Qn	Solution	Mark	Remarks
1(a)(i)	$\arg(a-\mathrm{i}b)=-\theta$		
(ii)	$\arg(b+\mathrm{i}a)=\arg(\mathrm{i})(a-\mathrm{i}b)=\frac{\pi}{2}-\theta$		
1(b)	$\mathrm{i}z+2w=0------(1)$ $z-w^*=3\Rightarrow z=3+w^*------(2)$ Sub (2) into (1) $\mathrm{i}(3+w^*)+2w=0$ $3\mathrm{i}+\mathrm{i}w^*+2w=0$ Let $w=x+\mathrm{i}y$ $3\mathrm{i}+\mathrm{i}(x-\mathrm{i}y)+2(x+\mathrm{i}y)=0$ $(y+2x)+\mathrm{i}(3+x+2y)=0$ Comparing real and imaginary parts, $y+2x=0------(3)$ $3+x+2y=0----(4)$ Sub (3) into (4) $3+x-4x=0$ $x=1$ $y=-2$ $\therefore w=1-2\mathrm{i}$ $z=3+1+2\mathrm{i}=4+2\mathrm{i}$		
2	Given $\frac{\mathrm{d}\theta}{\mathrm{d}t}\propto\theta-25$, $\therefore\ \frac{\mathrm{d}\theta}{\mathrm{d}t}=-k(\theta-25)$ $\int\frac{1}{\theta-25}\,\mathrm{d}\theta=-\int k\,\mathrm{d}t$ $\ln\lvert\theta-25\rvert=-kt+c$ $\lvert\theta-25\rvert=\mathrm{e}^{-kt+c}$ $\theta-25=\pm\mathrm{e}^{c}\mathrm{e}^{-kt}$ $\theta=A\mathrm{e}^{-kt}+25$, where $A=\pm\mathrm{e}^{c}$ When $t=0$, $\theta=32$ $32=A\mathrm{e}^{0}+25\Rightarrow A=7$ When $t=1$, $\theta=30$ $30=7\mathrm{e}^{-k}+25\Rightarrow k=-\ln\frac{5}{7}$ $\theta=7\mathrm{e}^{\left(\ln\frac{5}{7}\right)t}+25$ $\theta=7\left(\frac{5}{7}\right)^{t}+25$ (Shown) When $\theta=37$, $t=\frac{\ln\left(\frac{12}{7}\right)}{\ln\left(\frac{5}{7}\right)}=-1.60$ (3 sf)		

	The time of death is 10.24pm		
3(i)	$\mathrm{f}(r) = 2r^3+3r^2+r$ $\mathrm{f}(r)-\mathrm{f}(r-1)$ $=2r^3+3r^2+r-\left[2(r-1)^3+3(r-1)^2+(r-1)\right]$ $=2r^3+3r^2+r-\left[2\left(r^3-3r^2+3r-1\right)+3\left(r^2-2r+1\right)+(r-1)\right]$ $= 2r^3+3r^2+r-\left(2r^3-6r^2+6r-2+3r^2-6r+3+r-1\right)$ $=2r^3+3r^2+r-\left(2r^3-3r^2+r\right)= \underline{\underline{6r^2}}$ $\sum_{r=1}^{n} r^2 = \frac{1}{6}\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]$ $=\frac{1}{6}[\ \mathrm{f}(1)-\mathrm{f}(0)$ $+\mathrm{f}(2)-\mathrm{f}(1)$ $+\mathrm{f}(3)-\mathrm{f}(2)$ $\vdots$ $+\mathrm{f}(n)-\mathrm{f}(n-1)]$ $= \frac{1}{6}\left[\mathrm{f}(n)-\mathrm{f}(0)\right]$ $= \frac{1}{6}\left(2n^3+3n^2+n\right)$ $= \frac{1}{6}n\left(2n^2+3n+1\right)$ $= \underline{\underline{\frac{1}{6}n(n+1)(2n+1)}}$		
3(ii)	Let P_n be the statement "$\sum_{r=1}^{n}\left(4r^3+3r^2+r\right)=n(n+1)^3$" for all $n\in\mathbb{Z}^+$. LHS of P_1 $= 4+3+1=8$ RHS of P_1 $= (1)(1+1)^3=8$ $\therefore P_1$ is true. Assume P_k is true for some $k\in\mathbb{Z}^+$, i.e., $\sum_{r=1}^{k}\left(4r^3+3r^2+r\right) = k(k+1)^3$.		

	We want to prove that P_{k+1} is true, i.e., $\sum_{r=1}^{k}\left(4r^3+3r^2+r\right)=\left(k+1\right)\left[\left(k+1\right)+1\right]^3=\left(k+1\right)\left(k+2\right)^3$ LHS of P_{k+1} $=\sum_{r=1}^{k}\left(4r^3+3r^2+r\right)+4\left(k+1\right)^3+3\left(k+1\right)^2+\left(k+1\right)$ $=k\left(k+1\right)^3+4\left(k+1\right)^3+3\left(k+1\right)^2+\left(k+1\right)$ $=\left(k+1\right)\left[k\left(k+1\right)^2+4\left(k+1\right)^2+3\left(k+1\right)+1\right]$ $=\left(k+1\right)\left[k\left(k^2+2k+1\right)+4\left(k^2+2k+1\right)+3k+3+1\right]$ $=\left(k+1\right)\left(k^3+2k^2+k+4k^2+8k+4+3k+3+1\right)$ $=\left(k+1\right)\left(k^3+6k^2+12k+8\right)$ $=\left(k+1\right)(k+2)\left(k^2+4k+4\right)$ $=\left(k+1\right)\left(k+2\right)^3$ $=$ RHS of P_{k+1} Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, $\therefore$ by mathematical induction, P_n is true for all $n\in\mathbb{Z}^{+}$.		
3(iii)	$\sum_{r=1}^{n}\left(4r^3+3r^2+r\right) = n\left(n+1\right)^3$ $4\sum_{r=1}^{n}r^3+3\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}r = n\left(n+1\right)^3$ $4\sum_{r=1}^{n}r^3 = n\left(n+1\right)^3-3\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r$ $= n\left(n+1\right)^3-3\times\frac{1}{6}n\left(n+1\right)\left(2n+1\right)-\frac{n}{2}\left(n+1\right)$ $= n\left(n+1\right)\left[\left(n+1\right)^2-\frac{1}{2}\left(2n+1\right)-\frac{1}{2}\right]$ $= n\left(n+1\right)\left(n^2+2n+1-n-1\right) = n\left(n+1\right)\left(n^2+n\right)$ $= n\left(n+1\right)\left(n\right)\left(n+1\right)$ $= \left[n\left(n+1\right)\right]^2$ $\therefore \sum_{r=1}^{n}r^3 = \underline{\underline{\frac{1}{4}\left[n\left(n+1\right)\right]^2}}$ or $\underline{\underline{\left[\frac{1}{2}n\left(n+1\right)\right]^2}}$		

<table>
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<td>4(a) (i)
(ii)</td>
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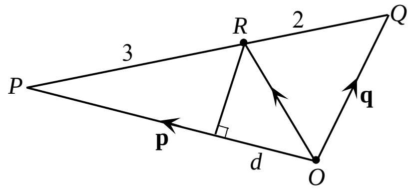
$\overrightarrow{OR} = \dfrac{2\mathbf{p}+3\mathbf{q}}{5}$

$d = \dfrac{\left|\overrightarrow{OR}\cdot\overrightarrow{OP}\right|}{\left|\overrightarrow{OP}\right|}$

$= \dfrac{\left|\frac{2\mathbf{p}+3\mathbf{q}}{5}\cdot\mathbf{p}\right|}{|\mathbf{p}|}$

$= \dfrac{|2\mathbf{p}\cdot\mathbf{p}+3\mathbf{q}\cdot\mathbf{p}|}{5|\mathbf{p}|}$



$= \dfrac{\left|2|\mathbf{p}|^2+3\mathbf{p}\cdot\mathbf{q}\right|}{5|\mathbf{p}|}$ (Shown)

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<td>4(b)(i)</td>
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<u>Method 1</u>

$l_1:\ \mathbf{r} = \begin{pmatrix}1\\0\\-3\end{pmatrix} + \lambda\begin{pmatrix}k\\-1\\1\end{pmatrix},\ \lambda\in\mathbb{R}$

$l_2:\ x-4 = y+1 = \dfrac{z+3}{2} \Rightarrow \dfrac{x-4}{1} = \dfrac{y-(-1)}{1} = \dfrac{z-(-3)}{2}$

$\Rightarrow l_2:\ \mathbf{r} = \begin{pmatrix}4\\-1\\-3\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix},\ \mu\in\mathbb{R}$

When l_1 and l_2 intersect,

$\begin{pmatrix}1\\0\\-3\end{pmatrix} + \lambda\begin{pmatrix}k\\-1\\1\end{pmatrix} = \begin{pmatrix}4\\-1\\-3\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix}$

$\begin{pmatrix}1+\lambda k\\-\lambda\\-3+\lambda\end{pmatrix} = \begin{pmatrix}4+\mu\\-1+\mu\\-3+2\mu\end{pmatrix}$

$\lambda k - \mu = 3$ (1)

$\lambda + \mu = 1$ (2)

$\lambda - 2\mu = 0$ (3)

Solving (2) and (3),

$\lambda = \dfrac{2}{3},\ \mu = \dfrac{1}{3}$

Sub. into (1),

$\dfrac{2}{3}k - \dfrac{1}{3} = 3$

$\dfrac{2}{3}k = \dfrac{10}{3}$

$k = 5$ (Shown)

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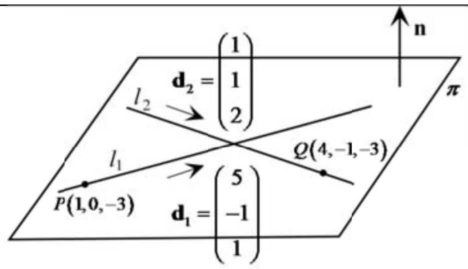
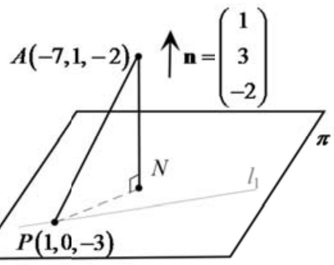
4(b)(i)	Method 2 $l_1: \quad \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} k \\ -1 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+\lambda k \\ -\lambda \\ -3+\lambda \end{pmatrix}$ $x = 1+\lambda k, \ y = -\lambda, \quad z = -3+\lambda \qquad (1)$ $l_2: \quad x-4 = y+1 = \frac{z+3}{2} \qquad (2)$ Sub. (1) into (2), $1+\lambda k-4 = -\lambda+1 = \frac{-3+\lambda+3}{2}$ $\Rightarrow \lambda k-3 = -\lambda+1 = \frac{\lambda}{2}$ $-\lambda+1 = \frac{\lambda}{2} \Rightarrow \frac{3}{2}\lambda = 1 \Rightarrow \lambda = \frac{2}{3}$ and $\lambda k+\lambda = 4 \qquad (3)$ Sub. $\lambda = \frac{2}{3}$ into (3), $\frac{2}{3}k+\frac{2}{3} = 4$ $\frac{2}{3}k = \frac{10}{3}$ $k = 5$ [Shown]		

4(b)(ii)	$\mathbf{d}_1 \times \mathbf{d}_2 = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$  $= \begin{pmatrix} -2-1 \\ -(10-1) \\ 5+1 \end{pmatrix}$ $= \begin{pmatrix} -3 \\ -9 \\ 6 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ A normal to π is $\mathbf{n} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$. Since $P(1, 0, -3)$ lies on l_1, $\therefore$ P also lies in π. $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 1 + 6 = 7$ [Note: Other points in π that can be used are $Q(4,-1,-3)$ on l_2, the pt of intersection of l_1 and l_2, $R\left(\frac{13}{3}, -\frac{2}{3}, -2\frac{1}{3}\right)$, etc.] An equation of π is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 7$.		
4(b)(iii)	Let N be the foot of perpendicular from $A(-7, 1, -2)$ to π. Method 1 Use dot product $\overrightarrow{AP} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} -7 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix}$ (recall $P(1, 0, -3)$ lies in π) Perpendicular distance from A to π $= \dfrac{\left\|\overrightarrow{AP} \cdot \mathbf{n}\right\|}{\|\mathbf{n}\|}$ $= \dfrac{\left\|\begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}\right\|}{\left\|\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}\right\|}$  $= \dfrac{\|8-3+2\|}{\sqrt{14}} = \dfrac{7}{\sqrt{14}}$ or $\dfrac{\sqrt{14}}{2}$ or 1.87 (3 sf)		

4(b)(iii)

<u>Method 2 Find N by solving eqns of AN and π</u>

Vector equation of AN: $\mathbf{r} = \begin{pmatrix} -7 \\ 1 \\ -2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ (1)

Equation of π: $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 7$ (2)

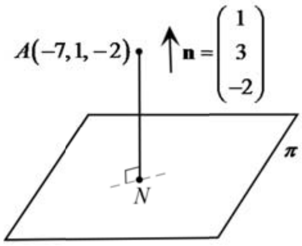
Sub. (1) into (2),

$$\begin{pmatrix} -7+\alpha \\ 1+3\alpha \\ -2-2\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 7$$

$$-7+\alpha+3+9\alpha+4+4\alpha = 7$$

$$14\alpha = 7$$

$$\alpha = \frac{1}{2}$$



Sub. into (1),

$$\overrightarrow{ON} = \begin{pmatrix} -7+\frac{1}{2} \\ 1+\frac{3}{2} \\ -2-2\left(\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} -\frac{13}{2} \\ \frac{5}{2} \\ -3 \end{pmatrix}$$

$$\overrightarrow{AN} = \begin{pmatrix} -\frac{13}{2} \\ \frac{5}{2} \\ -3 \end{pmatrix} - \begin{pmatrix} -7 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -1 \end{pmatrix}$$

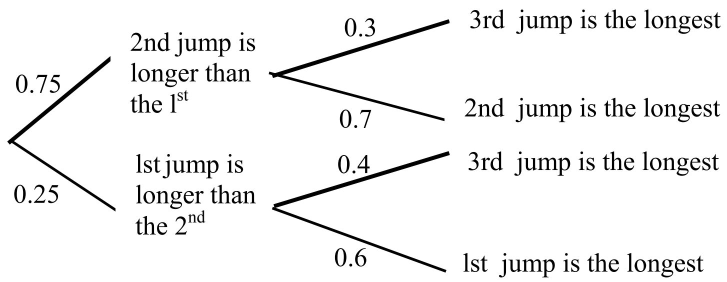
Perpendicular distance from A to π

$$= \left|\overrightarrow{AN}\right|$$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + (-1)^2}$$

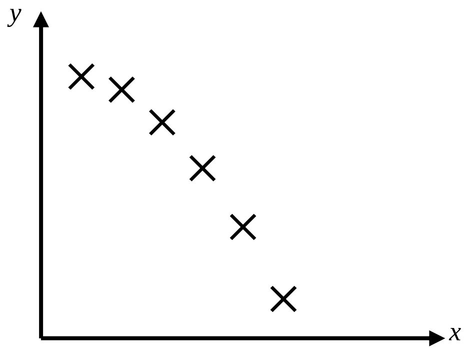
$$= \underline{\underline{\sqrt{\frac{7}{2}}}} \text{ or } \underline{\underline{\frac{\sqrt{14}}{2}}} \text{ or } \underline{\underline{1.87}} \text{ (3 sf)}$$

5(a)	Number all the sections from 1 to 90 and then generate 15 random numbers (from 1 to 90) to get a sample of 15 sections. Free from bias since every section has an equal chance of being selected/Easy to conduct		
5(b)	Systematic sampling. Teams are evenly spread out along the trial making supervision difficult.		
6(a)	Let X = Weight of a bag of oats. From GC, unbiased estimate of $\mu = \bar{x} = 991.2$ unbiased estimate of $\sigma^2 = s^2 = 23.967^2 = 574.42$ $H_0: \mu = 1000$ $H_1: \mu < 1000$ Since n = 10 is small, we need to assume that X has a normal distribution. Under H_0, Test statistic , $T = \dfrac{\bar{X} - 1000}{\sqrt{\frac{S^2}{10}}} \sim t(9)$ $\alpha = 0.05$ From GC, p-value = 0.13773 = 0.138 Since p-value = 0.138 > α = 0.05, we do not reject H_0 at the 5% level of significance and conclude there is insufficient evidence that Sheena's suspicion is valid/the mean weight is less than 1 kg. Assumption: The population distribution of the weight of a bag of oats is a normal distribution.		
6(b)	Because the sample mean is more than 1 kg, test statistic is positive and does not fall inside the critical region. The conclusion is unreliable because the sample is not random/may be biased/the manager may select relatively heavy bags.		
7(a)(i)	No. of numbers formed $= 4! \times 2! = 48$		
7(a) (ii)	No of numbers where I_1 is first = 4! No of numbers where I_5 is last $= 4!$ No. of numbers where I_1 is first and I_5 is last $= 3!$ No. of numbers formed $= 4! + 4! - 3! = 42$		

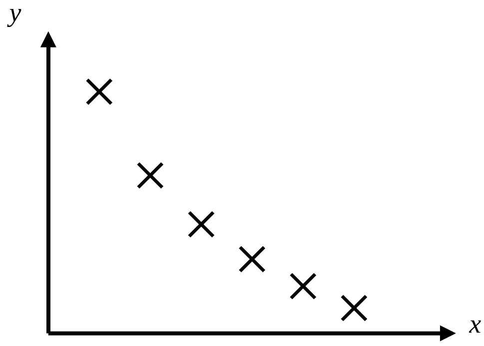
7(b) (i)	 P(Ivan improves on each subsequent jump) $= 0.75 \times 0.3 = 0.225$		
7(b) (ii)	P(Ivan improves on his lst jump) $= 0.75 + 0.25 \times 0.4$ or $0.75 \times 0.3 + 0.75 \times 0.7 + 0.25 \times 0.4$ $= 0.85$ [shown]		
7(b) (iii)	P(2nd jump is longest | Ivan improves on lst jump) $= \dfrac{\text{P(2nd jump is longest and improves on lst jump)}}{\text{P(Ivan improves on his lst jump)}}$ $= \dfrac{\text{P(2nd jump is the longest)}}{\text{P(Ivan improves on his lst jump)}}$ $= \dfrac{0.75 \times 0.7}{0.85} = \dfrac{21}{34}$ or 0.618		
8(a)	Let T = Time taken by Augustine to complete the course. Then $T \sim \text{N}(45.1,\ 1.9^2 + 1.5^2 + 1.3^2) = \text{N}(45.1,\ 7.55)$. $\text{P}(T < 47) = 0.75537 \approx 0.755$		
8(b) (i)	Let X = Time taken by Augustine to complete the running Section. & Y = Time taken by Anand to complete the running section. Then $X \sim \text{N}(10.6,\ 1.3^2)$ and $Y \sim \text{N}(11.0,\ 1.9^2)$. P(Augustine breaks the record) $= \text{P}(X < 9.8) = 0.26915$ P(Anand breaks the record) $= \text{P}(Y < 9.8) = 0.26383$ Since $\text{P}(X < 9.8) > \text{P}(Y < 9.8)$, Augustine is more likely to break the record.		
8(b) (ii)	$X - 4Y \sim \text{N}(10.6 - 4(11.0),\ 1.3^2 + 4^2(1.9^2)) = \text{N}(-33.4,\ 59.45)$ $\text{P}(34.2 + X > 4Y)$ $= \text{P}(X - 4Y > -34.2)$ $= 0.54132 \approx 0.541$		

9(i)

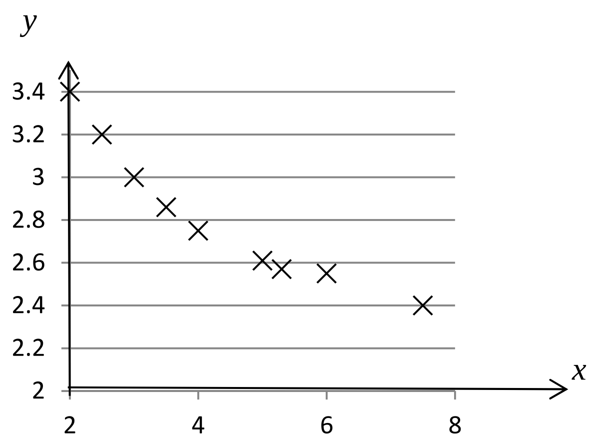
(i) (A) $y = a + bx^2$ where $a > 0,\ b < 0.$



(B) $y = c + d\ln x$ where $c > 0,\ d < 0.$



9(ii)



9(iii)

Case (B) is more appropriate since as x increases, y decreases at a decreasing rate.

$r \approx -0.988$.

9(iv)

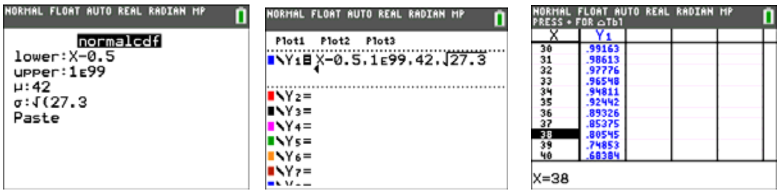
$y = 3.8616 - 0.75736\ln x$

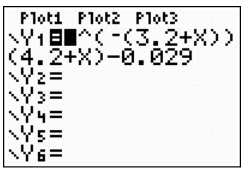
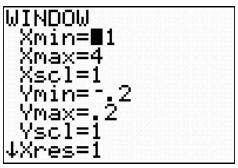
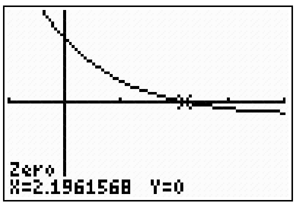
$y = 3.86 - 0.757\ln x$

When $x = 4.5,\ y = 3.8616 - 0.75736\ln(4.5)$

$\approx 2.7225 = 2.72\ (3\text{sf})$

The estimate will be reliable since r is close to -1 indicating a strong negative linear relationship between y and ln$x$, and x = 4.5 lies within the given data range.

10(i)	The two assumptions are: 1. Friends are logged on to the site independently. 2. The probability that any one friend is logged onto to the site is the same (0.35).		
10(ii)	The first assumption mentioned in part (i) may not be valid as the log-ons of her friends may not be independent as her friends may be out together or arrange to chat on the site etc. The second assumption mentioned in part (i) may not be valid as the probability of log-on may vary from person to person because of different lifestyles and schedules etc.		
10(iii)	Given X = No. of friends, out of 8, logged on to the site. Then $X \sim B(8, 0.35)$ $P(X > 4) = 1 - P(X \le 4) = 0.10609 \approx 0.106$		
10(iv)	Given L = No. of friends, out of 120, logged on to the site. $L \sim B(120, 0.35)$ Since n is large, $np = 120(0.35) = 42(>5)$ and $nq = 78\ (>5)$, $L \sim N(42, 27.3)$ approximately. Given $P(L \ge n) > 0.8 \xrightarrow{\text{c.c.}} P(L > n - 0.5) > 0.8$  From GC, greatest value of $n = 38$. <u>Alternatives:</u> $P\left(Z > \frac{n-0.5-42}{\sqrt{27.3}}\right) > 0.8$ Or $P(L > n-0.5) > 0.8$ $\Rightarrow P\left(Z \le \frac{n-42.5}{\sqrt{27.3}}\right) < 0.2$ $P(L \le n-0.5) < 0.2$ $\Rightarrow \frac{n-42.5}{\sqrt{27.3}} < -0.84162$ $n - 0.5 < 37.603$ $\Rightarrow n < 38.103$ $n < 38.103$ $\therefore$ Greatest value of $n = 38$.		
11(a)	Let W = No. of travellers arriving alone in a 2-minute interval. $W \sim Po(3.2 \times 2) = Po(6.4)$ $P(W < 6)$ $= P(W \le 5) = 0.38374 \approx 0.384$ (3 sf)		

11(b)	<u>Method 1:</u> Since n = 100 is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(3.2\ ,\dfrac{3.2}{100}\right)$ approximately. $\mathrm{P}(\overline{X} \le 3.6)$ $= 0.98733 \approx 0.987$ (3 sf) <u>Method 2:</u> Since n = 100 is large, by Central Limit Theorem, $X_1 + X_2 + ... + X_{100} \sim \mathrm{N}\left[3.2(100), 3.2(100)\right] = \mathrm{N}(320, 320)$ approx. $\mathrm{P}(\overline{X} \le 3.6)$ $= \mathrm{P}(X_1 + X_2 + ... + X_{100} \le 360) = 0.98733 \approx 0.987$ (3 sf) <u>Method 3:</u> $X_1 + X_2 + ... + X_{100} \sim \mathrm{Po}\left(3.2(100)\right) = \mathrm{Po}(320)$. Since mean = 320 is large, $X_1 + X_2 + ... + X_{100} \sim \mathrm{N}(320, 320)$ approximately. $\mathrm{P}(\overline{X} \le 3.6)$ $= \mathrm{P}(X_1 + X_2 + ... + X_{100} \le 360)$ $\xrightarrow{cc} \mathrm{P}(X_1 + X_2 + ... + X_{100} < 360.5)$ $= 0.98821 \approx 0.988$ (3 sf)		
11(c)	$T = X + Y \sim \mathrm{Po}(3.2 + \lambda)$ $\mathrm{P}(T \ge 2) = 0.971$ $\mathrm{P}(T \le 1) = 0.029$ $\Rightarrow \mathrm{P}(T = 0) + \mathrm{P}(T = 1) = 0.029$ $\Rightarrow \mathrm{e}^{-(3.2+\lambda)}\left[1 + (3.2 + \lambda)\right] = 0.029$ $\Rightarrow \mathrm{e}^{-(3.2+\lambda)}(4.2 + \lambda) - 0.029 = 0$    From GC, λ = 2.1962 ≈ 2.2 (1dp) [Shown]		

11(d) **(i)**	$T \sim \text{Po}(3.2+2.2)=\text{Po}(5.4)$ $\text{P}(T=6)$ $=0.15554 \approx 0.156$ (3 sf)		
(ii)	$\text{P}(Y<2 \mid T=6)$ $=\dfrac{\text{P}(Y=1 \text{ and } X=5)+\text{P}(Y=0 \text{ and } X=6)}{\text{P}(T=6)}$ $=\dfrac{\text{P}(Y=1)\text{P}(X=5)+\text{P}(Y=0)\text{P}(X=6)}{\text{P}(T=6)}$ $=\dfrac{0.24377\times 0.11398+0.11080\times 0.06079}{0.15554}$ $=0.22194 \approx 0.222$ (3 sf)		
11(e) **(i)**	The average number of travellers arriving alone at the station per minute may not remain constant for several hours because of specific train schedule.		
(ii)	$X+Y$ may not be appropriate because some travellers (whether alone or in family groups) may have bought the tickets earlier.		