TEMASEK JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 3

H3 MATHEMATICS **9820/01**

Paper 1 **22 September 2023**

3 hours

Additional Materials: Answer Booklet
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Write your Civics Group and name on all the work that you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 For each $i=1, 2, ..., n$, let a_i and b_i be non-zero real numbers.

(i) By considering $\sum_{i=1}^{n}(a_i t - b_i)^2 \geq 0$ for all real t, prove the Cauchy Schwarz's inequality

$$\left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2 .$$ [3]

(ii) Let p and q be positive real numbers. If for each $i=1, 2, ..., n$, $p \leq \dfrac{b_i}{a_i} \leq q$, show that $pqa_i^{\,2} - (p+q)a_i b_i + b_i^{\,2} \leq 0$ and deduce that

$$(p+q)\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} b_i^{\,2} + pq\sum_{i=1}^{n} a_i^{\,2} .$$ [3]

(iii) Let $m, M \in \mathbb{R}^+$ be such that $m \leq a_i \leq M$ and $m \leq b_i \leq M$ for each $i=1, 2, ..., n$. By using the result in part **(ii)**, or otherwise, show that

$$\left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right) \leq \frac{1}{4}\left(\frac{M}{m}+\frac{m}{M}\right)^2\left(\sum_{i=1}^{n} a_i b_i\right)^2 .$$ [3]

2 A differential equation of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$ is said to be homogeneous if $\mathrm{f}(tx, ty) = \mathrm{f}(x, y)$ for any real value t.

(i) Show that such a differential equation can be written in the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{g}\left(\dfrac{x}{y}\right)$, where g is a function of $\dfrac{x}{y}$. [2]

(ii) Show that $2xy\mathrm{e}^{\left(\frac{x}{y}\right)^2}\dfrac{\mathrm{d}x}{\mathrm{d}y} = y^2 + (y^2 + 2x^2)\mathrm{e}^{\left(\frac{x}{y}\right)^2}$ is a homogeneous differential equation. [2]

(iii) By using an appropriate substitution, find the general solution of the differential equation, leaving your answer in the form $y = \mathrm{h}\left(\dfrac{x}{y}\right)$. [4]

A solution curve of this differential equation has a tangent at the point $(4, -2)$ that is perpendicular to the line $y = mx$. Find the exact value of m. [3]

3 Let $\mathrm{f}^{(k)}$ denote the kth derivative of the function f and define $\mathrm{f}^{(0)} = \mathrm{f}$.

Taylor's Theorem states the following.

If f', f'', …, and $\mathrm{f}^{(n+1)}$ are all continuous on an interval containing a and x, then

$$\mathrm{f}(x) = \sum_{r=0}^{n} \frac{\mathrm{f}^{(r)}(a)}{r!}(x-a)^r + \frac{1}{n!}\int_a^x \mathrm{f}^{(n+1)}(t)(x-t)^n \,\mathrm{d}t .$$

(i) Write down the value of $\int_a^x \mathrm{f}'(t)\,\mathrm{d}t$ in terms of $\mathrm{f}(a)$ and $\mathrm{f}(x)$. Hence, show that Taylor's Theorem holds for the case $n = 0$. [2]

(ii) Prove Taylor's Theorem by induction on n. [4]

For a given positive integer n, the finite series $\mathrm{T}_n(x)$ is defined by $\mathrm{T}_n(x) = \sum_{r=0}^{n} \frac{\mathrm{f}^{(r)}(a)}{r!}(x-a)^r$ is called the nth degree Taylor polynomial of $\mathrm{f}(x)$ about a.

(iii) Use the 2nd degree Taylor polynomial of $\sin x$ about $\frac{\pi}{2}$ to show that

$$\sin 1.6 \approx 1 - \frac{1}{2}\left(1.6 - \frac{\pi}{2}\right)^2 .$$ [1]

An alternative version of Taylor's Theorem is

$$\mathrm{f}(x) = \sum_{r=0}^{n} \frac{\mathrm{f}^{(r)}(a)}{r!}(x-a)^r + \mathrm{R}_n(x),$$

where $\mathrm{R}_n(x) = \frac{\mathrm{f}^{(n+1)}(c)}{(n+1)!}x^{n+1}$ and c is a real number between a and x.

(iv) Let $x \in \mathbb{R}^+$ be fixed and let $\hat{x}$ denote an integer greater than x.

Show that for a positive integer n large enough such that $n > 2\hat{x}$,

$$\frac{x^n}{n!} < k\left(\frac{1}{2}\right)^{n-2\hat{x}} ,$$

where $k = \frac{\hat{x}}{1}\cdot\frac{\hat{x}}{2}\cdot\frac{\hat{x}}{3}\cdots\frac{\hat{x}}{2\hat{x}}$. [2]

(v) Deduce from part **(iv)** the value of $\lim_{n\to\infty}\left(\frac{x^n}{n!}\right)$ for $x \in \mathbb{R}$. [3]

(vi) Use the alternative version of Taylor's Theorem and part **(v)** to show that $\lim_{n\to\infty}\mathrm{R}_n(x) = 0$ and hence, explain why for each $x \in \mathbb{R}$, $\mathrm{e}^x = \sum_{r=0}^{\infty}\frac{x^r}{r!}$. [3]

4 Let p be a prime number.

(i) Explain why for any $k \in \mathbb{Z}$ with $2 \le k \le p$, there exist integers x_k and y_k with $y_k > 0$ such that

$$px_k + (k-1)y_k = 1.$$ [2]

(ii) With y_k established in part **(i)**, let $u_k = k(k-1)y_k$, for each $2 \le k \le p$. Explain why

$$u_k \equiv 0 \pmod{k-1} \text{ and } u_k \equiv k \pmod{p}.$$ [2]

(iii) Show that in modulo p, each $\frac{u_k}{k-1}$ for $2 \le k \le p$ are all distinct. [3]

A permutation of $\{1, 2, \ldots, p\}$, is a sequence $x_1, x_2, \ldots, x_p$, where each $x_i \in \{1, 2, \ldots, p\}$ and each x_i is unique.

(iv) Show that there exists a permutation $v_1, v_2, \ldots, v_p$ of $\{1, 2, \ldots, p\}$, such that each of the terms v_1, v_1v_2, $v_1v_2v_3$, …, and $v_1v_2 \ldots v_p$ leaves a different remainder when divided by p. [3]

(v) Write down a permutation of $\{1, 2, \ldots, 11\}$ as described in part **(iii)**. [2]

5 For $x \in \mathbb{R} \setminus \{0, 1\}$, the functions f and g satisfy the equations

$$\mathrm{f}(x) = \frac{1}{1-x} \text{ and } \mathrm{g}(x) = 1 - \frac{1}{x}.$$

(i) Show that the composite functions fg and gf exist. [2]

(ii) Show that $\mathrm{f}^2(x) = \mathrm{g}(x)$ and $\mathrm{g}^2(x) = \mathrm{f}(x)$. [2]

For $x \in \mathbb{R} \setminus \{0, 1\}$, the function h satisfies the equation

$$\mathrm{h}(x) + \mathrm{h}\left(\frac{1}{1-x}\right) = x.$$

(iii) Find $\mathrm{h}(x)$. [7]

6 **(a)** Let x, y and z be positive integers.

(i) If $xy \equiv 1 \pmod{z}$, show that $\gcd(x, z) = 1$. [2]

(ii) If $x \equiv 1 \pmod{y}$ and $x \equiv 1 \pmod{z}$ with $\gcd(y, z) = 1$, show that $x \equiv 1 \pmod{yz}$. [2]

(b) Let a, b and c be positive integers with $1 < a < b < c$. It is given that

$$ab \equiv 1 \pmod{c},\ ac \equiv 1 \pmod{b}, \text{ and } bc \equiv 1 \pmod{a}.$$

(i) Show that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. [5]

(ii) Find all the possible values of a, b and c. [3]

7 **(a)** A student-care centre has 6 classrooms, each assigned a level from Primary 1 to Primary 6. The principal needs to distribute 20 tables into the 6 classrooms. Find the number of ways of distributing the tables into the classrooms if

(i) the Primary 1 classroom has at most 3 tables, [2]

(ii) each of the Primary 5 and Primary 6 classrooms holds at least 3 tables. [3]

One student representing each level from Primary 1 to Primary 6 are queuing up in line to collect snacks for tea break.

(iii) Find the number of ways that these six students can arrange themselves in line such that no three consecutive students are in ascending order of level, from front to back. (For instance, $p_1p_3p_2p_6p_4p_5$ is allowed but $p_6\boxed{p_3p_4p_5}p_1p_2$ and $\boxed{p_3p_4p_6}\boxed{p_1p_2p_5}$ are not allowed.) [5]

(b) Given a set $\{m_1, m_2, m_3, ..., m_9\}$ of 9 distinct integers, show that there exists three integers m_α, m_β and m_γ, such that the difference between any two of them is divisible by 4. [3]

(c) The array below shows an arrangement of the letters contained in the word VICTORY.

```
                        V
                        |
                    V - I - V
                    |   |   |
                V - I - C - I - V
                |   |   |   |   |
            V - I - C - T - C - I - V
            |   |   |   |   |   |   |
        V - I - C - T - O - T - C - I - V
        |   |   |   |   |   |   |   |   |
    V - I - C - T - O - R - O - T - C - I - V
    |   |   |   |   |   |   |   |   |   |   |
V - I - C - T - O - R - Y - R - O - T - C - I - V
    |   |   |   |   |   |   |   |   |   |   |
    V - I - C - T - O - R - O - T - C - I - V
        |   |   |   |   |   |   |   |   |
        V - I - C - T - O - T - C - I - V
            |   |   |   |   |   |   |
            V - I - C - T - C - I - V
                |   |   |   |   |
                V - I - C - I - V
                    |   |   |
                    V - I - V
                        |
                        V
```

Find the number of paths in the array that spell out the word VICTORY. [2]

8 Given any real number x, the *Ceiling Function* of x, denoted by $\lceil x \rceil$, is defined to be the least integer greater than or equal to x. For example, $\lceil 2.6 \rceil = 3$ and $\lceil 4 \rceil = 4$.

For any positive integer n, the $3 \times n$ grid below contains all positive integers from n to $4n-1$.

n	$n+1$	$\cdots$	$2n-1$
$2n$	$2n+1$	$\cdots$	$3n-1$
$3n$	$3n+1$	$\cdots$	$4n-1$

John claims that the grid above always contains a cube number.

(i) Provide a counter-example to prove that John's claim is incorrect. [1]

Peter claims that by appending the number '$4n$' to John's grid above, it will always contain a cube number.

(ii) Sketch the graph of $y = 3x^3 - 3x^2 - 3x - 1$ and state the smallest integer value of k for which $3x^3 - 3x^2 - 3x - 1 > 0$ for all $x \geq k$. [2]

(iii) By assuming that there is no cube in Peter's grid for some positive integer n, use the method of contradiction and **(ii)** to prove that Peter's claim is correct. [5]

(iv) Find the range of values of n for which Peter's grid contains

(a) 2^3, [1]

(b) 3^3. [1]

(v) Find the exact range of values of n for which Peter's grid contains m^3 where m is a positive integer. You should give your answer in terms of m and define any functions used. [2]

(vi) Explain why for all $m \geq 2$, $m^3 \geq \left\lceil \frac{(m+1)^3}{4} \right\rceil$ and hence find in terms of m, the exact range of values of n for which Peter's grid contains both m^3 and $(m+1)^3$ for $m \geq 2$. [3]

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