1

(i)

$$\frac{1}{k!}-\frac{1}{(k+1)!}=\frac{(k+1)-1}{(k+1)!}=\frac{k}{(k+1)!}$$

Series

$$=\sum_{r=2}^{2n+1}\frac{r}{(r+1)!}$$
$$=\sum_{r=2}^{2n+1}\left(\frac{1}{r!}-\frac{1}{(r+1)!}\right)$$
$$=\frac{1}{2!}-\frac{1}{3!}$$
$$+\frac{1}{3!}-\frac{1}{4!}$$
$$\vdots$$
$$+\frac{1}{(2n)!}-\frac{1}{(2n+1)!}$$
$$+\frac{1}{(2n+1)!}-\frac{1}{(2n+2)!}$$
$$=\frac{1}{2!}-\frac{1}{(2n+2)!}$$
$$=\frac{1}{2}-\frac{1}{(2n+2)!}$$

(ii)

$$\sum_{r=2}^{2n}\frac{2}{(r+2)!}=\sum_{r=3}^{2n+1}\frac{2}{(r+1)!}$$
$$<\sum_{r=3}^{2n+1}\frac{r}{(r+1)!}=\left[\frac{1}{2}-\frac{1}{(2n+2)!}\right]-\frac{2}{(2+1)!}<\frac{1}{6}$$

2

(i)

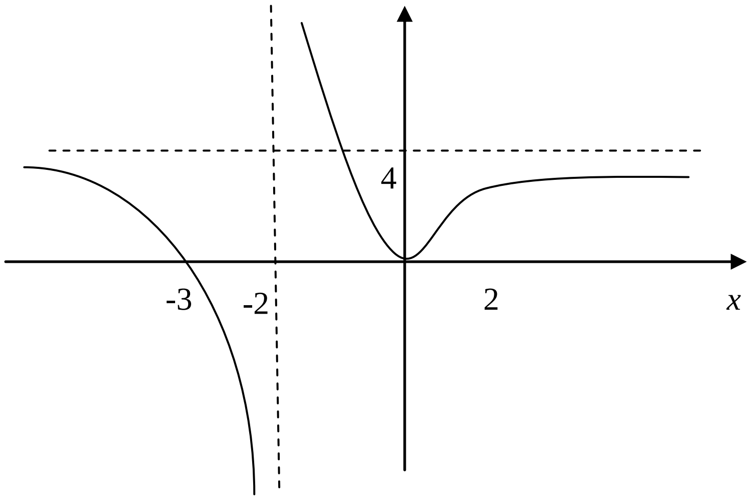
Distance travelled at the (n+1)$^{\text{th}}$ bounce

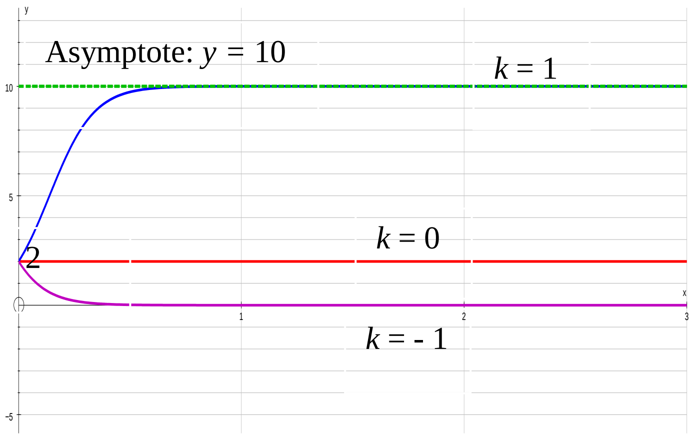
$$= 10+2(10)\left[\left(\frac{3}{4}\right)^{1}+\left(\frac{3}{4}\right)^{2}+\left(\frac{3}{4}\right)^{3}+\ldots\left(\frac{3}{4}\right)^{n}\right]$$

$$= 10 + 20\left[\frac{\frac{3}{4}\left(1-\frac{3}{4}^{n}\right)}{1-\frac{3}{4}}\right]$$

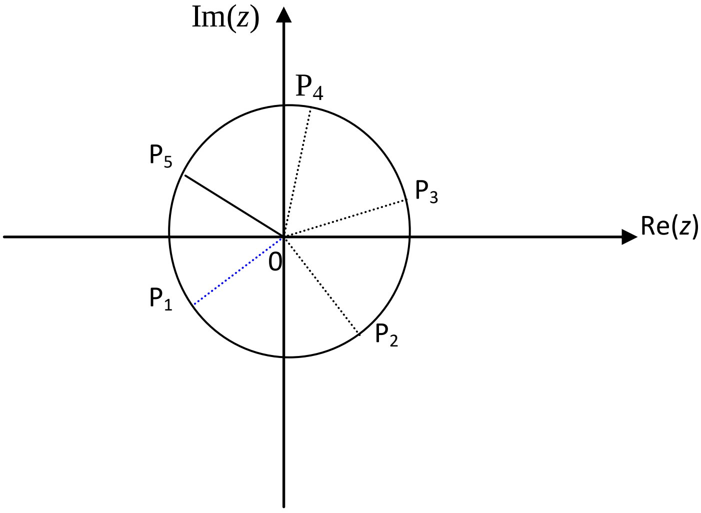
$$= 10 + 60\left(1-\frac{3}{4}^{n}\right)$$

	$= 70 - 60\left(\frac{3}{4}\right)^n$ **(ii)** Distance travelled at the $(n+1)^{th}$ bounce ≤ 55 $70 - 60\left(\frac{3}{4}\right)^n \leq 55$ $\frac{70-55}{60} \leq \left(\frac{3}{4}\right)^n$ $n \leq 4.82$ So the largest value of n is 4. In other words, at the $(4+1)^{th} = 5^{th}$ bounce, the distance travelled is less than 55m, while at the 6^{th} bounce, the distance travelled is more than 55m. Therefore, when the ball has travelled exactly 55m, it has only bounced 5 times. **(iii)** Total distance travelled at the maximum height after the 4^{th} bounce (at the exact moment when the floor is raised) $= 10 + 2\left[10\left(\frac{3}{4}\right)\right] + 2\left[10\left(\frac{3}{4}\right)^2\right] + 2\left[10\left(\frac{3}{4}\right)^3\right] + 10\left(\frac{3}{4}\right)^4$ $= 47.8515625\text{m}$ Height after 4^{th} bounce, $h = 10\left(\frac{3}{4}\right)^4 = 3.1640625$ m New height on platform, $h' = 3.1640625 - 2 = 1.1640625$ m Distance travelled on the raised floor $= h' + 2(h')\left[\left(\frac{3}{4}\right)^1 + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots\right]$ $= h' + 2h'\left(\frac{\frac{3}{4}}{1-\frac{3}{4}}\right) = h' + 2h'(3) = 7h' = 8.1484375$ m

	Total distance travelled = Dist. travelled before floor is raised + dist. travelled after floor is raised $= 47.8515625\text{m} + 8.1484375\text{ m} = 56\text{m}$
3 (a)	
(b)	Using long division $\dfrac{x^2 - 4x}{(x-2)^2} = 1 - \dfrac{4}{(x-2)^2}$ So A=1 and B = -4 Series of transformations: 1. Translate the graph of $y = \dfrac{1}{x^2}$ by 2 units in the positive x direction 2. Stretch the resulting graph parallel to the y-axis with a scale factor of -4. 3. Translate the resulting graph by 1 unit in the positive y direction.

4	$\frac{dx}{dt} = kx(10-x)$ $\int \frac{1}{10x-x^2}\,dx = \int k\,dt$ $\int \frac{1}{5^2-(x-5)^2}\,dx = \int k\,dt$ $\frac{1}{10}\ln\left(\frac{5+(x-5)}{5-(x-5)}\right) = kt + C$ $\ln\left(\frac{x}{10-x}\right) = 10kt + D$ $\frac{x}{10-x} = Be^{10kt}$, where $B = e^D$ Where t = 0, x = 2 $B = \frac{1}{4}$ $\frac{x}{10-x} = \frac{1}{4}e^{10kt}$ $4x = e^{10kt}(10-x)$ $x = \frac{10e^{10kt}}{4+e^{10kt}}$
	(i) 
	(ii) $k < 0$ A vaccination had been found to eliminate the virus OR The people infected by the virus had passed away OR any other reasonable answer.
5	$z - w = e^{i\alpha} - e^{i\beta}$ $= (\cos\alpha + i\sin\alpha) - (\cos\beta + i\sin\beta)$ $= (\cos\alpha - \cos\beta) + i(\sin\alpha - \sin\beta)$ $= -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2}) + i(2\sin(\frac{\alpha-\beta}{2})\cos(\frac{\alpha+\beta}{2}))$

	$= 2i\sin(\frac{\alpha-\beta}{2})\,[\mathrm{i}\sin(\frac{\alpha+\beta}{2})+\cos(\frac{\alpha+\beta}{2})\,]$ $= 2i\ \ \sin(\frac{\alpha-\beta}{2})\ e^{i(\frac{\alpha+\beta}{2})}$ (Proved). $(z-w)(z^*-w^*)$ $=(z-w)(z-w)^*$ $=\|z-w\|^2$ $=\|2i\ \ \sin(\frac{\alpha-\beta}{2})e^{i(\frac{\alpha+\beta}{2})}\|^2$ $=4\sin^2(\frac{\alpha-\beta}{2})$, since $\|i\|=1$ & $\|e^{i(\frac{\alpha+\beta}{2})}\|=1$ $=2(1-\cos(\alpha-\beta))$ Therefore, $k = 2$. Alternatively, Since arg(z^*) = -α and arg(w^*)=- β , we have $(z-w)(z^*-w^*)$ = $[2i\ \ \sin(\frac{\alpha-\beta}{2})\ e^{i(\frac{\alpha+\beta}{2})}][2i\ \ \sin(\frac{-\alpha+\beta}{2})\ e^{i(\frac{-\alpha-\beta}{2})}]$ $=4i^2(-\sin^2(\frac{\alpha-\beta}{2}))$ $=4\sin^2(\frac{\alpha-\beta}{2})$ $=\ 2(1-\cos(\alpha-\beta))$ Therefore, $k = 2$.
(b)	$\left(\frac{z}{1+\mathrm{i}\sqrt{3}}\right)^3+2\left(\frac{1+\mathrm{i}\sqrt{3}}{z^2}\right)=0$ $\frac{z^5+2(1+\sqrt{3}i)^4}{(1+\sqrt{3}i)^3z}=0$ $z^5+2(1+\sqrt{3}i)^4=0$ $z^5=-2(1+\sqrt{3}i)^4$ $=-2(2^4e^{i(\frac{4\pi}{3})})$ $=2^5e^{i\pi}e^{(\frac{4\pi}{3})}$ since $\mathrm{e}^{i\pi}=-1$ $=2^5e^{i(\frac{7\pi}{3})}$ $=2\,e^{i(\frac{\pi}{3})}$, correct to principal range.

	Therefore $z^5 = 2^5 e^{i(\frac{\pi}{3}+2k\pi)}, k = -2, -1, 0, 1, 2$ $z = 2e^{i(\frac{\pi+6k\pi}{15})}$, k = -2, -1,0,1,2 The roots are $z_1 = 2e^{i(-\frac{11\pi}{15})}, z_2 = 2e^{i(-\frac{5\pi}{15})}, z_3 = 2e^{i(\frac{\pi}{15})}, z_4 = 2e^{i(\frac{7\pi}{15})}, z_5 = 2e^{i(\frac{13\pi}{15})}$.
	 P_i represent the complex no z_i, where i= 1,2,3,4,5.
6	Three cases: Case 1: All three balls distinct (ABC) n(ABC) = ${}^6C_3$ (6 types of balls available) Case 2: two identical (AAB) n(AAB) = ${}^2C_1 \times {}^5C_1$.(2 types of balls available to choose the 2 identical balls from, then 5 types of balls available remaining to choose the last ball) Case 3: three identical (AAA). n(AAA) = 1 n(Total) = n(ABC) + n(AAB) + n(AAA) = 20+10+1=31
7	**(i)** It is necessary for the research company to collect a sample because it is impossible for the company to interview all eligible voters in the United States before the election as the population is too large (90 million) **(ii)** Quota Sampling. The method is not appropriate because it might be biased towards households that have eligible voters that are present only during the phone interviews. Moreover, choosing a particular telephone book means that the sample comes from a particular area/state of the United States, and the sample may not be representative of the

sentiments of all eligible voters in the United States.

(iii)
One could use stratified sampling. This could be conducted by ensuring that the sample of 1000 voters be sampled in the following manner

Educational Qualification	No degree	Bachelor Degree	Master degree or higher
Percentage / (Number)	58.61% 586	30.44% 304	10.95% 110

The research company could ensure that they sample 586 eligible voters that have no degrees, 304 eligible voters have Bachelor degrees and 110 eligible voters that have Master degrees or higher. The company can randomly choose the required number provided by the respective educational institutions (representing different regions/states in America) to ensure that each of the chosen interviewees within each stratum have an equal chance of being selected.

8

(i)

By symmetry, $\mu = \dfrac{250+300}{2} = 275$

P(X < 250) = 0.2

$P\left(Z < \dfrac{250-275}{\sigma}\right) = 0.2$

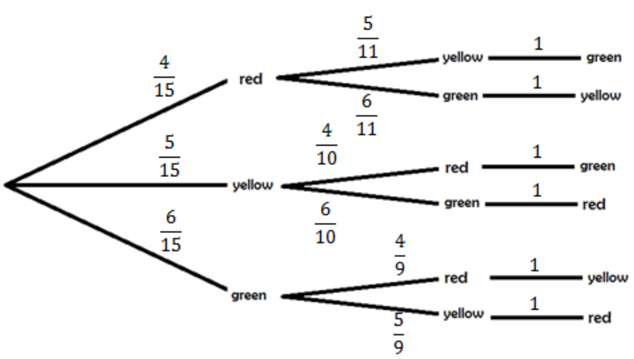
$\dfrac{-25}{\sigma} = -0.84162$

$\therefore \sigma = 29.7$

(ii)
P(X > c) = 0.01
$\therefore$ P(X < c) = 0.99

c = 344 (correct to 3 s.f.)

9



(i) P(last ball is green) = $\frac{4}{15} \times \frac{5}{11} + \frac{5}{15} \times \frac{4}{10} = \frac{14}{55}$

(ii)

$$P(first\ yellow|last\ green) = \frac{P(first\ yellow\ AND\ last\ green)}{P(last\ green)} = \frac{\frac{5}{15}\times\frac{4}{10}}{\frac{14}{55}} = \frac{11}{21}$$

10

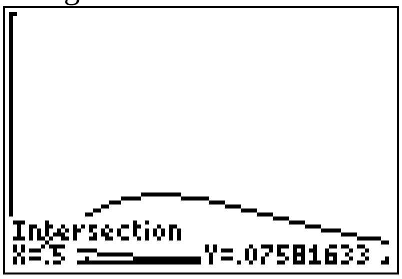
$X \sim \mathrm{P_o}(\lambda)$

$\mathrm{P}(X=2)=\frac{1}{8}\mathrm{P}(X=0)$

$\frac{e^{-\lambda}\lambda^2}{2!}=\frac{1}{8}\frac{e^{-\lambda}\lambda^0}{0!}$

$\lambda^2=\frac{1}{4} \Rightarrow \lambda=\frac{1}{2}=0.5$ (-ve rejected)

Alternative:
Using GC



$\mathrm{P}(1\le X<4) = \mathrm{P}(X=1)+\mathrm{P}(X=2)+\mathrm{P}(X=3)$
or $\mathrm{P}(X\le 3)-\mathrm{P}(X=0)$
$= 0.392$ (3 s.f.)

Let Y be the no. of defects in the carpet of area 500 m^2.

$Y \sim \mathrm{P_o}\left(\frac{500}{20}\times 0.5\right)$

$Y \sim \mathrm{P_o}(12.5)$

Since $\lambda=12.5>10$, $Y \sim \mathrm{N}(12.5, 12.5)$ approximately.

$\mathrm{P}(Y>10) \xrightarrow{c.c} \mathrm{P}(Y>10.5) = 0.714$ (3 s.f.)

11

(i)
Let X be the r.v. " volume of content of randomly chosen milk carton" and μ be the mean volume of content in milk cartons.
H_0: $\mu = 2$
H_1: $\mu < 2$
We carry out 1-tailed t-test at 10% significance level, since the population variance is unknown and sample size n is small.

Under H_0, $T=\frac{\overline{X}-\mu}{s/\sqrt{n}} \sim t(9)$

From GC,

$\bar{x} = 1.9834$

p-value $= 0.00882$

Since p-value < 0.10, we reject H_0 and conclude that at 10% significance level, there is sufficient evidence that the manufacturer has overestimated the volume of the content of the milk cartons.

(ii)

H_0: $\mu = 2$
H_1: $\mu \neq 2$
We carry out 2-tailed t-test at 10% significance level
p-value = 2(0.00882)=0.01764
Since p-value is still less than 10%, the conclusion in (i) does not change.

(iii)

Since standard deviation is known, we use z-test instead.
H_0: $\mu = 2$
H_1: $\mu < 2$
We carry out 1-tailed z-test at 10% significance level

Under H_0, $Z = \dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$

To reject H_0,

$$\frac{\bar{x} - \mu}{s / \sqrt{n}} < -1.2816$$

$$\Rightarrow \frac{1.998 - 2}{0.012 / \sqrt{n}} < -1.2816$$

$$\Rightarrow n > 59.13$$

Least number of milk cartons to be measured is 60.

Alternatively, using GC,
P($\bar{X}$ < 1.998) < 0.10
[Y=normcdf(-E99,1.998, 2, $\dfrac{0.012}{\sqrt{n}}$)]

X	Y1
55	.10822
56	.10616
57	.10414
58	.10217
59	.10024
60	.09835
61	.09651

X=60

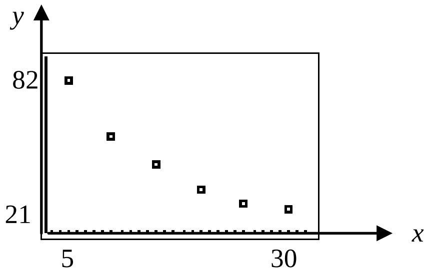
n = 59, P($\bar{X}$ < 1.998) = 0.10024
n = 60, P($\bar{X}$ < 1.998) = 0.09835

Least number of cartons used is 60.

12

(i)

The scatter diagram is



(ii)

From the scatter diagram, the regression line of y on x should have negative gradient. Therefore $y = 79 - 2x$ is the correct regression line of y on x.

(iii)

Since $(\bar{x}, \bar{y})$ lie on both regression lines, using GC to find the points of intersection,

$\bar{x} = 20.015 \approx 20$

$\bar{y} = 38.969 \approx 39$

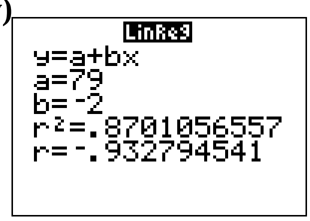
Let the lost pair of reading be (s, t)

s = 20×7 – (5 + 10 + 15 + 20 + 25 + 30) = 35

t = 39×7 – (82 + 56 + 42 + 30 + 24 + 21) = 18

The 7th pair of values is (35, 18).

(iv)



From GC : r = -0.932784541=-0.933

(v)

$e^y = ax^b$

$\Rightarrow y = \ln a + b\,(\ln x)$

$\ln a = 134.1077 \Rightarrow a = 1.746{'}\ 10^{58} = 1.75{'}\ 10^{58}$ (3 s.f.)

$b = -33.63857 \Rightarrow b = -33.6$ (to 3 s.f.)

13

(i)

1. Each of the students is equally likely to answer the Differential Equation question correctly (i.e. constant p throughout all trials)
2. The event of a student answering the Differential Equation question correctly is independent of the other students.

(ii)

Let X be the random variable "no of students out of 30 students who could do the Differential Equation question"

$$X \sim B(30, 0.3)$$

$P(X \geq 6) = 1 - P(X \leq 5)$

$\quad = 0.92341 \approx 0.923$

(iii)

Let S be the r.v. "no of students out of 8 who could do that question"
Let T be the r.v "no of students out of 22 who could do that question"

$$S \sim B(8, 0.3)$$

$$T \sim B(22, 0.3)$$

P(only 2 among first 8 could do that question| $X \geq 6$)

$= \dfrac{P(S=2)P(T \geq 4)}{P(X \geq 6)}$

$= 0.299$

(iv)

Let Y be the r.v. "no of students out of *n* who could do that question"

$$Y \sim B(n, 0.3)$$

$P(Y \leq 5) > 0.9$

From G.C,

X	Y1	
10	.95265	
11	.92178	
12	.88215	
13	.8346	
14	.78052	
15	.72162	
16	.65978	

X=11

Therefore, the largest possible value of *n* is 11.

Since sample size = 50 is large,

$$\overline{X} \sim N\left(9, \frac{6.3}{50}\right) \quad \text{approximately by Central Limit Theorem}$$

$P(\overline{X} \geq 10) = 0.00242$