NUMERICAL METHODS [FM]
EULER METHOD / IMPROVED EULER METHOD

1 A solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + \sqrt{xy}$ has $y = 1$ when $x = 1$.

(i) Use two iterations of Euler method of step size 0.5 to estimate the value of y when $x = 2$. [3]

(ii) Explain whether you would expect this value to be an under-estimate or an over-estimate of the true value. [2]

(iii) Explain why it is usually better to improve accuracy by using the improved Euler method rather than by simply using smaller step sizes in Euler method. [1]

[TJC/FM/2018/P1//Q2]

2 The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 \tan x = 1,$$

where $y = 1$ when $x = 1$, is to be solved numerically.

(i) Carry out two steps of Euler's method with step length 0.1 to estimate the value of y when $x = 1.2$, giving your answer to 4 decimal places. [3]

(ii) The method in part **(i)** is now replaced by the improved Euler method. The estimate obtained is 2.0156, given to 4 decimal places. State, with a reason, whether this estimate and the one found in part **(i)** are likely to be overestimates or underestimates of the actual value of y when $x = 1.2$. [2]

(iii) Explain why it would be inappropriate to continue the process in part **(i)** to estimate the value of y when $x = 1.6$. [2]

[VJC/FM/2019/P2/Q3]

3 Determine the maximum number of iterations, using the *Euler Method*, with step size 0.01 on the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y, \text{ where } y = 0 \text{ when } x = 1,$$

such that the error between the actual value of y and the approximated value of y is less than 0.001. [6]

[NYJC/FM/2018/P1//Q2]

4. The variables x and y are related by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$.

(i) Taking the initial value as $y(x_0) = y_0$, explain with the aid of a diagram, how the Euler method can be applied once on the differential equation to approximate the solution at $x = x_0 + h$. [3]

Given that $\mathrm{f}(x, y) = xy - \frac{y}{x}$ and $(x_0, y_0) =$ (1.5, 2).

(ii) Apply the following methods with a step size 0.5 to estimate y at $x = 2.5$.

(a) Euler method [2]

(b) Improved Euler method [3]

(iii) Comment on the accuracy of the estimates found in part (ii). [2]

(iv) State one advantage and one disadvantage in using the improved Euler method compared to the Euler method? [2]

[RVHS/FM/2018/P1/Q9]

5 A particle moves along a straight line which passes through a fixed point O. It is acted on by two resistive forces, one of which is proportional to its displacement x from O while the other is proportional to its speed v. As a result, the motion of the particle is governed by the differential equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -7x - 24v.$$

Given that $v = 121$ when $x = 0$, estimate the value of v when $x = 1$ using

(i) one iteration of Euler's Formula, [2]

(ii) one iteration of the improved Euler formula. [2]

Hence, explain why v is approximately a linear function of x for $0 \le x \le 1$. [2]

By considering the values of $\frac{x}{v}$ for $0 \le x \le 1$, use the given differential equation to find an expression for this linear function. [2]

[VJC/FM/2018/P1//Q5]

6 **(i)** Show that the substitution $z = y^2$ transforms the differential equation

$$\left(1 + x^2\right) y\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^2 = 3, \quad y \ge 0 \quad ---(1)$$

to

$$\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{4xz}{1 + x^2} = \frac{6}{1 + x^2}.$$ [2]

(ii) Hence find y in terms of x and sketch three members of the family of solution curves. [6]

(iii) A curve that has a y–intercept at 1 is defined by the differential equation in (1). Use Euler's method with step length 0.5 to estimate the value of y when $x=2$, giving your answer to 3 decimal places. [3]

(iv) An estimation is considered to be a good estimation if the error is less than 1% of the step length. Determine if the estimation in **(iii)** is good. [2]

[NJC/FM/2019/MYE/P1/Q10]

7 The function $y=y(x)$ satisfies $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{5}\left(\tan x+x^3y\right)$. The value of $y(h)$ is to be found, where h is a small positive number, and $y(0)=0$.

(i) Use one step of the improved Euler formula to find an alternative approximation to $y(h)$ in terms of h. [2]

(ii) It can be shown that $y=y(x)$ satisfies $y(h)=\mathrm{e}^{0.05h^4}\int_0^h \frac{\tan x}{5}\mathrm{e}^{-0.05x^4}\,\mathrm{d}x$. Assume that h is small and hence find another approximation to $y(h)$ in terms of h. [2]

(iii) Discuss the relative merits of these two methods employed to obtain these approximations. [2]

[EJC/FM/2018/P1/Q1]

8 A differential equation is given by $(x+4)\frac{\mathrm{d}y}{\mathrm{d}x}+(x+5)y+2\mathrm{e}^x=0$, where $x\neq -4$.

(i) By using the substitution $u=(x+4)y$, show that the differential equation can be reduced to $\frac{\mathrm{d}u}{\mathrm{d}x}+u=-2\mathrm{e}^x$. Hence, find y in terms of x, given that $y=0$ when $x=0$. Hence obtain the value of y when $x=0.2$. [6]

Consider the differential equation $(x+4)\frac{\mathrm{d}y}{\mathrm{d}x}+(x+5)y+2+2x+x^2=0$, where $y=0$ when $x=0$.

(ii) Use the Euler method with step length 0.1 to estimate the value of y when $x=0.2$. [3]

(iii) Use the improved Euler method with step length 0.1 to estimate the value of y when $x=0.2$. [3]

(iv) Comment on your numerical answers for the values of y from parts **(i)**, **(ii)** and **(iii)**. [2]

[HCI/FM/2019/MCT//Q7]

Answers

1 (i) 4.47 (correct to 3 s.f.)

2 (i) 1.6656 (4d.p)

3 max no of iterations = 20

4 (ii) a. $y(2.5) \approx 4.96$ b. $y(2.5) \approx 7.98$

5 (i) 97 (ii) 97.0; $v \approx 121 - 24x$.

6 (ii) $y = \dfrac{\sqrt{6x + 2x^3 + C}}{1 + x^2}$ (iii) 1.07422

7 (i) $\dfrac{h}{10}\tan(h)$ (ii) $y(h) \approx \mathrm{e}^{0.05h^4}\dfrac{1}{5}\left[\dfrac{h^2}{2} - \dfrac{h^6}{120}\right]$ or $y(h) \approx \left(1 + \dfrac{1}{20}h^4\right)\cdot\dfrac{1}{5}\left[\dfrac{h^2}{2} - \dfrac{h^6}{120}\right]$

8 (i) $y = \dfrac{\mathrm{e}^{-x} - \mathrm{e}^{x}}{x + 4}$; -0.0959 (3 s.f.) (ii) $-0.0977(3\text{sf})$ (iii) $-0.0958(3\text{sf})$