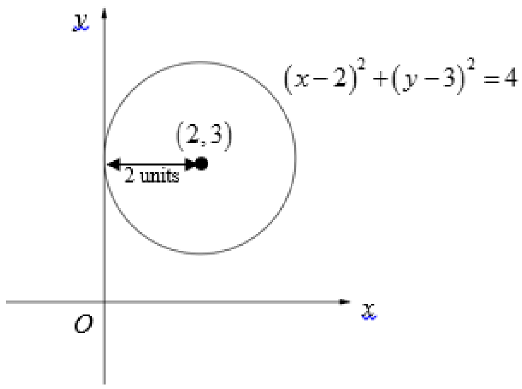
National Junior College
2016 – 2017 H2 Further Mathematics
Further Differential Equations (Assignment Solutions)

1 The normal at any point on a certain curve always passes through the point $(2,3)$. Form a differential equation to express this property and hence find the equation of the family of curves that possess this property. Sketch a typical member of this family. [5]

	Solution	Mark Scheme
1	$y-3=\frac{-1}{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}(x-2)$ $\frac{\mathrm{d}y}{\mathrm{d}x}(y-3)=-(x-2)$ $\int y-3\ \mathrm{d}y=\int -(x-2)\ \mathrm{d}x$ $\frac{y^2}{2}-3y=-\frac{x^2}{2}+2x+C'$ $x^2+y^2-6y-4x+2C'=0$ $(x-2)^2+(y-3)^2+2C'-13=0$ $(x-2)^2+(y-3)^2=C$, where $C=13-2C'>0$  	B1: DE for normal M1: Integrating both sides A1: Correct solution B1: Circle B1: Centre labelled

2 Spruce budworm is a serious pest in eastern Canada and northern Minnesota. It consumes the leaves of coniferous tree and excess consumption can damage and kill the tree. In the absence of predators, the worm's population, P (in millions), satisfies the logistic growth,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right).$$

One scientist proposed that the worm's predators eat the worm at a rate proportional to the worm's population.

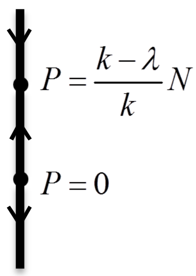
(i) Explain the significance of the constant N in this model. [1]

(ii) Write down the differential equation of the population growth of the worm in the presence of predators. [1]

(iii) Given that the population of worm increases towards an equilibrium value in the long run, draw the phase line diagram and comment on the stability of the equilibrium values. [3]

(iv) Under what condition will the population of the worms decrease and become extinct eventually. [1]

[2014 H3 Prelim/HCI/Modified]

	Solution	**Mark Scheme**
2(i)	N represents the carrying capacity of the worm's population.	B1
2(ii)	$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - \frac{P}{N}) - \lambda P$, where λ is a positive constant	B1
2(iii)	$kP(1 - \frac{P}{N}) - \lambda P = 0$ $P(k - \frac{kP}{N} - \lambda) = 0$ $P = 0$ or $P = \frac{k-\lambda}{k}N$  Since the population increases to an equilibrium value in the long run, $\frac{k-\lambda}{k}N > 0$	M1: Solving for equilibrium values A1: Phase line diagram B1: Converging to the correct equilibrium value
2(iv)	When $\lambda > k$, the worm will become extinct in the long run.	B1

3 The current in a particular electrical circuit is described by the equation

$$\frac{d^2I}{dt^2}+25\frac{dI}{dt}+100I=-170\sin 20t,$$

where I is the current in amperes and t is the time in seconds after the power source is switched on.

Find the solution for which $\frac{dI}{dt}=I=0$ when $t=0$. [8]

Given that $I\to I_1$ as $t\to\infty$, find the maximum value of I_1. [2]

	Solution	Mark Scheme
3	$\frac{d^2I}{dt^2}+25\frac{dI}{dt}+100I=-170\sin 20t$ Characteristics equation: $m^2+25m+100=0\Rightarrow m=-5$ or $m=-20$ $\therefore I_c=Ae^{-5t}+Be^{-20t}$. Let $I_p=a\cos 20t+b\sin 20t$. Then $I_p'=-20a\sin 20t+20b\cos 20t$, $I_p''=-400a\cos 20t-400b\sin 20t$ Substitute into DE: $-300a+500b=0$ and $-300b-500a=-170$ $\therefore\ a=\frac{1}{4},b=\frac{3}{20}$.Thus $I_p=\frac{1}{4}\cos 20t+\frac{3}{20}\sin 2t$. General solution is $I=Ae^{-5t}+Be^{-20t}+\frac{1}{4}\cos 20t+\frac{3}{20}\sin 20t$. When $t=0, I=0$. Thus $A+B=-\frac{1}{4}$. $\frac{dI}{dt}=-20A-5B+3=0$. Solving simultaneously, we get $A=\frac{17}{60},B=-\frac{8}{15}$. $I=\frac{17}{60}e^{-5t}-\frac{8}{15}e^{-20t}+\frac{1}{4}\cos 20t+\frac{3}{20}\sin 20t$ As $t\to\infty, I\to\frac{1}{4}\cos 20t+\frac{3}{20}\sin 20t=I_1$. Thus $I_1=R\cos(20t+\alpha)$, where $R=\sqrt{\left(\frac{1}{4}\right)^2+\left(\frac{3}{20}\right)^2}=2.92$ and $\alpha=\tan^{-1}\left(\frac{3}{5}\right)$. Hence maximum value of $I_1=2.92$.	M1: C Equation A1: Complementary solution M1: Suggesting the particular solution M1: Differentiating 2 times M1: Solving for unknown constants by comparing coefficient or otherwise A1: Particular solution M1: Solving arbitrary constants using initial conditions A1: Correct solution B1: Solution approaches $\frac{1}{4}\cos 20t+\frac{3}{20}\sin 20t$. (e.c.f.) B1: 2.92 (e.c.f.)

4 A 2 kg object stretches a vertical spring 80cm beyond its natural length to reach the equilibrium position. Now the spring is stretched further by 5cm and released from that position with zero velocity.

(a) Assume that the air resistance is negligible. Write down the differential equation that models the behaviour of the spring. Define all the variables used in the equation. [2]

(b) It is known that air resistance is proportional to the velocity of the object with the proportionality constant being k .

(i) Take $k=1$. Find the exact position of the spring at any time *t*. [4]

(ii) Describe the damping effect of the air resistance. [2]

(c) In order for the system to be critically damped, it is now placed in a viscous liquid with damping coefficient λ. What should be the value of λ? You may assume the buoyancy is negligible. [2]

	Solution	Mark Scheme
4(a)	Let x be the displacement in m of the object from the equilibrium position and t be the time from the object is released. $2g=k(0.8)$ $k=\dfrac{2\times 9.81}{0.8}=24.525$ $2\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}=-24.525x$ $2\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}=-24.5x$	M1: Solving the coefficient of proportionality A1: Correct DE
4(b) (i)	$2\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+\dfrac{\mathrm{d}x}{\mathrm{d}t}+24.525x=0$ $2r^2+r+24.525=0$ $r=\dfrac{-1\pm\sqrt{1-4(2)(24.525)}}{4}$ $r=-\dfrac{1}{4}\pm 3.4928\mathrm{i}$ $x=\mathrm{e}^{-0.25t}(c_1\cos 3.4928t+c_2\sin 3.4928t)$ When $t=0, x=0.05$ (take the downward direction to be positive) $0.05=c_1$ $x=\mathrm{e}^{-0.25t}(0.05\cos 3.4928t+c_2\sin 3.4928t)$ When $t=0,\ \dfrac{\mathrm{d}x}{\mathrm{d}t}=0$	M1: C Eqn A1: General Solution M1: Solving for arbitrary constants

	$\frac{\mathrm{d}x}{\mathrm{d}t}=(-0.25)\mathrm{e}^{-0.25t}(0.05\cos 3.4928t+c_2\sin 3.4928t)$ $+\mathrm{e}^{-0.25t}(3.4928)(-0.05\sin 3.4928t+c_2\cos 3.4928t)$ $0=0.05(-0.25)+3.4928c_2$ $c_2=0.0035788$ Hence, we have $x=\mathrm{e}^{-0.25t}(0.05\cos 14.0t+0.00358\sin 14.0t)$	A1: Particular Solution
4(b) (ii)	$1^2-4(2)(24.525)<0$ So the system is under-damped. That means the system oscillates with the amplitude gradually decreasing to zero.	M1: Explaining using discriminant or the particular solution A1: Oscillate with amplitude decreasing to 0.
4(c)	$\lambda^2-4(2)(24.525)=0$ $\lambda=14.0$	M1: Equating discriminant to 0 A1: Correct value