But this contradicts $\gcd(x,y,z)=1$. Similarly, $\gcd(x,z)=\gcd(y,z)=1$. (So, (x,y,z) are pairwise coprime.) Claim: $\gcd(xy,yz,zx)=1$. Proof: Let *p* be a prime. We need to prove that if *p* divides any one of the terms, it does not divide another. WLOG, Let $p\mid xy$. We need to prove that *p* does not divide *yz* or *p* does not divide <u>*xz*</u>. Since *p* is prime, $p\mid xy \Rightarrow p\mid x \text{ or } p\mid y$ by Euclid's Lemma. WLOG, $p\mid x$. By earlier claim, $\gcd(x,y)=1$, so *p* does not divide *y*. Similarly, *p* does not divide *z*. Since *p* is prime, and *p* divides neither *y* nor *z*, *p* does not divide *yz* (shown). <u>Alternatively</u> Suppose for contradiction that $\gcd(x,y,z)\neq 1$. Then $p\mid xy, xz, yz$ for some prime *p*. Since *p* is prime, $p\mid xy \Rightarrow p\mid x \text{ or } p\mid y$ by Euclid's Lemma. WLOG, $p\mid x$. Since *p* is prime, $p\mid yz \Rightarrow p\mid y \text{ or } p\mid z$ by Euclid's Lemma. WLOG, $p\mid y$ So $p\mid x^2+y^2=z^2 \Rightarrow p\mid z$ by Euclid's Lemma This means that $p\mid \gcd(x,y,z)$, which contradicts $\gcd(x,y,z)=1$. (The other cases are similar.)
c	$\left(z^4-x^2y^2\right)^2=z^8-2x^2y^2z^4+(xy)^4$, so it suffices to prove that $(yz)^4+(zx)^4=z^8-2x^2y^2z^4$, or equivalently, that $x^4+y^4=z^4-2x^2y^2$. Substituting $x^2+y^2=z^2$ into RHS, $z^4-2x^2y^2=\left(x^2+y^2\right)^2-2x^2y^2$ $=x^4+y^4=$ LHS (shown)

<table>
<tr><td>d</td><td>By (c) and (bii), each primitive Pythagorean triple (x, y, z) gives rise to an integer solution $\left(xy, yz, zx, z^4 - x^2y^2\right)$ to $u^4 + v^4 + w^4 = t^2$ such that $\gcd(u, v, w) = 1$.
By (aii), there are infinitely many primitive Pythagorean triples of the form $\left(2k+1, 2k^2+2k, 2k^2+2k+1\right)$.
Different primitive triples are associated with different integer solutions, since each component of $\left(2k+1, 2k^2+2k, 2k^2+2k+1\right)$ is an increasing function of k, so products of them are also increasing functions.
(E.g. $k_1 < k_2 \Rightarrow \left(2k_1^2+2k_1\right)\left(2k_1^2+2k_1+1\right) < \left(2k_2^2+2k_2\right)\left(2k_2^2+2k_2+1\right)$)

<u>Alternatively</u>
Suppose $(xy, yz, zx) = (ab, bc, ca) = (p, q, r)$, then
$x = a = \frac{\sqrt{pqr}}{q}, y = b = \frac{\sqrt{pqr}}{r}, z = c = \frac{\sqrt{pqr}}{p}$.
This implies that if $(x, y, z) \neq (a, b, c) \Rightarrow (xy, yz, zx) \neq (ab, bc, ca)$
So there are infinitely many integer solutions to $u^4 + v^4 + w^4 = t^2$ such that $\gcd(u, v, w) = 1$ as required.</td></tr>
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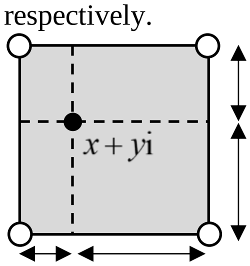
<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>8a</td><td>Let $z = x + y\text{i}$ where x and y are real. Let x_w and y_w be the integers closest to x and y respectively.


Clearly $\left|x - x_w\right| \le 0.5$ and $\left|y - y_w\right| \le 0.5$, so $\left|z - w\right| \le \sqrt{0.5^2 + 0.5^2} = \frac{1}{\sqrt{2}}$.
[Or: Since $\left(\lceil x \rceil - x\right) + \left(x - \lfloor x \rfloor\right) = 1 \text{ or } 0$, the smaller of these is always ≤ 0.5.]

Alternative presentation:

Divide the complex plane into unit squares centred on the Gaussian integers.
Each point in the complex plane must lie in one of these squares. (Use any of the adjoining squares for points on the edges or vertices.)
Since the maximum distance of a point on a square from the Gaussian integer at its centre is $\frac{1}{2}\sqrt{1^2 + 1^2} = \frac{1}{\sqrt{2}}$, applying this to the point representing z, we obtain the result.</td></tr>
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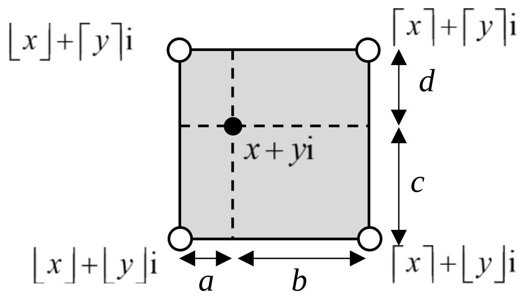
b	Applying the result from (a) to the complex number $\frac{s}{t}$, there is a complex number *q* such that $\left\lvert\frac{s}{t}-q\right\rvert \le \frac{1}{\sqrt{2}} < 1$. Multiplying both sides by $\lvert t\rvert$, $\left\lvert\frac{s}{t}-q\right\rvert\lvert t\rvert < \lvert t\rvert$ $\lvert s-qt\rvert < \lvert t\rvert$ Let $r = s - qt$, which is a Gaussian integer because *s*, *q* and *t* are Gaussian integers. The Gaussian integers *q* and *r* defined, meet the required conditions.
c	$\frac{s}{t} = \frac{5+4\mathrm{i}}{1+2\mathrm{i}} = 2.6-1.2\mathrm{i}$ Consider Gaussian integers *w* with $\mathrm{Re}(w) = 2$ or 3, and $\mathrm{Im}(w) = -2$ or -1. All other Gaussian integers will have either $\lvert\mathrm{Re}(w)-2.6\rvert > 1$ or $\lvert\mathrm{Im}(w)-(-1.2)\rvert > 1$, and thus $\left\lvert\frac{s}{t}-w\right\rvert > 1$. We have $\left\lvert\frac{s}{t}-(2-\mathrm{i})\right\rvert = 0.6^2+0.2^2 < 1$ $\left\lvert\frac{s}{t}-(3-\mathrm{i})\right\rvert = 0.4^2+0.2^2 < 1$ $\left\lvert\frac{s}{t}-(2-2\mathrm{i})\right\rvert = 0.6^2+0.8^2 = 1$ $\left\lvert\frac{s}{t}-(3-2\mathrm{i})\right\rvert = 0.4^2+0.8^2 < 1$ so there are exactly 3 pairs of Gaussian integers (q, r), namely $(2-\mathrm{i}, 1+\mathrm{i})$, $(3-\mathrm{i}, -\mathrm{i})$ and $(3-2\mathrm{i}, -2)$.
d	The problem is equivalent to showing that for all Gaussian integers *s*, *t* such that $\frac{s}{t}$ is not a Gaussian integer, there exist at least two Gaussian integers *q* such that $\left\lvert\frac{s}{t}-q\right\rvert < 1$, i.e. less than 1 unit away from $\frac{s}{t}$. Let $z = \frac{s}{t} = x + y\mathrm{i}$ where *x* and *y* are real. If *x* is integer, then *y* is not an integer. Then $q = x + \lceil y \rceil\mathrm{i}$ and $q = x + \lfloor y \rfloor\mathrm{i}$ are distinct Gaussian integers satisfying the condition. Similarly, if *y* is integer, then *x* is not an integer, so $q = \lceil x \rceil + y\mathrm{i}$ and $q = \lfloor x \rfloor + y\mathrm{i}$ are distinct Gaussian integers satisfying the condition. The last case is where neither *x* nor *y* are integer. Then:

Method 1

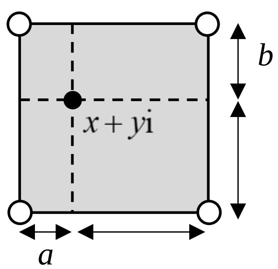
Let $a = x - \lfloor x \rfloor$, $b = \lceil x \rceil - x$, $c = y - \lfloor y \rfloor$, $d = \lceil y \rceil - y$ (see diagram)



Observe that $a^2 + b^2 < a^2 + 2ab + b^2 = (a+b)^2 = 1$ and similarly $c^2 + d^2 < 1$, so that $a^2 + b^2 + c^2 + d^2 < 2$.

Grouping the terms, we get $\left(a^2 + c^2\right) + \left(b^2 + d^2\right) < 2$. This implies that either $a^2 + c^2 < 1$ or $b^2 + d^2 < 1$, so either $\lfloor x \rfloor + \lfloor y \rfloor \mathrm{i}$ or $\lceil x \rceil + \lceil y \rceil \mathrm{i}$ is less than 1 unit away from $x + y\mathrm{i}$.
Similarly, by grouping the sum as $\left(a^2 + d^2\right) + \left(b^2 + c^2\right) < 2$, either $\lceil x \rceil + \lfloor y \rfloor \mathrm{i}$ or $\lfloor x \rfloor + \lceil y \rceil \mathrm{i}$ is less than 1 unit away from $x + y\mathrm{i}$.
So we have found two Gaussian integers less than 1 unit away from z.

Method 2



Consider the unit square containing z with vertices on Gaussian integers. Define a to be the shortest distance from z to an edge of the square, and define b to be the next shortest such distance, such that the distance from z to the nearest Gaussian integer at a vertex of the square is $\sqrt{a^2 + b^2}$. Clearly $0 < a \le b \le \dfrac{1}{2}$, so this distance is less than 1.

But observe that

	$\sqrt{a^2+(1-b)^2} \le \sqrt{b^2+(1-b)^2}$ $< \sqrt{b^2+2b(1-b)+(1-b)^2}$ $= \sqrt{[b+(1-b)]^2}$ $= 1$ i.e. there is a second vertex (Gaussian integer) less than distance 1 away from *z*, which is what we wanted. <u>Method 3</u> Consider the unit square containing *z* with vertices on Gaussian integers. Draw a quadrant of a unit circle centred at the lower-left vertex, and another centred at the upper-right vertex. Since every point in the interior of the square lies in the interior of at least one of these two quadrants, at least one of the two Gaussian integers represented by the vertices at the lower-left and upper-right lies within distance 1 of *z*.  Similarly, by considering quadrants centred at the upper-left and lower-right vertices, at least one of these is a Gaussian integer within distance 1 of *z*. So there are at least 2 such Gaussian integers.