A-Level H3 Mathematics Solutions (2017-2026)

Thang Pang Ern

October 7, 2024

Contents

1 2017 Specimen Paper Solutions

Question 1

(a) Using the Cauchy-Schwarz inequality,

$$\left[\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2\right]\left[\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2+\left(\frac{x}{y}\right)^2\right] \geq \left[\left(\frac{x}{y}\right)\left(\frac{y}{z}\right)+\left(\frac{y}{z}\right)\left(\frac{z}{x}\right)+\left(\frac{z}{x}\right)\left(\frac{x}{y}\right)\right]^2$$

$$\left[\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2\right]^2 \geq \left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)^2$$

$$\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2 \geq \frac{x}{z}+\frac{y}{x}+\frac{z}{y}$$

so we have proven the upper bound for $\frac{x}{z}+\frac{y}{x}+\frac{z}{y}$.

Next, using the AM-GM inequality, we have

$$\frac{x}{z}+\frac{y}{x}+\frac{z}{y} \geq 3\sqrt[3]{\left(\frac{x}{z}\right)\left(\frac{y}{x}\right)\left(\frac{z}{y}\right)} = 3$$

so we have proven the lower bound for $\frac{x}{z}+\frac{y}{x}+\frac{z}{y}$.

(b) **(i)** By definition of the scalar product, for any two vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

we have $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the two vectors. Since $|\cos\theta| \leq 1$, then $\mathbf{a}\cdot\mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$, which implies that

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\cdot\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \leq \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right|\left|\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right|$$

and the result follows. For equality to hold, we must have $a_i = kb_i$ for all $1 \leq i \leq 3$ and some $k \in \mathbb{R}\setminus\{0\}$.

(ii) Using the Cauchy-Schwarz inequality,

$$\left[\left(\frac{x}{\sqrt{y+z}}\right)^2+\left(\frac{y}{\sqrt{z+x}}\right)^2+\left(\frac{z}{\sqrt{x+y}}\right)^2\right]\left[\left(\sqrt{y+z}\right)^2+\left(\sqrt{z+x}\right)^2+\left(\sqrt{x+y}\right)^2\right] \geq (x+y+z)^2$$

$$\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right)(y+z+z+x+x+y) \geq (x+y+z)^2$$

$$2\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right) \geq x+y+z$$

and equality holds if and only if $x=y=z$.

Question 2

(i) Using the substitution $u = x^2$, the integral becomes

$$\int_2^3 \frac{x^2}{x-1}\,dx = \int_2^3 x+1+\frac{1}{x-1}\,dx = \ln 2+\frac{7}{2}.$$

The substitution is motivated by the presence of the square root in the denominator and the fact that 4 and 9 are square numbers.

(ii) Using the substitution $y = xu$, we have

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$
$$\frac{1}{x}\frac{dy}{dx} = \frac{du}{dx} + \frac{u}{x}$$

so the differential equation becomes $\dfrac{du}{dx} = f(u)$.

(iii) Dividing both sides by x, we have

$$\frac{1}{x}\frac{dy}{dx} = \sqrt{\frac{x}{y}} - \frac{x}{y} + \frac{y}{x^2}$$

so

$$f\left(\frac{y}{x}\right) = \sqrt{\frac{x}{y}} - \frac{x}{y}.$$

As such,

$$f(u) = \frac{1}{\sqrt{u}} - \frac{1}{u} = \frac{\sqrt{u}-1}{u}.$$

The differential equation becomes $\dfrac{du}{dx} = \dfrac{\sqrt{u}-1}{u}$. From **(i)**, we have

$$\frac{2u\sqrt{u} + 3u + 6\sqrt{u} + 6\ln|\sqrt{u}-1|}{3} = x + c,$$

where c is a constant. As the solution curve passes through $\left(\dfrac{1}{3}, \dfrac{4}{3}\right)$, then $u = 4$. Substituting these into the above equation yields $c = 13$. Hence,

$$2\frac{y}{x}\sqrt{\frac{y}{x}} + 3\left(\frac{y}{x}\right) + 6\sqrt{\frac{y}{x}} + 6\ln\left|\sqrt{\frac{y}{x}} - 1\right| = 3x + 39.$$

When $y = 9x$, we have $60 + 6\ln 2 = 3x$, so $x = 20 + 2\ln 2$, which is the required x-coordinate.

Question 3

(i) **(a)** Let $S = \{a, 2a, \ldots (p-1)a\}$. For all $1 \le i \le p-1$, none of the $ia \in S$ is divisible by p because a is not divisible by p. Suppose $ai \equiv aj \pmod p$. Then, there exists $\lambda \in \mathbb{Z}$ such that $ai = \lambda p + aj$, so $a(i-j) = \lambda p$. However, p does not divide a so p must divide $i - j$. That is, $i \equiv j \pmod p$. As $1 \le i, j \le p-1$, then $i = j$ so all the elements in S are distinct. In $\bmod p$, the elements in S are a permutation of T, where $T = \{1, 2, \ldots, p-1\}$.

(b) In $\bmod p$, the product of the elements in S is congruent to the product of the elements in T. That is,

$$a \cdot 2a \cdot 3a \cdot (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdot (p-1) \pmod p.$$

So, $a^{p-1} \equiv 1 \pmod p$.

(ii) By the binomial theorem,

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$
$$= x^5 + y^5 + 5k$$

where $k \in \mathbb{Z}$. So, $x^5 + y^5 = (x+y)^5 - 5k$. As $x^5 + y^5 \equiv 0 \pmod 5$, then $(x+y)^5 \equiv 0 \pmod 5$.

We use the method of contraposition to prove that $x + y \equiv 0 \pmod 5$. That is to say, given that $x + y$ is not a multiple of 5, we wish to prove that $(x+y)^5$ is also not a multiple of 5. We write $x + y = 5k + r$, where $1 \le r \le 4$. So,

$$(x+y)^5 = (5k+r)^5 = 3125k^5 + 3125k^4r + 1250k^3r^2 + 250k^2r^3 + 25kr^4 + r^5$$

so $(x+y)^5 \equiv r^5 \pmod 5$ but because $1 \le r \le 4$, then r^5 is not a multiple of 5 and so we have proven that $x + y \equiv 0 \pmod 5$. As such, there exists $\alpha \in \mathbb{Z}$ such that $x + y = 5\alpha$. Then, $x = 5\alpha - y$. Hence,

$$x^5 + y^5 = (5\alpha - y)^5 + y^5 = 3125\alpha^5 - 3125\alpha^4y + 1250\alpha^3y^2 - 250\alpha^2y^3 + 25\alpha y^4.$$

It follows that $x^5 + y^5$ is divisible by 25.

Remark for Question 3: This deals with a well-known result in number theory called Fermat's little theorem. It states that if $\gcd(a, p) = 1$ (i.e. a is not divisible by p), then $a^{p-1} \equiv 1 \pmod{p}$. An alternative representation says that for any integer a, $a^p \equiv a \pmod{p}$. Our method of proving Fermat's little theorem was using modulo inverse.

Question 4

(i) 5^n

(ii) **(a)** $B_1 = 5$ and $B_2 = 24$; B_2 can be calculated easily by considering the complement of the event 'never chooses Scrambled eggs on consecutive days' so $B_2 = 5^2 - 1$.

(b) We consider two cases.

- **Case 1 (Scrambled eggs on the 1st day):** On the 2nd day, she has 4 choices remaining. There would be no restrictions on what she has on the remaining $n-2$ days. This contributes to $4B_{n-2}$.
- **Case 2 (no Scrambled eggs on the 1st day):** On the 1st day, she has 4 choices. Thereafter, she has no restrictions on what she has on the remaining days. This contributes to $4B_{n-1}$.

Since the 2 cases are mutually exclusive, the result follows.

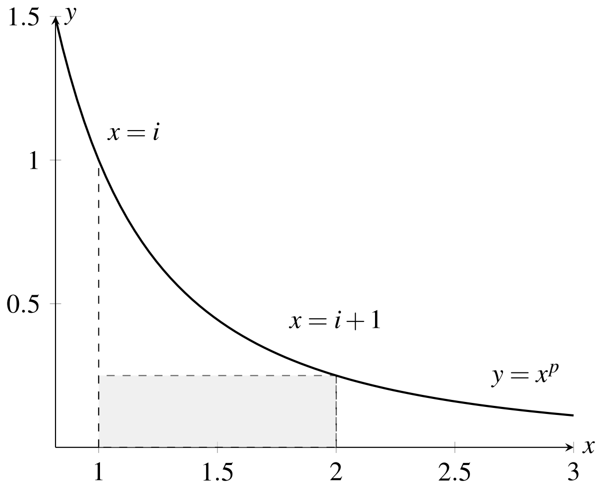
(c) Let P_k be the proposition that $B_{3k+1} \equiv 0 \pmod{5}$ for all $k \in \mathbb{Z}_{\geq 0}$.
When $k = 0$, we have $B_1 = 5$, which is divisible by 5. So, P_0 is true.
Assume P_r is true for some $r \in \mathbb{Z}_{\geq 0}$. Then, $B_{3r+1} \equiv 0 \pmod{5}$. We are required to show $B_{3r+4} \equiv 0 \pmod{5}$.
Using the relation in **(iib)**, as $B_k = 4B_{k-1} + 4B_{k-2}$, then

$$\begin{aligned}
B_{3r+4} &= 4B_{3r+3} + 4B_{3r+2} \\
&= 4\left(4B_{3r+2} + 4B_{3r+1}\right) + 4B_{3r+2} \\
&= 20B_{3r+2} + 16B_{3r+1} \\
&\equiv 16B_{3r+1} \pmod{5} \\
&\equiv 0 \pmod{5} \quad \text{by induction hypothesis}
\end{aligned}$$

Since P_0 is true and P_r is true implies P_{r+1} is true, by mathematical induction, P_k is true for all $k \in \mathbb{Z}_{\geq 0}$.

Question 5

(i) **(a)** Consider the following graph of $y = x^p$ for $p < 0$ and $x > 0$ (we set $i = 2$ here but actually, i is arbitrary):

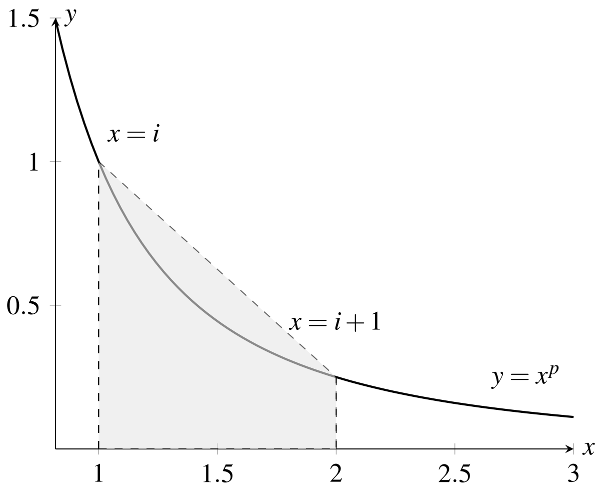


$\int_i^{i+1} x^p \, dx$ denotes the area bounded by the curve, the x-axis and the ordinates $x = i$ and $x = i+1$. We construct the rectangle above which has a base of 1 unit and a height of $(i+1)^p$. Its area is $(i+1)^p$ units^2, which is less than the given integral.

(b) It suffices to prove that

$$\int_i^{i+1} x^p \, dx < \frac{i^p + (i+1)^p}{2}.$$

Naturally, we would think of the right side of the inequality as the area of another figure other than a rectangle. Consider the following graph:



We construct a trapezium bounded by the x-axis and the ordinates $x = i$ and $x = i+1$. Its area is $\frac{i^p + (i+1)^p}{2}$. The integral is less than the area of the trapezium and the result follows.

(ii) Using **(ia)**,

$$\begin{aligned}
(i+1)^p &< \int_i^{i+1} x^p \, dx \\
2^p + 3^p + \ldots + n^p &< \int_1^2 x^p \, dx + \int_2^3 x^p \, dx + \ldots + \int_{n-1}^{n} x^p \, dx \\
\sum_{k=1}^{n} k^p &< 1 + \int_1^n x^p \, dx
\end{aligned}$$

The required sum is $\sum_{k=1}^{\infty} k^p$ so as $n \to \infty$, we have

$$\sum_{k=1}^{\infty} k^p < 1 + \left[\frac{x^{p+1}}{p+1}\right]_0^{\infty} = \lim_{n\to\infty}\left(1 + \frac{n^{p+1} - 1}{p+1}\right) = 1 - \frac{1}{p+1} = \frac{p}{1+p}.$$

(iii) In **(ii)**, we used **(ia)** to show that

$$2^p + 3^p + \ldots + n^p < \int_1^n x^p \, dx.$$

Considering the integral on the right side of the equation, we have

$$\int_1^n x^p \, dx = \frac{n^{p+1} - 1}{p+1}.$$

Adding $1^p = 1$ to both sides, we establish an upper bound for $1^p + 2^p + 3^p + \ldots + n^p$.

Using **(ib)**, we have

$$\begin{aligned}
\int_i^{i+1} x^p \, dx &< \frac{i^p + (i+1)^p}{2} \\
\int_1^2 x^p \, dx + \int_2^3 x^p \, dx + \ldots + \int_{n-1}^n x^p \, dx &< \frac{1^p + 2^p}{2} + \frac{2^p + 3^p}{2} + \ldots + \frac{(n-1)^p + n^p}{2} \\
\int_1^n x^p \, dx &< \frac{1^p}{2} + \frac{n^p}{2} + \sum_{k=2}^{n-1} k^p \\
\frac{n^{p+1} - 1}{p+1} &< \frac{1^p}{2} + \frac{n^p}{2} + \sum_{k=2}^{n-1} k^p \\
\frac{1^p + n^p}{2} + \frac{n^{p+1} - 1}{p+1} &< \sum_{k=1}^{n} k^p
\end{aligned}$$

so we have established a lower bound for $1^p + 2^p + 3^p + \ldots + n^p$.

Therefore,

$$\begin{aligned}
\frac{1 + n^p}{2n^{p+1}} + \frac{n^{p+1} - 1}{n^{p+1}(p+1)} &< \frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}} < \frac{1}{n^{p+1}} + \frac{n^{p+1} - 1}{n^{p+1}(p+1)} \\
\frac{1}{2n^{p+1}} + \frac{1}{n} + \frac{1}{p+1} - \frac{1}{n^{p+1}(p+1)} &< \frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}} < \frac{1}{n^{p+1}} + \frac{1}{p+1} - \frac{1}{n^{p+1}(p+1)}
\end{aligned}$$

As $p > -1$, then $p + 1 > 0$. As $n \to \infty$ on both sides, by the squeeze theorem, the upper and lower bounds will tend to $\dfrac{1}{p+1}$. Therefore,

$$\lim_{n \to \infty} \left(\frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}} \right) = \frac{1}{p+1}.$$

Remark for Question 5: There is a formula for the sum $1^p + 2^p + \ldots + n^p$ which is known as Faulhaber's formula. It states that

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{r=0}^{p} \binom{p+1}{r} B_r n^{p-r+1},$$

where B_r denotes the sequence of Bernoulli numbers.

Question 6

(i) Since f is continuous on $[0, 0.4]$, $f(0) = 1 > 0$ and $f(0.4) = -0.136 < 0$, then there exists a root in $(0, 0.4)$.
Next, since f is continuous on $[0.4, 2]$, $f(0.4) = -0.136 < 0$ and $f(2) = 3 > 0$, then there exists a root in $(0.4, 2)$.
Lastly, since f is continuous on $[-2, 0]$, $f(-2) = -1 < 0$ and $f(0) = 1 > 0$, then there exists a root in $(-2, 0)$.

The above shows that f has at least three distinct real roots. To show that there are only three distinct real roots, consider $f'(x) = 3\left(x^2 - 1\right)$ so f is strictly increasing for $x > 1$ and strictly decreasing for $x < -1$.

(ii) Note that

$$fg(x) = f\left(\frac{1}{1-x}\right) = \left(\frac{1}{1-x}\right)^3 - 3\left(\frac{1}{1-x}\right) + 1 = -\frac{1 - 3x + x^3}{(1-x)^3}$$

so $g(\alpha), g(\beta)$ and $g(\gamma)$ are the roots of f. From **(i)**, we know that $\alpha \in (-2,0)$, $\beta \in (0,0.4)$ and $\gamma \in (0.4,2)$. We have $g(\gamma) < 0$, which implies that $g(\gamma) = \alpha$. Suppose on the contrary that $g(\beta) = \beta$. Then,

$$\frac{1}{1-\beta} = \beta.$$

That is, $\beta^2 - \beta + 1 = 0$. However, the roots of this equation are not real, which is a contradiction. As such, $g(\beta) = \gamma$, leaving us with $g(\alpha) = \beta$.

(iii) Write $h(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Then for $x = \alpha, \beta, \gamma$, we have

$$\begin{aligned}
ax^2 + bx + c &= \frac{1}{1-x} \\
ax^2(1-x) + bx(1-x) + c(1-x) - 1 &= 0 \\
ax^2 - ax^3 + bx - bx^2 + c - cx - 1 &= 0 \\
-ax^3 + (a-b)x^2 + (b-c)x + c - 1 &= 0
\end{aligned}$$

Comparing the last line with $f(x)$, we see that $a = -1$, $b = -1$ and $c = 2$. So, $h(x) = -x^2 - x + 2$.

Remark for Question 6: For **(i)**, to put it more rigorously, the justification of the existence of a root is due to the intermediate value theorem. It states that given a continuous function f on an interval $[a,b]$ such that $f(a)$ and $f(b)$ have different polarities (i.e. either $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$), then there exists some $c \in (a,b)$ such that $f(c) = 0$.

Question 7

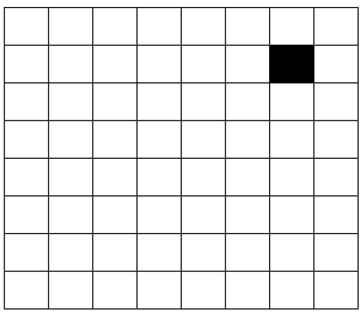
(i) Consider a square board with 4^n unit squares. Without a loss of generality, suppose a unit square in the 1st quadrant is covered. Then, consider the 4 unit squares at the centre. Cover all the squares except that in the 1st quadrant. As the board can be rotated in any direction, regardless of which unit square is originally covered, the result follows.

(ii) First, note that the area of the board is 4^n units^2, then the length must be 2^n units.
Let P_n be the proposition that on a $2^n \times 2^n$ square board, if one unit square is initially covered, then the remaining unit squares can be covered by triominoes, and the total number of triominoes required is $\frac{1}{3}(4^n - 1)$ for all $n \in \mathbb{N}$.

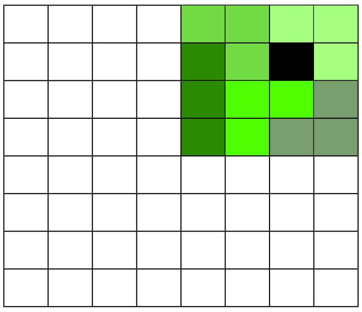
When $n = 1$, we have a 2×2 square board. If one unit square is initially covered, then we have 3 unit squares remaining. They are arranged in an L-shape. The remaining unit squares can be covered by triominoes. The total number of triominoes is 1. Hence, P_1 is true.

Assume that P_k is true for some $k \in \mathbb{N}$. That is, on a $2^k \times 2^k$ square board, if one unit square is initially covered, the remaining unit squares can be covered by $\frac{1}{3}\left(4^k - 1\right)$ triominoes. We wish to prove that P_{k+1} is true. That is, on a $2^{k+1} \times 2^{k+1}$ square board, if one unit square is initially covered, the remaining unit squares can be covered by $\frac{1}{3}\left(4^{k+1} - 1\right)$ triominoes.

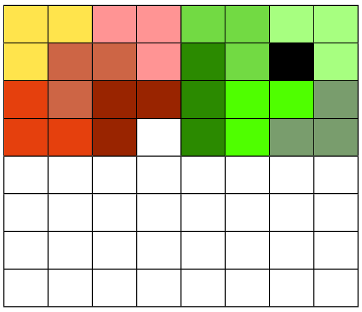
Consider a $2^{k+1} \times 2^{k+1}$ square board. Divide it into four $2^k \times 2^k$ square boards. Without a loss of generality, suppose a unit square in the 1st quadrant is initially covered as shown.



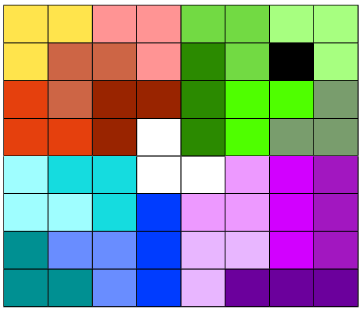
By the induction hypothesis, that $2^k \times 2^k$ board can be covered by $\frac{1}{3}\left(4^k - 1\right)$ triominoes.



For the $2^k \times 2^k$ board in the 2nd quadrant, we can cover it with $\frac{1}{3}\left(4^k-1\right)$ triominoes such that a unit square remains in the bottom-right corner.



Repeat this process for the $2^k \times 2^k$ boards in the 3rd and 4th quadrants and do not occupy the top-right and top-left corners respectively. This can be covered with $2 \times \frac{1}{3}\left(4^k-1\right)$ triominoes.



Finally, the 2×2 square board in the centre can be covered by one triomino.

We see that the total number of triominoes required is $\frac{4}{3}\left(4^k-1\right)+1=\frac{1}{3}\left(4^{k+1}-1\right)$. Since P_1 is true and P_k is true implies P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{N}$.

Question 8

(a) Let S_1 and S_2 be the following sets:

$$S_1=\{x \in \mathbb{Z}: a \leq x \leq b\} \quad \text{and} \quad S_2=\{x \in \mathbb{Z}: c \leq x \leq d\}$$

One can sketch a number line and come up with two cases.

- **Case 1:** Suppose $c \leq a \leq b \leq d$. Then, $S_1 \subseteq S_2$. Since $|S_1| \leq |S_2|$, then the number of integers x is $b-a+1$.
- **Case 2:** Suppose $a \leq c \leq d \leq b$. Then, $S_2 \subseteq S_1$. Since $|S_2| \leq |S_1|$, the number of integers x is $d-c+1$.

The result follows.

(b) Consider $x+y=n$ and $0 \leq y \leq b$. Then, $0 \leq n-x \leq b$, so $n-b \leq x \leq n$. Thus, the equation $x+y=n$ is restricted to the conditions $0 \leq x \leq a$ and $n-b \leq x \leq n$.

Since $a+b \geq n$, then $a \geq n-b$. Consider $x+y=n$ and $0 \leq x \leq a$. Then, $0 \leq n-y \leq a$, so $n-a \leq y \leq n$. Thus, the equation $x+y=n$ is restricted to the conditions $0 \leq y \leq b$ and $n-a \leq y \leq n$. The result follows.

(c) Let A, B and C denote the following sets:

$$A = \{x \in \mathbb{Z}_{\geq 0} : x > a\}, \quad B = \{y \in \mathbb{Z}_{\geq 0} : y > a\} \quad \text{and} \quad C = \{z \in \mathbb{Z}_{\geq 0} : z > a\}$$

So,

$$\begin{aligned} \left|A' \cap B' \cap C'\right| &= |\xi| - |A \cup B \cup C| \quad \text{by de Morgan's law} \\ &= |\xi| - 3\,|A| + 3\,|A \cap B| - |A \cap B \cap C| \quad \text{by the principle of inclusion and exclusion} \end{aligned}$$

Note that $|A \cap B \cap B| = 0$. If the cardinality was positive, it would imply that $x + y + z > 3a$ but this contradicts the fact that $x + y + z \leq 3a$.

Hence,

$$|A| = \binom{n-a+1}{2}, \; |A \cap B| = \binom{n-2a}{2} \text{ and } |A \cap B \cap C| = 0.$$

Therefore,

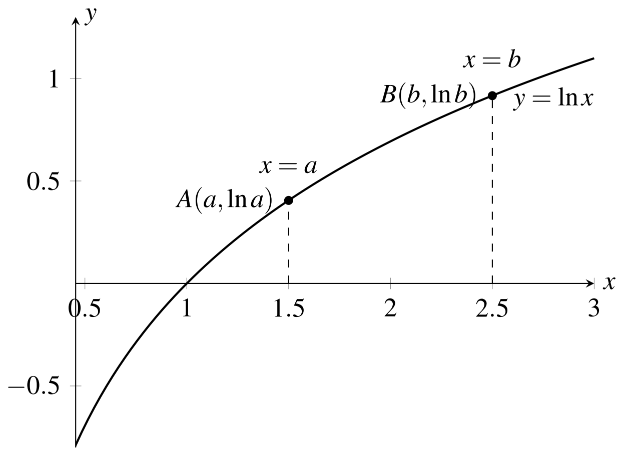
$$\left|A \cap B' \cap C'\right| = \binom{n+2}{2} - 3\binom{n-a+1}{2} + 3\binom{n-2a}{2}.$$

Remark for Question 8: Here is an interactive solution to **(ii)**.

2 2017 Paper Solutions

Question 1

(i) Consider the graph of $y = \ln x$. Plot the points $A(a, \ln a)$ and $B(b, \ln b)$ and without loss of generality, assume $0 < a \leq b$. Here, a and b are set to be 1.5 and 2.5 respectively but they can be arbitrarily chosen such that $0 < a \leq b$.



Let M be on the graph such that its x-coordinate is the average of A and B. So, M has coordinates $\left(\frac{a+b}{2}, \ln\left(\frac{a+b}{2}\right)\right)$. Also, let C be such that its y-coordinate is the average of A and B. Then, the y-coordinate of C is $\frac{1}{2}(\ln a + \ln b)$.

As $y = \ln x$ is concave down, the y-coordinate of C is less than or equal to that of M. As such, the result follows with equality attained if and only if $a = b$.

(ii) We have

$$\ln\left(\frac{a+b}{2}\right) \geq \ln\sqrt{ab}.$$

Since $y = \ln x$ is an injective function, then $\dfrac{a+b}{2} \geq \sqrt{ab}$, which is the AM-GM inequality for two variables.

(iii) Let $y = x\ln x$. Then, $\dfrac{d^2y}{dx^2} = \dfrac{1}{x}$ so for all $x > 0$, $\dfrac{d^2y}{dx^2} > 0$. This shows that y is concave up (alternatively, one can use a graphing calculator to verify this). Let A and B have coordinates $(a, a\ln a)$ and $(b, b\ln b)$ respectively. So,

$$\begin{aligned}
\frac{a\ln a + b\ln b}{2} &\geq \frac{a+b}{2}\ln\left(\frac{a+b}{2}\right) \quad \text{since } y = x\ln x \text{ is concave down} \\
a\ln a + b\ln b &\geq (a+b)\ln\left(\frac{a+b}{2}\right) \\
\ln\left(a^a b^b\right) &\geq \ln\left(\left(\frac{a+b}{2}\right)^{a+b}\right) \\
a^a b^b &\geq \left(\frac{a+b}{2}\right)^{a+b} \quad \text{by injectivity of } \ln x
\end{aligned}$$

Question 2

(i) Let P_n be the proposition that

$$\frac{d^n}{dx^n}(xy) = x\frac{d^n y}{dx^n} + n\frac{d^{n-1}y}{dx^{n-1}}$$

for all positive integers n.

When $n = 1$, the LHS is $\frac{d}{dx}(xy)$, which is equal to $x\frac{dy}{dx} + y$. This expression is equal to the RHS.

Assume that P_k is true for some positive integer k. That is,

$$\frac{d^k}{dx^k}(xy) = x\frac{d^k y}{dx^k} + k\frac{d^{k-1}y}{dx^{k-1}}.$$

To show that P_{k+1} is true, we need to prove that

$$\frac{d^{k+1}}{dx^{k+1}}(xy) = x\frac{d^{k+1}y}{dx^{k+1}} + (k+1)\frac{d^k y}{dx^k}.$$

So,

$$\begin{aligned}
\text{LHS} &= \frac{d^{k+1}}{dx^{k+1}}(xy) \\
&= \frac{d}{dx}\left(\frac{d^k}{dx^k}(xy)\right) \\
&= \frac{d}{dx}\left(x\frac{d^k y}{dx^k} + n\frac{d^{k-1}y}{dx^{k-1}}\right) \quad \text{by induction hypothesis} \\
&= x\frac{d^{k+1}y}{dx^{k+1}} + \frac{d^k y}{dx^k} + n\frac{d^k y}{dx^k} \\
&= x\frac{d^{k+1}y}{dx^{k+1}} + (k+1)\frac{d^k y}{dx^k} = \text{RHS}
\end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, by mathematical induction, P_n is true for all positive integers n.

(ii) **(a)** $y_0 = 1$

$y_1 = e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right) = e^{x^2}\left(-2xe^{-x^2}\right) = -2x$

$y_2 = e^{x^2}\frac{d^2}{dx^2}\left(e^{-x^2}\right) = e^{x^2}\frac{d}{dx}\left(-2xe^{-x^2}\right) = 4x^2 - 2$

(b)

$$\begin{aligned}
y_{n+2} + 2xy_{n+1} + 2(n+1)y_n &= e^{x^2}\frac{d^{n+2}}{dx^{n+2}}\left(e^{-x^2}\right) + 2xe^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) + 2(n+1)e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right) \\
&= e^{x^2}\frac{d^{n+2}}{dx^{n+2}}\left(e^{-x^2}\right) + 2e^{x^2}\left[x\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) + (n+1)\frac{d^n}{dx^n}\left(e^{-x^2}\right)\right] \\
&= e^{x^2}\frac{d^{n+2}}{dx^{n+2}}\left(e^{-x^2}\right) + 2e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(xe^{-x^2}\right) \quad \text{using } \textbf{(i)} \text{ by setting } y = e^{-x^2} \\
&= e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left[\frac{d}{dx}\left(e^{-x^2}\right) + 2xe^{-x^2}\right] \\
&= 0
\end{aligned}$$

(c) From **(b)**, it follows that $y_{n+2} + 2xy_{n+1} = -2(n+1)y_n$. Hence,

$$\begin{aligned}
\frac{d}{dx}(y_{n+1}) &= \frac{d}{dx}\left[e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)\right] \\
&= e^{x^2}\frac{d^{n+2}}{dx^{n+2}}\left(e^{-x^2}\right) + 2xe^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) \\
&= y_{n+2} + 2xy_{n+1} \\
&= -2(n+1)y_n
\end{aligned}$$

Question 3

(a) As $\gcd(1591, 3913, 9331) = 43$, factorising 43 from both sides of the equation yields $37x + 91y = 217$, or rather, $37x = 7(31 - 13y)$. So, $37x$ is a multiple of 7, which forces x to be a multiple of 7. The only prime that is a multiple of 7 is 7, but if $x = 7$, then $y = -\frac{6}{13} \notin \mathbb{Z}$. So, we conclude that there are no integer solutions with x prime.

(b) **(i)** As a and b are factors of n, there exist $\lambda, \mu \in \mathbb{Z}$ such that $n = \lambda a = \mu b$.
Given that $ra + sb = 1$, then $ran + sbn = n$. So, $ab(r\mu + s\lambda) = n$, which asserts that ab is a factor of n.

(ii) Suppose $x \equiv u \pmod{a}$. Then, there exists $k \in \mathbb{Z}$ such that $x = ka + u$.
Write $k = k_1 b + q$ for some $k_1, q \in \mathbb{Z}$. So, $x = ak_1 b + aq + u$, which implies that $x = aq + u \pmod{b}$.
As $ra + sb = 1$, then

$$\begin{aligned} r(v-u)a + s(v-u)b &= v - u \\ r(v-u)a &\equiv v - u \pmod{b} \quad (*) \end{aligned}$$

By choosing $k = k_1 b + r(v-u)$, i.e. $q = r(v-u)$, we have

$$\begin{aligned} x &\equiv ar(v-u) + u \pmod{b} \\ &= v - u + u \pmod{b} \quad \text{using } (*) \\ &\equiv v \pmod{b} \end{aligned}$$

Hence, we have constructed a number $x = (b + r(v-u))a + u = ab + ar(v-u) + u$ such that $x \equiv u \pmod{a}$ and $x \equiv v \pmod{b}$.

Question 4

(i)

$$\begin{aligned} I_n + I_{n-2} &= \int_0^{\frac{\pi}{4}} \tan^n x + \tan^{n-2} x \, dx \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(1 + \tan^2 x\right) \, dx \\ &= \int_0^{\frac{\pi}{4}} \sec^2 x \tan^{n-2} x \, dx \\ &= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} = \frac{1}{n-1} \end{aligned}$$

(ii) $y = \tan x$ is strictly increasing on $[0, \frac{1}{4}\pi]$. Substituting the x-coordinates of the endpoints, $0 \le \tan x \le 1$.
Consider the y-coordinates of a linear function $y = mx$ to be upper bounds for all y values of $y = \tan x$. For $0 \le x \le \frac{\pi}{4}$, it must satisfy $0 \le mx \le 1$.

Hence, $0 \le mx \le \frac{m}{4}\pi$, implying that $m = \frac{4}{\pi}$. The required linear function is $y = \frac{4}{\pi}x$. The inequality $\tan x \le \frac{4}{\pi}x$ is true and equality holds if and only if $x = 0$ or $x = \frac{\pi}{4}$.

(iii) Since $\tan x \ge 0$ on $[0, \frac{1}{4}\pi]$, combining this with **(ii)** yields $0 \le \tan x \le \frac{4}{\pi}x$.

So,

$$\begin{aligned} 0 &\le \int_0^{\frac{\pi}{4}} \tan^n x \, dx \le \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi}x\right)^n dx \\ 0 &\le I_n \le \left(\frac{4}{\pi}\right)^n \int_0^{\frac{\pi}{4}} x^n \, dx \\ 0 &\le I_n \le \left(\frac{4}{\pi}\right)^n \left[\frac{x^{n+1}}{n+1}\right]_0^{\frac{\pi}{4}} \\ 0 &\le I_n \le \frac{\pi}{4(n+1)} \end{aligned}$$

As $\lim\limits_{n\to\infty} \dfrac{\pi}{4(n+1)} = 0$, by the squeeze theorem, I_n tends to zero as well.

(iv) Note that $I_2+I_0=\frac{1}{1}$, $I_4+I_2=\frac{1}{3}$ and $I_6+I_4=\frac{1}{5}$, which are the magnitudes of the first three terms of the series. By the method of difference,

$$\begin{aligned}\frac{1}{1}-\frac{1}{3}+\frac{1}{5}+\ldots &= (I_2+I_0)-(I_4-I_2)+(I_6-I_4)+\ldots \\ &= I_0=\frac{\pi}{4}\end{aligned}$$

Remark for Question 4: This question involves proving Madhava's formula for π. It is an example of a Madhava series which is a collection of infinite series believed to have been discovered by Madhava of Sangamagrama in the 1200s. James Gregory and Gottfried Wilhelm Leibniz discovered the series much later in the 1670s. For most of the Western world, the series is known as the Leibniz series.

Question 5

(i) Suppose there are no restrictions. For each object, it can go into either box. There are 2^r ways to do this. As there are 2 cases where either box is empty, the result follows.

(ii) **(a)** Note that $S(r,n)$ represents a Stirling number of the second kind.
Let the set M comprise the r objects. So, we write $M=\{a_1,\ldots,a_r\}$.

- **Case 1:** Suppose for some $1\leq j\leq r$, a_j is the only object in a box. There is 1 way as the boxes are identical. The remaining $r-1$ objects can be distributed into the remaining 2 boxes. The number of ways is $2^{r-2}-1$.
- **Case 2:** Suppose for some $1\leq j\leq r$, a_j is mixed with other objects. We first distribute the remaining $r-1$ objects into 3 boxes. Then, a_j can enter either one of the 3 boxes in $3S(r-1,3)$ ways.

Since the two cases are mutually exclusive, the result follows by the addition principle.

(b) As $S(r,3)=2^{r-2}-1+3S(r-1,3)$, then $S(r+2,3)=9S(r,3)+5\left(2^{r-1}\right)-4$.
Let P_r be the proposition that

$$S(r,3)\equiv\begin{cases}0 \pmod 6 & \text{if } r \text{ is even}\\ 1 \pmod 6 & \text{if } r \text{ is odd}\end{cases}$$

for all positive integers r such that $r\geq 3$.

When $r=3$, there is only 1 way to distribute 1 object into 1 box, so $S(3,3)=1\equiv 1 \pmod 6$.
When $r=4$, using the recurrence relation established in **(iia)**, we have $S(4,3)=3+3S(3,3)\equiv 0 \pmod 6$.
These assert that the base cases P_3 and P_4 are true.

Assume P_k is true for some positive integer k such that $k\geq 3$. That is,

$$S(k,3)\equiv\begin{cases}0 \pmod 6 & \text{if } k \text{ is even}\\ 1 \pmod 6 & \text{if } k \text{ is odd}\end{cases}$$

We wish to prove that P_{k+2} is true. That is,

$$S(k+2,3)\equiv\begin{cases}0 \pmod 6 & \text{if } k+2 \text{ is even}\\ 1 \pmod 6 & \text{if } k+2 \text{ is odd}\end{cases}$$

Suppose k is even. Then, $k+2$ is also even, so

$$\begin{aligned}S(k+2,3) &= 9S(k,3)+5\left(2^{k-1}\right)-4\\ &\equiv 5\left(2^{k-1}\right)-4 \pmod 6 \quad \text{by induction hypothesis}\\ &\equiv 0 \pmod 6\end{aligned}$$

Now, suppose k is odd. Then, $k+2$ is also odd, so

$$\begin{aligned}
S(k+2,3) &= 9S(k,3)+5\left(2^{k-1}\right)-4 \\
&\equiv 5+5\left(2^{k-1}\right) \pmod 6 \quad \text{by induction hypothesis} \\
&= 5\left(2^{k-1}+1\right) \\
&\equiv 5(-1) \pmod 6 \\
&\equiv 1 \pmod 6
\end{aligned}$$

Since P_3 and P_4 are true and P_k is true implies P_{k+1} is true, by mathematical induction, P_r is true for all positive integers r such that $r \geq 3$.

Question 6

(a) **(i)** Label the beads $a_1,\ldots,a_n$.
We first consider a linear permutation, which can be done in $n!$ ways.
As the circle can be rotated, then suppose a_1 goes to the old position of a_2, a_2 goes to the old position of a_3, and so on. We obtain a permutation of the same configuration as before. So, the number of arrangements in a circle is $\dfrac{n!}{n}=(n-1)!$.

(ii) When there are no restrictions, there are $(n-1)!$ ways to arrange the beads.
If two beads are adjacent, there are $2((n-1)-1)!$ ways to arrange them.
Hence, the required number of ways is $(n-1)!-2(n-2)!=(n-2)!(n-3)$.

(iii) First, note that the result holds if and only if $n>5$.
There are $\binom{n}{3}$ ways to choose 3 beads out of n and n ways to choose 3 adjacent beads. For 2 fixed but adjacent beads, there are n ways to choose and $n-4$ ways to choose the 3rd bead so that the 3rd bead is not adjacent to the first two beads.

Hence, the answer is

$$\begin{aligned}
\binom{n}{3}-n-n(n-4) &= \frac{n(n-1)(n-2)}{6}-n-n(n-4) \\
&= n\left[\frac{(n-1)(n-2)+6(3-n)}{6}\right] \\
&= \frac{n(n-4)(n-5)}{6}
\end{aligned}$$

(b) Let A and B denote the following sets:

$A=\{$all 4-tuples denoting all collections of 4 points on the perimeter of the circle$\}$

$B=\{$all interior points in the circle when the maximum possible number of interior points is achieved$\}$

Let $f:A\to B$ be a function. Suppose $a\in A$. Then there exists $b\in B$ such that $f(a)=b$ as we can always find 4 points that form the 2 chords on which b is the intersection of the 2 chords.

To show f is injective, suppose $f(a)=f(a')$. Suppose on the contrary that $a\neq a'$. Then, we can shift 2 of the 4 points in a' such that 2 additional interior points are formed instead of 1, which is a contradiction. So, $a=a'$.

To show f is surjective, as every $b\in B$ is formed by the intersection of 2 chords, it corresponds to 4 distinct points on the perimeter of the circle.

Since f is injective and surjective, it is thus bijective, so by the bijection principle, $|A|=|B|=\binom{n}{4}$.

Question 7

(i) Let $x=4k+3$ for some $k\in\mathbb{Z}_{\geq 0}$. As $x\equiv 1 \pmod 2$, then x is odd, so its divisors are also odd.
Suppose on the contrary that all the prime divisors of x are of the form 1 mod 4. For any two integers of the form

1 mod 4, say $4m+1$ and $4n+1$, where $m,n \in \mathbb{Z}$, their product, $16mn+4m+4n+1$ is also 1 mod 4. Hence, the product of any number of integers of the form 1 mod 4 is also of the form 1 mod 4.

Thus, there exists at least one prime factor of the form 1 mod 3, which is in Q.

(ii) Suppose on the contrary that there are finitely many primes in Q. Then, $Q = \{q_1, \ldots, q_n\}$ with $q_1 = 3$, etc. From **(i)**, $N = 4q_2 \ldots q_n + 3$ is divisible by some prime in Q. However, none of the q_i's, for $1 \le i \le n$, divides N. Thus, there are infinitely many primes in Q.

Remark for Question 7: The infinitude of primes of the form $4k+3$ is a particular case of Dirichlet's theorem on arithmetic progressions. It states that if $\gcd(a,b) = 1$, then there are infinitely many primes of the form $an+b$.

Question 8

(i) **(a)** By considering the sequence $1,1,2,0,2,2,1,0,1,1,2,0,2,2,1,0,\ldots$, the period is 8

(b) By considering the sequence $1,1,2,3,1,0,1,1,2,3,1,0,1,\ldots$, the period is 6

(ii) Modulo m, there are m possible values which are $0,1,2,\ldots,m-1$. So, there are m^2 distinct pairs. As $1 \le j < k \le m^2+1$, by considering m^2+1 pairs of (F_i, F_{i+1}) modulo m, where $1 \le i \le m^2+1$, the result follows by the pigeonhole principle.

(iii) Here, we would use the method of strong induction. Unlike the conventional method of mathematical induction, strong induction uses more statements in the induction hypothesis.

Let P_n be the proposition that there exists $j,k \in \mathbb{N}$, where $j<k$, such that $F_{j+n} \equiv F_{k+n} \pmod{m}$ for all $n \in \mathbb{Z}_{\ge 0}$.
When $n=0$, then $F_j \equiv F_k \pmod{m}$.
When $n=1$, then $F_{j+1} \equiv F_{k+1} \pmod{m}$.
The base cases P_0 and P_1 are true because in **(ii)**, we established that $\left(F_j, F_{j+1}\right) \equiv \left(F_k, F_{k+1}\right) \pmod{m}$.

Assume that P_r and P_{r+1} are true for some $r \in \mathbb{Z}_{\ge 0}$. That is,

$$F_{j+r} \equiv F_{k+r} \pmod{m} \text{ and } F_{j+r+1} \equiv F_{k+r+1} \pmod{m}.$$

To show P_{r+2} is true, we need to prove $F_{j+r+2} \equiv F_{k+r+2} \pmod{m}$.
This is true because

$$\begin{aligned} F_{j+r+2} &= F_{j+r+1} + F_{j+r} && \text{by definition of Fibonacci sequence} \\ &\equiv F_{k+r+1} + F_{k+r} \pmod{m} && \text{by induction hypothesis} \\ &\equiv F_{k+r+2} \pmod{m} && \text{by definition of Fibonacci sequence} \end{aligned}$$

Since P_0 and P_1 are true and P_r and P_{r+1} are true imply P_{r+2} is true, then by strong induction, P_n is true for all $n \in \mathbb{Z}_{\ge 0}$.

(iv) By **(iii)**, for any positive integer m, the Fibonacci sequence modulo m is periodic.
Hence, there exists a pair (F_i, F_{i+1}) such that $F_i \equiv F_1 \equiv 1 \pmod{m}$ and $F_{i+1} \equiv F_2 \equiv 2 \pmod{m}$.
So, $F_{i-1} \equiv 0 \pmod{m}$.

Remark for Question 8: I found an interesting post on StackExchange which is related to **(iii)**.

3 2018 Paper Solutions

Question 1

(i)

$$F_n(0) = \sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1},$$

which follows by using partial fractions and the method of difference.
As n tends to infinity, $F_n(0)$ increases and tends to 1.

(ii) (a)

$$\begin{aligned} F_n(x) &= \sum_{r=1}^{n} \left[\frac{1}{r} - \frac{1}{r+1} + \frac{2}{(r-1)x+1} - \frac{2}{rx-1} \right] \\ &= \underbrace{\sum_{r=1}^{n} \left[\frac{1}{r} - \frac{1}{r+1} \right]}_{\mathrm{F}_n(0)} + 2\sum_{r=1}^{n} \left[\frac{1}{(r-1)x+1} - \frac{1}{rx-1} \right] \\ &= 1 - \frac{1}{n+1} + 2\left(1 - \frac{1}{nx+1}\right) \quad \text{by the method of difference} \\ &= 3 - \frac{1}{n+1} - \frac{2}{nx+1} \end{aligned}$$

(b)

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & \text{if } x = 0; \\ 3 & \text{if } x \neq 0 \end{cases}$$

Question 2

(i) Starting with the RHS, let $t = a - x$. When $x = 0$, then $t = a$; when $x = a$, then $t = 0$. Also, $dt = -dx$.
The RHS becomes

$$-\int_0^a f(t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx.$$

(ii) Since f is symmetrical about $x = \frac{1}{2}a$, then $f\left(x + \frac{1}{2}a\right) = f\left(-x + \frac{1}{2}a\right)$.
Replacing x with $x - \frac{1}{2}a$, we have $f(x) = f(a-x)$.
Considering the LHS,

$$\begin{aligned} \int_0^a xf(x)\ dx &= \int_0^a (a-x) f(a-x)\ dx \\ &= a\int_0^a f(a-x)\ dx - \int_0^a xf(a-x)\ dx \end{aligned}$$

Using **(i)**, the integrals become

$$a\int_0^a f(x)\,dx - \int_0^a xf(x)\,dx.$$

So,

$$2\int_0^a xf(x)\,dx = a\int_0^a f(x)\,dx.$$

Dividing both sides by 2 yields the result.

(iii) Let

$$g(x) = \frac{x\sin x}{1+\cos^2 x}.$$

Then, g is even because $g(-x) = g(x)$, so the integrand g is symmetrical about $x = 0$.
Setting $a = 0$ in **(ii)**, we have

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx.$$

For the integral on the right, let $u = \cos x$, so $du = -\sin x\,dx$. The integral becomes

$$-\frac{\pi}{2}\int_1^{-1} \frac{1}{1+u^2}\,du = \frac{\pi}{2}\int_{-1}^{1} \frac{1}{1+u^2}\,du = \frac{\pi}{2}\left[\tan^{-1} u\right]_{-1}^{1} = \frac{\pi^2}{4}.$$

Question 3

(i) Without loss of generality, it suffices to show that

$$\frac{a}{1+a}+\frac{b}{1+b}-\frac{c}{1+c}\geq 0,$$

where $a+b\geq c$, which is a consequence of the triangle inequality.

The LHS can be written as

$$\frac{a}{1+a}+\frac{b}{1+b}-\frac{c}{1+c}=\frac{a(1+b)(1+c)+b(1+a)(1+c)-c(1+a)(1+b)}{(1+a)(1+b)(1+c)}.$$

Since the denominator is always positive and the numerator can be expanded and simplified as $a+b-c+2ab+abc\geq 0$, the result follows.

(ii) Without a loss of generality, it suffices to show that $\sqrt{a}+\sqrt{b}-\sqrt{c}\geq 0$, where $a+b\geq c$.

Think of $\sqrt{a}+\sqrt{b}-\sqrt{c}$ as the difference of $\sqrt{a}+\sqrt{b}$ and $\sqrt{c}$. By multiplying and dividing by its 'conjugate', the LHS can be written as

$$\sqrt{a}+\sqrt{b}-\sqrt{c}=\frac{\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}{\sqrt{a}+\sqrt{b}+\sqrt{c}}.$$

Similar to **(i)**, we consider the numerator, which can be written as $\left(\sqrt{a}+\sqrt{b}\right)^2-c=a+b-c+2\sqrt{ab}\geq 0$. The result follows.

(iii) Without a loss of generality, it suffices to show that

$$\sqrt{a(b+c-a)}+\sqrt{b(c+a-b)}-\sqrt{c(a+b-c)}\geq 0,$$

where $a+b\geq c$ and a,b,c are the lengths of a triangle.

Let the triangle's perimeter be P, so $P=a+b+c$. So,

$$\sqrt{a(b+c-a)}+\sqrt{b(c+a-b)}-\sqrt{c(a+b-c)}=\sqrt{a(P-2a)}+\sqrt{b(P-2b)}-\sqrt{c(P-2c)}.$$

Let $x=\sqrt{a(P-2a)}$, $y=\sqrt{b(P-2b)}$ and $z=\sqrt{c(P-2c)}$.
Using the cosine rule, say we have a triangle XYZ with $XY=z$, $YZ=x$, $ZX=y$ and $\angle XYZ=\theta$. Then,

$$\begin{aligned}\cos\theta&=\frac{x^2+y^2-z^2}{2xy}\\&=\frac{a(P-2a)+b(P-2b)-c(P-2c)}{2\sqrt{ab(P-2a)(P-2b)}}\\&=\sqrt{\frac{(c-a+b)(c+a-b)}{4ab}}\end{aligned}$$

As $|\cos\theta|\leq 1$, it follows that

$$\begin{aligned}\frac{(c-a+b)(c+a-b)}{4ab}&\leq 1\\c^2&\leq a^2+2ab+b^2\\c&\leq a+b\end{aligned}$$

and the result follows.

Question 4

(i) **(a)** Number of ways is $\binom{7+4-1}{4-1}=120$

(b) The question is equivalent to asking the number of integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7, \text{ where all the } x_i\text{'s} \geq 1.$$

Letting $x_i = 1 - y_i$, we have

$$y_1 + y_2 + y_3 + y_4 = 3, \text{ where all the } y_i\text{'s} \geq 0.$$

So, the number of ways is $\binom{3+4-1}{3} = 20$.

(a) Number of ways is $4^7 = 16384$

(b) Fix any T-shirt in the 1st slot, then the 2nd slot can contain either of the remaining 3 types of T-shirts. Repeating this process up to the 7th slot, we see there are $4\left(3^{7-1}\right) = 2916$ ways.

(c) Let A_i denote the event that the T-shirt of the i^{th} colour is not used, where $1 \leq i \leq 4$.
So,

$$\sum_{i=1}^{4} |A_i| = \binom{4}{1} 3^7$$

$$\sum_{i<j} |A_i \cap A_j| = \binom{4}{2} 2^7$$

$$\sum_{i<j<k} |A_i \cap A_j \cap A_k| = \binom{4}{3} 1^7$$

By the principle of inclusion and exclusion, the answer is $4^7 - \binom{4}{1}3^7 + \binom{4}{2}2^7 - \binom{4}{3}1^7 = 8400$.

Question 5

(i) **(a)** An $a \times b$ rectangle and a $p \times q$ rectangle have ab and pq squares respectively. Since some number of rectangles are used to tessellate the large board, the result follows.

(b) Suppose the base and height of the rectangle are denoted by a and b respectively. If the large board is tessellated from left to right with α vertical and β horizontal rectangles, then the bottom row of the board has $\alpha a + \beta b$ squares. As each row has q squares, then $q = \alpha a + \beta b$. Similarly, if we tessellate the board from the bottom to the top, we have $p = \gamma a + \delta b$. Since $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_{\geq 0}$, the result follows.

(c) In each $a \times b$ tile, along each row, there is only one shaded square. Since there are b rows, there will be b shaded squares in each tile. If k tiles are used in the tessellation, there will be kb shaded squares on the large board. From **(a)**, as ab is a factor of pq, then there exists $k \in \mathbb{N}$ such that $kab = pq$. Hence, kb refers to the number of shaded squares on the board.

(ii) **(a)** Given that

$$p \equiv r \pmod{a},\ 0 \leq r < a$$
$$q \equiv s \pmod{a},\ 0 \leq s < a$$

then there exists $m, n \in \mathbb{N}$ such that

$$p = ma + r \text{ and } q = na + s.$$

Consider a large $p \times q$ rectangle. So, $pq = a^2mn + ams + anr + rs$. We remove an $r \times s$ rectangle from the bottom-right corner so the remaining figure has

$$\frac{pq - rs}{a} = mna + ms + nr$$

non-overlapping rows or columns of $a \times 1$ rectangles. So, this figure comprises $\frac{pq - rs}{a}$ shaded blocks.
As $t = \min\{r, s\}$ and there are t shaded blocks in the $r \times s$ rectangle, the result follows.

(b) From **(iia)**, the number of shaded squares in the $p \times q$ rectangle is $\frac{pq - rs}{a} + t$.
From **(ic)**, the number of shaded squares in the tessellated $p \times q$ rectangle is $\frac{pq}{a}$.
Equating the two, we have $at = rs$.
If $t = r$, then $r(a - s) = 0$. However, if $r \neq 0$, then $s = a$, which is a contradiction as $s < a$. So, $r = 0$ and $a|p$.
If $t = s$, then $s(a - r) = 0$, which implies that $s = 0$ and so $a|q$.

Question 6

(a) Let $A = \{a_1, \dots, a_n\}$ be a group of n students, and for each i, the number of students a_i knows is $f(i)$. Also, let $B = \{1, \dots, n\}$. Then, for all $1 \le i \le n$, we have $0 \le f(i) \le n-1$.

- **Case 1:** Suppose there exist $i, j \in B$, where $i \neq j$, such that $f(i) = f(j) = 0$. Then, a_i and a_j both have no friends, so the result is trivial.
- **Case 2:** Suppose there is precisely one $i \in [1,n]$ such that $f(i) = 0$. Then, for all $j \in [1,n] \setminus \{i\}$, we have $1 \le f(j) \le n-2$. By the pigeonhole principle, we have $j, k \in [1,n] \setminus \{i\}$, where $j \neq k$, such that $f(j) = f(k)$.
- **Case 3:** Suppose $f(i) > 0$ for all $1 \le i \le n$. Then, we have $1 \le f(i) \le n-1$. By the pigeonhole principle, for $1 \le i, j \le n$, where $i \neq j$, we have $f(i) = f(j)$.

We assume that friendship is a symmetric relation, meaning if a_1 is a friend of a_2, then a_2 is also a friend of a_1.

(b) Define the fractional part of x, $\{x\}$, to be $x - \lfloor x \rfloor$.
Consider $\{kx\}$, where $1 \le k \le n$, and subintervals of $[0,1)$ each of length $\frac{1}{n}$. These are

$$I_1 = \left[0, \frac{1}{n}\right), \ I_2 = \left[\frac{1}{n}, \frac{2}{n}\right), \dots, I_n = \left[\frac{n-1}{n}, 1\right).$$

- **Case 1:** Suppose some $\{kx\}$ falls in I_1. As $\{kx\} < \frac{1}{n}$, then

$$\begin{aligned} kx - \lfloor kx \rfloor &< \frac{1}{n} \\ \left| x - \frac{\lfloor kx \rfloor}{k} \right| &< \frac{1}{kn} \end{aligned}$$

  so by setting $a = \lfloor kx \rfloor$ and $b = k$, we establish the desired inequality.
- **Case 2:** Suppose none of the $\{kx\}$ falls in I_1. By the pigeonhole principle, at least two $\{kx\}$ fall in the same I_i, where $2 \le i \le n$. Let

$$\frac{i-1}{n} \le \{px\} < \frac{i}{n} \text{ and } \frac{i-1}{n} \le \{qx\} < \frac{i}{n}.$$

  Then,

$$\begin{aligned} |\{px\} - \{qx\}| &< \frac{1}{n} \\ |px - \lfloor px \rfloor - qx + \lfloor qx \rfloor| &< \frac{1}{n} \\ |(p-q)x - (\lfloor px \rfloor - \lfloor qx \rfloor)| &< \frac{1}{n} \\ \left| x - \frac{\lfloor px \rfloor - \lfloor qx \rfloor}{p-q} \right| &< \frac{1}{(p-q)n} \end{aligned}$$

  so by setting $a = \lfloor px \rfloor - \lfloor qx \rfloor$ and $b = p - q$, we establish the desired inequality.

Remark for Question 6: For **(b)**, a faster method without considering the pigeonhole principle is as such. We start off by noting that $nx - \lfloor nx \rfloor = \{nx\}$, so

$$\begin{aligned} x - \frac{\lfloor nx \rfloor}{n} &= \frac{\{nx\}}{n} \\ \left| x - \frac{\lfloor nx \rfloor}{n} \right| &= \left| \frac{\{nx\}}{n} \right| < \frac{1}{n} \end{aligned}$$

As $a, b \in \mathbb{Z}$ and $1 \le b \le n$, we can set $a = \lfloor nx \rfloor$ and $b = 1$ and the result follows.

The interested can look up Diophantine approximation.

Question 7

(i) Using the substitution $t = \dfrac{dy}{dx}$, we have $\dfrac{dt}{dx} = \dfrac{d^2y}{dx^2}$.
Differentiating (1) with respect to x, we have

$$\begin{aligned} y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 &= 2x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2 \\ y &= 2x\frac{dy}{dx} \end{aligned}$$

Solving the differential equation yields $\frac{1}{2}\ln|x| = \ln|y| + c$.
So $y^2 = Ax$, where $A = e^{-2c}$. Thus, c and A are constants.

Differentiating $y^2 = Ax$ with respect to x, we have

$$2y\frac{dy}{dx} = A,$$

and substituting this into the original differential equation, we have $A = 4$.
Hence, the equation of S is $y^2 = 4x$.

(ii) First, we show that if a straight line is tangent to S, then it is a solution to equation (1).
Note that

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$

Suppose the line is tangent to the curve at $P\left(\frac{1}{4}a^2, a\right)$. Then, the equation of the tangent at P is

$$\begin{aligned} y - a &= \frac{2}{a}\left(x - \frac{a^2}{4}\right) \\ y &= \frac{2}{a}x + \frac{a}{2} \end{aligned}$$

Substituting this into the original differential equation, the result follows.

Next, we show that if a straight line is a solution to (1), then it is tangent to S.
Note that any line satisfies $y = mx + c$. Substituting this into the original differential equation yields $m(mx+c) = xm^2 + 1$. Since this holds for any $x \in \mathbb{R}$, then $c = \frac{1}{m}$.
The equation of the line becomes

$$y = mx + \frac{1}{m}.$$

Note that $2y\dfrac{dy}{dx} = 4$ and as $\dfrac{dy}{dx} = m$, then $y = \dfrac{2}{m}$. As such, $x = \dfrac{1}{m^2}$.
Since $y^2 = 4x$, then

$$\begin{aligned} \left(mx + \frac{1}{m}\right)^2 &= 4x \\ m^2x^2 - 2x + \frac{1}{m^2} &= 0 \end{aligned}$$

The discriminant of the above quadratic equation is 0, implying that $y = mx + \dfrac{1}{m}$ is tangent to the curve.
In particular, the line is tangential at the point $\left(\frac{1}{m^2}, \frac{2}{m}\right)$.

Question 8

(i)

$$\begin{aligned} \sum_{r=1}^{3} n\left(\frac{11}{7}r\right) &= n\left(\frac{11}{7}\right) + n\left(\frac{22}{7}\right) + n\left(\frac{33}{7}\right) \\ &= 2 + 3 + 5 = 10 \end{aligned}$$

Line	Lattice points underneath $y = \frac{7}{11}x + \frac{1}{2}$	Number of lattice points
$x = 1$	$(1,1)$	1
$x = 2$	$(2,1)$	1
$x = 3$	$(3,1)$ and $(3,2)$	2
$x = 4$	$(4,1)$, $(4,2)$ and $(4,3)$	3
$x = 5$	$(5,1)$, $(5,2)$ and $(5,3)$	3

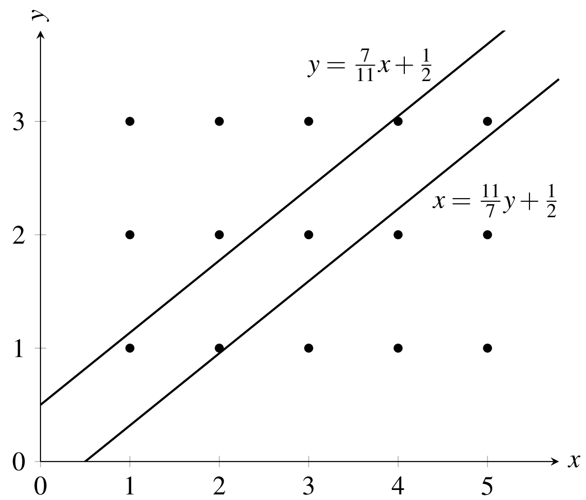
(ii) We have $\sum_{r=1}^{5} n\left(\frac{7}{11}r\right) = 1+1+2+3+3 = 10.$

From the table, we see that there are also 10 points underneath the line $y = \frac{7}{11}x + \frac{1}{2}$.

(iii) Suppose $\left(0, \frac{1}{2}\right)$ is mapped to (a,b) by the rotation. Since $(3,2)$ is the midpoint of $\left(0, \frac{1}{2}\right)$ and (a,b), by the midpoint formula, $a = 6$ and $b = \frac{7}{2}$.
Hence, $\left(6, \frac{7}{2}\right)$ lies on the rotated line.

Suppose $\left(-\frac{11}{14}, 0\right)$ is mapped to (c,d) by the rotation. In a similar fashion, $c = \frac{95}{14}$ and $d = 4$.
Hence, $\left(\frac{95}{14}, 4\right)$ lies on the rotated line.

The equation of the line joining $\left(6, \frac{7}{2}\right)$ and $\left(\frac{95}{14}, 4\right)$ is $x = \frac{11}{7}y + \frac{1}{2}$.



As a rotation by 180° about $(3,2)$ leaves all the lattice points unchanged, then by symmetry,

$$\begin{aligned}
\sum_{r=1}^{3} n\left(\frac{11}{7}r\right) &= \text{total number of lattice points to the left of the line } x = \frac{11}{7}y + \frac{1}{2} \text{ for } y = 1,2,3 \\
&= \text{total number of lattice points underneath the line } y = \frac{7}{11}x + \frac{1}{2} \text{ for } x = 1,2,3,4,5 \\
&= \sum_{r=1}^{5} n\left(\frac{7}{11}r\right)
\end{aligned}$$

(iv) Note that

$$\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)$$

denotes the number of integer points in the rectangle bounded by $1 \le x \le \frac{1}{2}(p-1), 1 \le y \le \frac{1}{2}(q-1)$.
Also, $n\left(\frac{q}{p}r\right)$ denotes the number of integer points underneath the line $y = \frac{q}{p}x + \frac{1}{2}$ for $x = r$.
Then,

$$\sum_{r=1}^{\frac{p-1}{2}} n\left(\frac{q}{p}r\right) = \text{total number of integer points underneath the line } y = \frac{q}{p}x + \frac{1}{2} \text{ for } 1 \le x \le \frac{p-1}{2}$$

and

$$\sum_{r=1}^{\frac{q-1}{2}} n\left(\frac{p}{q}r\right) = \text{total number of integer points to the left of the line } x = \frac{p}{q}y+\frac{1}{2} \text{ for } 1 \leq y \leq \frac{q-1}{2}.$$

Let A and B be the set of integer points underneath the line $y = \frac{q}{p}x + \frac{1}{2}$ and to the left of $x = \frac{p}{q}y + \frac{1}{2}$ respectively. Thus,

$$|A| = \sum_{r=1}^{\frac{p-1}{2}} n\left(\frac{q}{p}r\right) \text{ and } |B| = \sum_{r=1}^{\frac{q-1}{2}} n\left(\frac{p}{q}r\right).$$

As such,

$$\begin{aligned} N &= |A \cap B| \\ &= |A| + |B| - |A \cup B| \\ &= 2|A| - \left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right) \end{aligned}$$

and therefore, $N + \left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right) \equiv 0 \pmod{2}$.

Remark for Question 8: For **(iii)**, those who have background knowledge of linear algebra would find the use of the rotation matrix extremely helpful in the early part of this question. Overall, this question deals with a geometric proof of the law of quadratic reciprocity, which was established by Gotthold Eisenstein.

4 2019 Paper Solutions

Question 1

(i) By the Cauchy-Schwarz inequality,

$$\begin{aligned}\left(x^2+y^2+z^2\right)\left(2^2+3^2+6^2\right) &\geq (2x+3y+6z)^2 \\ (2x+3y+6z)^2 &\leq 49\left(x^2+y^2+z^2\right) \\ 2x+3y+6z &\leq 7 \quad \text{since } x^2+y^2+z^2=1\end{aligned}$$

(ii) From **(i)**, by setting up the inequality and noting that $2^2+3^2+6^2=7^2$, we have $x=\frac{2}{7}$, $y=\frac{3}{7}$ and $z=\frac{6}{7}$.

(iii) By the Cauchy-Schwarz inequality,

$$n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2.$$

Since $\sum_{i=1}^{n} x_i^2 = 1$, we have $n \geq \left(\sum_{i=1}^{n} x_i\right)^2$ so the required maximum value is $\sqrt{n}$.

(iv) Let the length of each square be l_i, where $i \geq 1$ and suppose there are n squares.
Since $\sum_{i=1}^{n} 4l_i = 18$, then $\sum_{i=1}^{n} l_i = \dfrac{9}{2}$.
Also, the area of the large unit square is 1 so $\sum_{i=1}^{n} l_i^2 = 1$.
By the Cauchy-Schwarz inequality, $n \geq \frac{81}{4}$ so there are more than 20 such squares.

Question 2

(i) **(a)** Number of ways is $\binom{8+4-1}{4-1} = 165$

(b) The question is equivalent to asking the number of integer solutions to the equation

$$x_1+x_2+x_3+x_4=8, \text{ where all the } x_i\text{'s} \geq 1.$$

Letting $x_i = 1-y_i$, we have

$$y_1+y_2+y_3+y_4=4, \text{ where all the } y_i\text{'s} \geq 0.$$

So, the number of ways is $\binom{4+4-1}{3} = 35$

(ii) **(a)** Number of ways is $4^8 = 65536$

(b) Number of ways is $4 \times 3^7 = 8748$

(c) We assign each base to an arbitrary index i, where $1 \leq i \leq 4$. Let A_i denote the event that base i is not used. So,

$$\begin{aligned}\sum_{i=1}^{4} |A_i| &= \binom{4}{1}3^8 \\ \sum_{i<j} |A_i \cap A_j| &= \binom{4}{2}2^8 \\ \sum_{i<j<k} |A_i \cap A_j \cap A_k| &= \binom{4}{3}1^8\end{aligned}$$

By the principle of inclusion and exclusion, the answer is $4^8 - \binom{4}{1}3^8 + \binom{4}{2}2^8 - \binom{4}{3}1^8 = 40824$.

Remark for Question 2: This question is very similar to Question 4 of the 2018 A-Level paper.

Question 3

(i) **(a)** Let P_i be the proposition that $x_i \geq \frac{1}{i}$ for all $i \in \mathbb{N}$.
By definition, $x_1 = 1$ so P_1 is true.
Assume P_k is true for some $k \in \mathbb{N}$. That is, $x_k \geq \frac{1}{k}$.
To prove P_{k+1} is true, we need to show that $x_k \geq \frac{1}{k+1}$.
So,

$$\begin{aligned}
x_{k+1} &= \frac{k+a}{k+1} x_k \quad \text{by definition of the recurrence relation} \\
&\geq \frac{k+a}{k(k+1)} \quad \text{by induction hypothesis} \\
&= \frac{1}{k+1}\left(1+\frac{a}{k}\right) \\
&\geq \frac{1}{k+1}
\end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, then P_i is true for all $i \in \mathbb{N}$.

(b)

$$\begin{aligned}
\sum_{i=n+1}^{2n} x_i &\geq \sum_{i=n+1}^{2n} \frac{1}{i} \quad \text{by } \textbf{(i)} \\
&\geq \sum_{i=n+1}^{2n} \frac{1}{2n} \quad \text{since } i \leq 2n \\
&= \frac{1}{2}
\end{aligned}$$

(c) Suppose on the contrary that $\sum_{i=1}^{\infty} x_i$ is bounded.

Note that $\sum_{i=1}^{n} x_i$ is strictly increasing and bounded above. By the monotone convergence theorem, it converges to some finite number, say N.
From **(b)**, we established that

$$\sum_{i=n+1}^{2n} x_i \geq \frac{1}{2}$$

so

$$\sum_{i=1}^{2n} x_i \geq \frac{1}{2} + \sum_{i=1}^{n} x_i.$$

As $n \to \infty$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} x_i &\geq \frac{1}{2} + \sum_{i=1}^{\infty} x_i \\
N &\geq \frac{1}{2} + N
\end{aligned}$$

which is a contradiction.

(ii) **(a)** The recurrence relation can be written as $(i+1)x_{i+1} = (i+a)x_i$ so

$$\begin{aligned}
(i+1)x_{i+1} - ix_i &= ax_i \\
\sum_{i=m}^{n} [(i+1)x_{i+1} - ix_i] &= \sum_{i=m}^{n} ax_i \\
a\sum_{i=m}^{n} x_i &= (n+1)x_{n+1} - mx_m \text{ by the method of difference}
\end{aligned}$$

(b) Let $b = -a > 0$. Note that

$$\begin{aligned}
\frac{i-b}{i+1} &< 0 \quad \text{if } i < b; \\
&> 0 \quad \text{if } i > b
\end{aligned}$$

Hence, the non-zero terms of the sequence x_i alternates in signs for $1 \leq i \leq \lfloor b \rfloor$.
If $x_{\lfloor b \rfloor} < 0$, then $x_{\lfloor b \rfloor+1} \geq 0$, $x_{\lfloor b \rfloor+2} \geq 0$, and so on. Hence, for all $k \geq 1$, $x_{\lfloor b \rfloor+k} \geq 0$.
Similarly, for all $k \geq 1$, if $x_{\lfloor b \rfloor} > 0$, then $x_{\lfloor b \rfloor+k} \leq 0$.
If $b \in \mathbb{N}$, for all $k \geq 1$, then $x_{b+k} = 0$.
Hence, for sufficiently large $m, n \in \mathbb{N}$, in particular $n > m > \lfloor b \rfloor$, x_m and x_n will have the same sign and the result follows.

Question 4

(i) **(a)** Consider the n-digit number having 1st digit 2. There are Y_n such numbers.
Then, consider the $(n-1)$-digit number from the 2nd to the last digit. The 2nd digit has to be 1 or 3.

Define Z_n to be the number of n-digit numbers with first digit 3. By symmetry, $Y_n = Z_n$.

- **Case 1:** If the 2nd digit is 1, there are X_{n-1} $(n-1)$-digit numbers.
- **Case 2:** If the 2nd digit is 3, there are Z_{n-1} $(n-1)$-digit numbers. As $Y_{n-1} = Z_{n-1}$, there are Y_{n-1} $(n-1)$-digit numbers.

Since the two cases are mutually exclusive, then $Y_n = X_{n-1} + Y_{n-1}$.

(b) Consider the n-digit number having 1st digit 1. There are X_n such numbers.
Consider the $(n-1)$-digit number from the 2nd to the last digit. There is no restriction on the 2nd digit.

- **Case 1:** If the 2nd digit is 1, there are X_{n-1} $(n-1)$-digit numbers.
- **Case 2:** If the 2nd digit is 2, there are Y_{n-1} $(n-1)$-digit numbers.
- **Case 3:** If the 2nd digit is 3, there are Z_{n-1} $(n-1)$-digit numbers. As $Y_{n-1} = Z_{n-1}$, then there are Y_{n-1} $(n-1)$-digit numbers.

Since the three cases are mutually exclusive, then $X_n = X_{n-1} + 2Y_{n-1}$.

(c) We have

$$\begin{aligned} X_{n+1} &= X_n + 2Y_n \quad \text{by } \textbf{(b)} \\ &= X_n + 2\left(X_{n-1} + Y_{n-1}\right) \quad \text{by } \textbf{(a)} \\ &= X_n + 2X_{n-1} + X_n - X_{n-1} \quad \text{by } \textbf{(b)} \\ &= 2X_n + X_{n-1} \end{aligned}$$

(ii) Let P_n be the proposition that $X_n \equiv n^2 - n + 1 \pmod 4$ for all $n \in \mathbb{N}$.
When $n = 1$, we have $X_1 = 1 \equiv 1 \pmod 4$ so P_1 is true.
When $n = 2$, we have $X_2 = 3 \equiv 3 \pmod 4$ so P_2 is true.

Assume P_{k-1} and P_k are true for some $k \in \mathbb{N}$, where $k \geq 2$. That is,

$$X_{k-1} \equiv (k-1)^2 - (k-1) + 1 \pmod 4 \quad \text{and} \quad X_k \equiv k^2 - k + 1 \pmod 4$$

respectively.

To show that P_{k+1} is true, we need to prove that $X_{k=1} \equiv (k+1)^2 - (k+1) + 1 \pmod 4$.
Note that $(k+1)^2 - (k+1) + 1 = k^2 + k + 1$.

Using the recurrence relation,

$$\begin{aligned} X_{k+1} &= 2X_k + X_{k-1} \\ &\equiv 2\left(k^2 - k + 1\right) + (k-1)^2 - (k-1) + 1 \pmod 4 \\ &= 2k^2 - 2k + 2 + k^2 - 2k + 1 - k + 2 \\ &= 3k^2 - 5k + 5 \\ &\equiv k^2 + k + 1 + 2(k-1)(k-2) \pmod 4 \end{aligned}$$

As $k-1$ and $k-2$ are of opposite parities, it implies that $(k-1)(k-2)$ is even so $X_{k+1} \equiv k^2+k+1 \pmod 4$.

Since P_1 and P_2 are true and P_{k-1} and P_k are true imply that P_{k+1} is true, by strong induction, P_n is true for all $n \in \mathbb{N}$.

(iii)

$$\begin{aligned} T_n &= X_n + Y_n + Z_n \\ &= X_n + 2Y_n \\ &= X_{n+1} \quad \text{by } \textbf{(ib)} \\ &\equiv n^2+n+1 \pmod 4 \end{aligned}$$

Question 5

(i) Using the substitution, we have

$$\frac{dt}{dx} = \frac{d^2u}{dx^2}.$$

The differential equation becomes $\dfrac{dt}{dx} = t$.

So, $\displaystyle\int \frac{1}{t}\,dt = \int dx$, which implies $\ln|t| = x + c$, where c is a constant. So, $t = Ae^x$, where $A = e^c$.

Since $\dfrac{du}{dx} = Ae^x$, then $\displaystyle\int du = \int Ae^x\,dx$, implying that $u = Ae^x + k$, where k is a constant too.

(ii) Let $u = e^{-\int f(x)y\,dx}$.

Then,

$$\begin{aligned} \frac{du}{dx} &= -e^{-\int f(x)y\,dx} f(x)\,y \\ &= -uyf(x) \end{aligned}$$

Differentiating one more time yields

$$\begin{aligned} \frac{d^2u}{dx^2} &= -uyf'(x) + f(x)\left(-u\frac{dy}{dx} - y\frac{du}{dx}\right) \\ &= -uyf'(x) - f(x)\left(u\frac{dy}{dx} + y\frac{du}{dx}\right) \\ &= -uyf'(x) - uf(x)\frac{dy}{dx} - yf(x)\frac{du}{dx} \end{aligned}$$

Rearranging,

$$uf(x)\frac{dy}{dx} = -uyf'(x) - yf(x)\frac{du}{dx} - \frac{d^2u}{dx^2}.$$

As $\dfrac{dy}{dx} = f(x)y^2 + g(x)y$, then

$$\begin{aligned} uy^2[f(x)]^2 + uyf(x)g(x) + uyf'(x) + yf(x)\frac{du}{dx} + \frac{d^2u}{dx^2} &= 0 \\ uy^2[f(x)]^2 + uyf(x)g(x) + uyf'(x) + yf(x)[-uyf(x)] + \frac{d^2u}{dx^2} &= 0 \\ \frac{d^2u}{dx^2} + uyf(x)g(x) + uyf'(x) &= 0 \\ \frac{d^2u}{dx^2} - g(x)\frac{du}{dx} + uyf'(x) &= 0 \\ f(x)\frac{d^2u}{dx^2} - f(x)g(x)\frac{du}{dx} + uyf(x)f'(x) &= 0 \\ f(x)\frac{d^2u}{dx^2} - \left[f'(x) + f(x)g(x)\right]\frac{du}{dx} &= 0 \end{aligned}$$

(iii) As $\dfrac{dy}{dx} = e^{-2x}y^2 + 3y$, then $f(x) = e^{-2x}$ and $g(x) = 3$.
Using **(ii)**,

$$\begin{aligned} f(x)\frac{d^2u}{dx^2} - \left[f'(x) + f(x)g(x)\right]\frac{du}{dx} &= 0 \\ e^{-2x}\frac{d^2u}{dx^2} - e^{-2x}\frac{du}{dx} &= 0 \end{aligned}$$

As e^{-2x} is non-zero for all $x \in \mathbb{R}$, then $\dfrac{d^2u}{dx^2} = \dfrac{du}{dx}$.
From **(i)**, the solution is of the form $u = e^{x+c} + k$ for constants c and k. So,

$$\begin{aligned} e^{-\int e^{-2x}y\,dx} &= e^{x+c} + k \\ -\int e^{-2x}y\,dx &= \ln\left(e^{x+c} + k\right) \\ -e^{-2x}y &= \frac{e^{x+c}}{e^{x+c} + k} \\ y &= -\frac{e^{3x+c}}{e^{x+c} + k} \end{aligned}$$

When $x = 0$, $y = -\frac{1}{4}$, so $k = 3e^c$. Therefore, $y = -\dfrac{e^{3x}}{e^x + 3}$.

Question 6

(i) Let $x_1, x_2, \ldots, x_{2n}$ denote the positions of the $(+1)$'s and (-1)'s and each $+1$ precedes a corresponding -1. Denote x_i, where $1 \le i \le 2n$ to be the starting point.

There exists $i \in [1, 2n]$ such that $(x_i, x_{i+1}) = (+1, -1)$ or $(-1, +1)$, meaning there are two adjacent points of opposite polarity. Delete x_i and x_{i+1}, so we would have $2n - 2$ positions remaining. Repeat this process until we have 2 points remaining, say x_j and x_k. As such, $(x_j, x_k) = (+1, -1)$.

Suppose x_j is the final position of the $+1$. Restoring all the positions and moving in a clockwise manner, the next position must be either $+1$ or -1. If we proceed with the former, then $T_i = 2$. For the latter, $T_i = 0$. Subsequently, the next position must be either -1 or $+1$ respectively and repeating this process, we conclude that there does not exist $i \in [1, 2n]$ such that $T_i < 0$.

(ii) Regardless of the polarity of the first position, $T_1 \equiv 1 \pmod 2$. As i increases by 1, then the polarity of T_i changes. If n is odd, then $T_i + T_{i+1}$ is odd so

$$n + \sum_{i=1}^{2n} T_i = 2\lambda + 1 + 2\mu + 1 \equiv 0 \pmod 2.$$

If n is even, then $T_i + T_{i+1}$ is even so

$$n + \sum_{i=1}^{2n} T_i = 2\lambda + 2\mu \equiv 0 \pmod 2.$$

Question 7

(i) $c\cos\theta + d\sin\theta < a$ and $c\sin\theta + d\cos\theta < b$

(ii) We first prove the forward direction by contraposition. Suppose $d \ge b$. Then, $a > c \ge d \ge b$ and

$$\begin{aligned} c\sin\theta + d\cos\theta &\ge b(\sin\theta + \cos\theta) \\ &= b\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) \end{aligned}$$

As $0 < \theta < \dfrac{\pi}{2}$, then $\cos\left(\theta - \dfrac{\pi}{4}\right) \ge \dfrac{1}{\sqrt{2}}$, so $b\sqrt{2}\cos\left(\theta - \dfrac{\pi}{4}\right) > b$, which completes the proof.

Next, we prove the backward direction. Choose θ sufficiently small such that

$$c\sin\theta < \varepsilon = \min\{a - c, b - d\}.$$

Then,
$$c\cos\theta + d\sin\theta \le c\cos\theta + c\sin\theta < c+\varepsilon \le a \quad \text{and} \quad d\cos\theta + c\sin\theta < d+\varepsilon \le b,$$
which completes the proof.

(iii) Let θ_0 be the angle for which the $c\times d$ rectangle is strictly contained in the $a\times b$ rectangle. By **(i)**, we must have
$$c\cos\theta_0 + d\sin\theta_0 < a \quad \text{and} \quad c\sin\theta_0 + d\cos\theta_0 < b.$$
From **(ii)**, by considering $c\cos\theta + d\sin\theta \le c\cos\theta + c\sin\theta < c+\varepsilon \le a$ and substituting it into the above inequalities, we have
$$c\sin\left(\frac{\pi}{2}-\theta_0\right) + d\cos\left(\frac{\pi}{2}-\theta_0\right) < b \le a.$$
Let $f(\theta) = c\cos\theta + d\sin\theta$, where $0 \le \theta \le \frac{\pi}{2}$, $\theta_1 = \min\left\{\theta_0, \frac{\pi}{2}-\theta_0\right\}$ and $\theta_2 = \max\left\{\theta_0, \frac{\pi}{2}-\theta_0\right\}$.
Then, $f(\theta_1), f(\theta_2) < a$ and $\theta_1 \le \frac{\pi}{4} \le \theta_2$.
Since $f(\theta) > 0$ and $f''(\theta) = -(c\cos\theta + d\sin\theta) < 0$ for $0 \le \theta \le \frac{\pi}{2}$, then f attains a maximum at θ_{max}, where
$$f'(\theta_{\max}) = -c\sin\theta_{\max} + d\cos\theta_{\max} = 0.$$
This implies that
$$\theta_{\max} = \tan^{-1}\left(\frac{d}{c}\right) \le \frac{\pi}{4}.$$
Since $f(0) = c \ge a$ and f is increasing on $[0, \theta_{\max}]$, then $\theta_{\max} < \theta_1$. Moreover, as f is decreasing on $\left[\theta_{\max}, \frac{\pi}{2}\right]$, it is also increasing on $[\theta_1, \theta_2]$.

(iv) The condition is that $a > c$ or $a\sqrt{2} > c+d$.

First, we will prove the necessary statement. Suppose a $c\times d$ rectangle can be strictly contained in an $a\times a$ square. If $a > c$, we are done. Otherwise, if $a \le c$, then by **(iii)**, we have $\sqrt{2} > c+d$.

Next, we will prove the sufficiency statement. If $a > c$, by **(ii)**, a $c\times d$ rectangle can be strictly contained in an $a\times a$ square if and only if $a > d$. However, as $a > c \ge d$, then the rectangle can always be contained in the square.

If $\sqrt{2} > c+d$, by considering the inequalities in **(i)** which are
$$c\cos\theta + d\sin\theta < a \quad \text{and} \quad c\sin\theta + d\cos\theta < b,$$
setting $\theta = \dfrac{\pi}{4}$ into each both yield $\dfrac{c+d}{\sqrt{2}} < a$.

(i) For any $x \in \mathbb{N}$, we have $(12-x)^2 \equiv x^2 \pmod{12}$ so we only consider the first 6 non-negative square numbers.
$$\begin{aligned}
0^2 &\equiv 0 \pmod{12}\\
1^2 &\equiv 1 \pmod{12}\\
2^2 &\equiv 4 \pmod{12}\\
3^2 &\equiv 9 \pmod{12}\\
4^2 &\equiv 4 \pmod{12}\\
5^2 &\equiv 1 \pmod{12}
\end{aligned}$$
and the result follows.

(ii) Let $N = 9$. We have $5^2 \equiv 7 \pmod 9$ and $7 \in S(9)$ but 7 is non-square.

(iii) For all $N \in \mathbb{N}$, $n[S(N)]$ is greater than or equal to the number of distinct non-negative integers m satisfying $m^2 < N$.

- **Case 1:** Suppose N is non-square. Here, $\sqrt{N} < \lfloor\sqrt{N}\rfloor \notin \mathbb{N}$ so $m < \sqrt{N}$. This implies $0 \le m \le \lfloor\sqrt{N}\rfloor$ so there are $1+\lfloor\sqrt{N}\rfloor$ distinct non-negative integers m satisfying $m < \sqrt{N}$. Thus, there are at least $1+\lfloor\sqrt{N}\rfloor$ distinct elements of $S(N)$ and the result follows.

- **Case 2:** Suppose N is square. Here, $\lfloor\sqrt{N}\rfloor = \sqrt{N} \in \mathbb{N}$. This implies that $m \leq \sqrt{N}$, so $0 \leq m \leq \sqrt{N}-1$. There are $\lfloor\sqrt{N}\rfloor$ distinct non-negative integers m satisfying $m < \sqrt{N}$. Thus, there are at least $\lfloor\sqrt{N}\rfloor$ distinct elements of $S(N)$.

(iv) We consider 2 cases – when λ is even and when λ is odd.

- **Case 1:** If λ is even, then $\frac{\lambda}{2}$ would still be an integer. Thus,

$$x^2 = 17 + 2^{n+1}\left(\frac{\lambda}{2}\right)$$

and in modulo 2^{n+1}, we have $x^2 \equiv 17 \pmod{2^{n+1}}$ and the result follows.

- **Case 2:** If λ is odd, let $\mu \in \mathbb{Z}$ such that

$$\mu = \frac{\lambda + x + 2^{n-2}}{2}$$

and the above equation is valid since λ and x are odd, so their sum would be even.

Hence, $2^{n+1}\mu + 17 = (x+2^{n-1})^2$ and so $\left(x+2^{n-1}\right)^2 \equiv 17 \pmod{2^{n+1}}$.

(v) From **(iii)**, $S(2^n)$ has at least $\sqrt{2^n}$ elements.
From **(iv)**, there exist $x, \lambda \in \mathbb{Z}$ such that $x^2 = 17 + 2^n$ for $n \geq 5$.
Note that all squares p^2, where $0 \leq p^2 \leq 2^n$, are elements of $S(2^n)$ and there are at least $\sqrt{2^n}$ of these elements.
Moreover, from **(iv)**, $17 \in S(2^n)$ and 17 is non-square, and the result follows.

5 2020 Paper Solutions

Question 1

(i) Observe that on the LHS of the inequality, there are $n-1$ copies of x and 1 copy of y. It is most plausible to apply the AM-GM inequality.
Hence,

$$\frac{(n-1)x+y}{n} \geq \sqrt[n]{x^{n-1}y}.$$

Multiplying both sides by n and raising them to the n^{th} power yields the desired result.
Equality holds if and only if $x = y$.

(ii) For the term $(1+a)^2$, it hints that $a = y$.
Comparing with **(i)**, we have

$$[(n-1)x+a]^n \geq n^n x^{n-1} a.$$

Observe that the power on the LHS must be 2, so $n = 2$. Consequently, $x = 1$.
Thus,

$$(1+a)^2 \geq 4a.$$

Next, for the term $(1+b)^3$, it hints that $b = y$.
Comparing it with **(i)**, we have

$$[(n-1)x+b]^n \geq n^n x^{n-1} b.$$

Observe that the power on the LHS is 3, so $n = 3$. Consequently, $x = \frac{1}{2}$.
Thus,

$$(1+b)^3 \geq \frac{27}{4}b.$$

Lastly, in a similar fashion, one can show that

$$(1+c)^4 \geq \frac{256}{27}c.$$

In each scenario, for equality to be obtained, we must have $a = 1$, $b = \frac{1}{2}$, and $c = \frac{1}{3}$ by the AM-GM inequality.
Multiplying the inequalities

$$(1+a)^2 \geq 4a,\ (1+b)^3 \geq \frac{27}{4}b \text{ and } (1+c)^4 \geq \frac{256}{27}c$$

yields

$$(1+a)^2(1+b)^3(1+c)^4 \geq (4a)\left(\frac{27}{4}b\right)\left(\frac{256}{27}c\right) = 256.$$

However, the original inequality in **(ii)** is strict and $abc = 1$ by the constraint in the question. Previously, we mentioned that $abc = \frac{1}{6}$ by the AM-GM inequality. This contradiction implies that the inequality is strict.

Question 2

(i) Let $y = \dfrac{1}{ax+b}$. Then, $x = \dfrac{1}{a}\left(\dfrac{1-by}{y}\right)$, which implies that $f^{-1}(x) = \dfrac{1}{a}\left(\dfrac{1-bx}{x}\right)$, where $x \neq 0$.

(ii) Let $p \in \mathbb{R}$ be arbitrary.
Clearly, p, $f(p)$ and $f^2(p)$ are all not equal to $-\frac{b}{a}$ so $f^3(p)$ exists.
Moreover,

$$\begin{aligned} p &= f^3(p) \\ &= f(f^2(p)) \\ &= \frac{1}{af^2(p)+b} \end{aligned}$$

which is non-zero.
Thus,

$$\begin{aligned}
f^2(p) &= f^{-1}(p) \\
\frac{1}{a\left(\dfrac{1}{ap+b}\right)+b} &= \frac{1-bp}{ap} \\
ap(ap+b) &= (1-bp)\left(a+abp+b^2\right) \\
\left(a+b^2\right)\left(1-bp-ap^2\right) &= 0
\end{aligned}$$

We now consider two cases.

- **Case 1:** Suppose $a=-b^2$. Then,

$$\begin{aligned}
f(x) &= \frac{1}{b(1-bx)} \\
f^2(x) &= f\left[\frac{1}{b(1-bx)}\right] \\
&= \frac{bx-1}{b^2x} \\
f^3(x) &= f\left(\frac{bx-1}{b^2x}\right) \\
&= x
\end{aligned}$$

  Since x was arbitrary, then f^3 fixes all x for which f^3 exists.
- **Case 2:** Suppose $1-bp-ap^2=0$. Then, $\dfrac{1}{ap+b}=p$, which implies that $f(p)=p$. As such, p is a fixed point of f.

(iii) Note that

$$x_{n+1}=\frac{1}{Ax_n+B}.$$

We have

$$f(x_n)=\frac{1}{Ax_n+B}, \quad \text{where } x_n \neq -\frac{B}{A}.$$

Setting $A=1$ and $B=0$ yields

$$x_{n+1}=\frac{1}{x_n} \quad \text{and} \quad x_n \neq 0 \quad \text{for all } n \geq 1.$$

Hence,

$$x_{n+2}=x_n \quad \text{for all } n \geq 1.$$

The required recurrence relation which generates a periodic sequence of period 2 is

$$x_nx_{n+1}=1, \quad \text{where} \quad x_1 \neq 0.$$

Next, from **(ii)**, as $A=-B^2$, then

$$f^3(x_n)=x_n, \quad \text{where} \quad x_n \neq \frac{1}{B}.$$

Setting $A=-1$ and $B=1$ yields the recurrence relation

$$-x_nx_{n+1}+x_{n+1}=1, \quad \text{where } x_1 \neq 1.$$

Hence,

$$\begin{aligned}
x_{n+1} &= \frac{1}{1-x_n} \\
&= \frac{1}{1-\dfrac{1}{1-x_{n-1}}} \\
&= 1-\frac{1}{x_{n-1}} \\
&= 1-\frac{1}{\dfrac{1}{1-x_{n-2}}} \\
&= x_{n-2}
\end{aligned}$$

which shows that the recurrence relation generates a periodic sequence of period 3.

Question 3

(i) Let Q_n denote the proposition

$$\int_0^t x^n e^{-x}\,dx = n!\left(1-e^{-t}P_n(t)\right)$$

for all non-negative integers n.

When $n=0$, we have

$$\text{LHS} = \int_0^t e^{-x}\,dx = 1-e^{-t} = \text{RHS}$$

so Q_0 is true.
Assume that Q_k is true for some non-negative integer k. That is,

$$\int_0^t x^k e^{-x}\,dx = k!\left(1-e^{-t}P_k(t)\right).$$

To prove that Q_{k+1} is true, we need to show

$$\int_0^t x^{k+1} e^{-x}\,dx = (k+1)!\left(1-e^{-t}P_{k+1}(t)\right).$$

Consider Q_{k+1}. Then,

$$\begin{aligned}
\text{LHS} &= \int_0^t x^{k+1}e^{-x}\,dx \\
&= \left[x^{k+1}\left(-e^{-x}\right)\right]_0^t + (k+1)\int_0^t x^k e^{-x}\,dx \\
&= -t^{k+1}e^{-t} + (k+1)\left[k!\left(1-e^{-t}\mathrm{P}_k(t)\right)\right] \quad \text{by induction hypothesis} \\
&= -t^{k+1}e^{-t} + (k+1)! - e^{-t}(k+1)\sum_{i=0}^{k}\frac{t^i}{i!} \\
&= (k+1)! - e^{-t}\left[t^{k+1} + (k+1)\sum_{i=0}^{k}\frac{t^i}{i!}\right] \\
&= (k+1)! - e^{-t}\left[(k+1)!\sum_{i=0}^{k+1}\frac{t^i}{i!}\right] \\
&= (k+1)!\left(1-e^{-t}P_{k+1}(t)\right)
\end{aligned}$$

Since Q_0 is true and Q_k is true implies Q_{k+1} is true, then by induction, Q_n is true for all non-negative integers n.

(ii) Note that $\sum_{i=0}^{n}\frac{t^i}{i!}$ is the partial sum of the Maclaurin series of e^t so

$$\sum_{i=0}^{n}\frac{t^i}{i!} < e^t.$$

Hence,

$$\begin{aligned}\int_0^{\infty} x^n e^{-x}\, dx &= \lim_{t\to\infty}\left[n!\left(1-e^{-t}P_n(t)\right)\right]\\ &= n! - \lim_{t\to\infty} e^{-t}\sum_{i=0}^{n}\frac{t^i}{i!}\\ &= n!\end{aligned}$$

(iii) By the binomial theorem,

$$\begin{aligned}\left(1+\frac{t}{n}\right)^n &= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{t}{n}\right)^i\\ &= \sum_{i=0}^{n}\frac{n(n-1)(n-2)\dots(n-i+1)}{i!}\left(\frac{t}{n}\right)^i\\ &= \sum_{i=0}^{n}(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{i-1}{n}\right)\left(\frac{t}{n}\right)^i\\ &\leq \sum_{i=0}^{n}\left(\frac{t}{n}\right)^i\\ &\leq \sum_{i=0}^{n}\frac{t^i}{i!}\\ &= P_n(t)\end{aligned}$$

Now, we ascertain the upper bound for $P_n(t)$.

$$\begin{aligned}\left(1-\frac{t}{n}\right)^{-n} &= 1+(-n)\left(-\frac{t}{n}\right)+\frac{(-n)(-n-1)}{2!}\left(-\frac{t}{n}\right)^2+\frac{(-n)(-n-1)(-n-2)}{3!}\left(-\frac{t}{n}\right)^3+\dots\\ &= 1+t+\frac{n(n+1)}{2!}\left(\frac{t}{n}\right)^2+\frac{n(n+1)(n+2)}{3!}\left(\frac{t}{n}\right)^3+\dots\\ &= 1+t+\frac{t^2}{2!}\left(1+\frac{1}{n}\right)+\frac{t^3}{3!}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\dots\\ &> 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots \quad \text{since } 1+\frac{k}{n}>1 \text{ for all } k\in\mathbb{Z}^+\\ &= e^t\\ &> \sum_{i=0}^{n}\frac{t^i}{i!}\\ &= \mathrm{P}_n(t)\end{aligned}$$

Question 4

(i) We have $23y = 13(144 - 3x - 5z)$, which implies $23y$ is a multiple of 13. However, as $\gcd(23, 13) = 1$, then y is a multiple of 13. Since y is prime, then $y = 13$.

(ii) **(a)** We have $3x + 5z = 121$. In modulo 5, this equation becomes $3x \equiv 1 \pmod 5$. Consider the following table:

$x \pmod 5$	$3x \pmod 5$
0	0
1	3
2	1
3	4
4	2

It follows that $x \equiv 2 \pmod 5$.

In modulo 3, $3x + 5z = 121$ can be written as $2z \equiv 1 \pmod 3$. Consider the following table:

$z \pmod 3$	$2z \pmod 3$
0	0
1	2
2	1

It follows that $z \equiv 2 \pmod 3$.

(b) From **(a)**, there exists $s, t \in \mathbb{Z}$ such that $x = 5s + 2$ and $z = 3t + 2$.
Since $3x + 5z = 121$, then $3(5s + 2) + 5(3t + 2) = 121$. As such, $s + t = 7$.
So, $|z - x| = |3t - 5s| = |3(7 - s) - 5s| = |21 - 8s|$.
The minimum value of $|z - x|$ is obtained when $s = 3$, so $x = 17$. Consequently, $z = 14$. Therefore, $(x, y, z) = (17, 13, 14)$.

(iii) From **(i)**, since y is prime, then $y = 13$. We now find solutions to $3x + 5z = 121$ such that x and z are prime.
As $x = \dfrac{121 - 5z}{3}$, then $x \leq 40\dfrac{1}{3}$. We consider the primes of the form 2 modulo 5 and are less than 40, which are 2, 7, 17, and 37.
If $x = 2$, then $z = 23$ which is prime. If $x = 7$, then $z = 20$ which is not prime. If $x = 17$, then $z = 14$ which is not prime. Lastly, if $x = 37$, then $z = 2$ which is prime.
We conclude that $(x, y, z) = (2, 13, 23), (37, 13, 2)$ are the only solutions.

Question 5

(a) **(i)** Replace x with $\frac{1}{x}$ and the result follows.

(ii) We have

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x \quad (1)$$

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x} \quad (2)$$

$2 \times (2)$ yields $2f\left(\dfrac{1}{x}\right) + 4f(x) = \dfrac{6}{x}$. So, $3f(x) = \dfrac{6}{x} - 3x$. It follows that $f(x) = \dfrac{2}{x} - x$.

(b) As mentioned in the question,

$$g(x) + g(-x) + g\left(\frac{1}{x}\right) = x \quad (3).$$

Replacing x with $-x$ in (3) yields

$$g(-x) + g(x) + g\left(-\frac{1}{x}\right) = -x \quad (4).$$

Adding ③ and ④ yields

$$g\left(\frac{1}{x}\right)+g\left(-\frac{1}{x}\right)=-2g(x)-2g(-x) \quad ⑤.$$

Replacing x with $\frac{1}{x}$ in ③ yields

$$\begin{aligned} g\left(\frac{1}{x}\right)+g\left(-\frac{1}{x}\right)+g(x)&=\frac{1}{x} \\ -g(x)-2g(-x)&=\frac{1}{x} \quad \text{by ⑤} \end{aligned}$$

Denote $-g(x)-2g(-x)=\frac{1}{x}$ by ⑥. Replacing x with $-x$ in ⑥ yields $-g(-x)-2g(x)=-\frac{1}{x}$. Let this equation be ⑦. So, ⑦ $-2\times$ ⑥ yields $3g(-x)=-\frac{3}{x}$. It follows that $g(x)=\frac{1}{x}$.

Question 6

(i) Note that

$$\begin{aligned} x_{n+1}^2-x_nx_{n+2}&=x_{n+1}^2-x_n\left(dx_{n+1}-x_n\right) \\ &=x_{n+1}^2+x_n^2-dx_nx_{n+1} \end{aligned}$$

Now, we prove by induction that $x_{n+1}^2-x_nx_{n+2}=D$ for all positive integers n. Let P_n denote this proposition.
P_1 is true as $x_2^2-x_1x_3=D$ as mentioned in the question.
Assume P_k is true for some positive integer k. That is, $x_{k+1}^2-x_kx_{k+2}=D$.
To show P_{k+1} is true, we need to show $x_{k+2}^2-x_{k+1}x_{k+3}=D$.

$$\begin{aligned} x_{k+2}^2-x_{k+1}x_{k+3}&=x_{k+2}^2-x_{k+1}\left(dx_{k+2}-x_{k+1}\right) \quad \text{by definition of recurrence relation} \\ &=x_{k+2}^2+x_{k+1}^2-dx_{k+1}x_{k+2} \\ &=x_{k+2}^2+x_{k+1}^2-x_kx_{k+2}+x_kx_{k+2}-dx_{k+1}x_{k+2} \\ &=x_{k+2}^2+D+x_kx_{k+2}-dx_{k+1}x_{k+2} \quad \text{by induction hypothesis} \\ &=D+x_{k+2}^2+x_kx_{k+2}-dx_{k+1}x_{k+2} \\ &=D+x_{k+2}\left(x_{k+2}+x_k-dx_{k+1}\right) \\ &=D \end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, then P_n is true for all positive integers n.

(ii) Set $x_n=0$ and we obtain $x_nx_{n+2}=0$. Hence, $D=x_{n+1}^2$, which is a perfect square.

(iii)
- **Case 1:** Suppose the sequence contains a zero term. By **(ii)**, D is a perfect square.
- **Case 2:** Suppose the sequence does not contain any zero terms. So, it contains both positive and negative terms. Then, there exists a positive integer n such that x_n and x_{n+1} have different signs.

  To justify this, we prove by contradiction. Suppose on the contrary that $x_1,x_2,x_3,\ldots$ all have the same sign. Then, the sequence contains either only positive or negative terms, which is a contradiction. As such, $x_nx_{n+1}\leq -1$, implying that $-dx_nx_{n+1}\geq d$ since $d>0$.

  Since $x_n,x_{n+1}\in\mathbb{Z}$, then the sum of their squares is at least 2. Hence,

$$\begin{aligned} D&=x_n^2+x_{n+1}^2-dx_nx_{n+1} \\ &\geq 2+d \end{aligned}$$

(iv) As $x_nx_{n+1}\leq -1$, set $x_n=1$ and $x_{n+1}=1$.
It is easy to show that the five successive terms, by substitution, are $1,-1,-4,-11,-29$.

Question 7

(i) Take some element in X. It can be mapped to Y via n ways. Repeat this for the remaining $m-1$ elements in X. It follows that the number of functions that map X to Y is n^m.

(ii) Take some element in X, which can be mapped to Y via n ways.
Take another element in X, which can be mapped to one of the remaining $n-1$ elements in Y.
Repeating this process, the last element in X can be mapped to either of the remaining $n-m+1$ elements in Y.
It follows that the number of one-to-one functions from X to Y is $n(n-1)(n-2)\ldots(n-m+1)=\dfrac{n!}{(n-m)!}$.

(iii) Let A_i be the event that $y_i \in Y$ does not get mapped from any element in X, where $1 \leq i \leq n$. Note that $\{y_1,\ldots,y_n\}$ is a permutation of $\{1,\ldots,n\}$.

We wish to find $|A_1' \cap \ldots \cap A_n'|$, for which by de Morgan's law, is

$$n(S)-\left|\bigcup_{i=1}^{n} A_i\right|.$$

From **(i)**, $n(S)=n^m$.
Also,

$$\begin{aligned}
\sum_{i=1}^{n}|A_i| &= \binom{n}{1}(n-1)^m \\
\sum_{i<j}|A_i \cap A_j| &= \binom{n}{2}(n-2)^m \\
\sum_{i<j<k}|A_i \cap A_j \cap A_k| &= \binom{n}{3}(n-3)^m
\end{aligned}$$

By the principle of inclusion and exclusion,

$$\begin{aligned}
n(S)-\left|\bigcup_{i=1}^{n} A_i\right| &= n^m-\binom{n}{1}(n-1)^m+\binom{n}{2}(n-2)^m-\binom{n}{3}(n-3)^m+\ldots+(-1)^{n-1}\binom{n}{n-1}1^m \\
&= \sum_{r=0}^{n-1}(-1)^k\binom{n}{r}(n-r)^m
\end{aligned}$$

(iv) Since $m=n=5$, the number of one-to-one functions is $5!$.
First, we subtract all functions where each element is mapped to itself, for which there are $\binom{5}{1}(5-1)!$ of them.
Then, add all functions consisting of two elements that are mapped to themselves due to overcounting previously. We thus add $\binom{5}{2}(5-2)!$.

It follows by the principle of inclusion and exclusion that the required number of one-to-one functions mapping X to Y which map no element to itself is

$$5!-\binom{5}{1}(5-1)!+\binom{5}{2}(5-2)!-\binom{5}{3}(5-3)!+\binom{5}{4}(5-4)!-\binom{5}{5}(5-5)!=44.$$

Remark for Question 7: For **(iv)**, this can be also thought of as the number of derangements of a set with 5 elements. It is a known result that the number of derangements of an n-element set is given by $\displaystyle\sum_{r=0}^{n}\frac{(-1)^r}{r!}$ which follows by the principle of inclusion and exclusion. Substituting $n=5$, the result follows.

Question 8

(a) An ellipse has two lines of symmetry which are along its major axis and along its minor axis. So, if we rotate the point with position vector $\mathbf{x}$ (which lies in F) about the origin by $180°$, we obtain the point with position vector $-\mathbf{x}$ which also lies in F.

As

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\mathbf{y}-\mathbf{x}}{2} = \frac{\mathbf{y}+(-\mathbf{x})}{2},$$

it implies that the point with position vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is the midpoint of the points with position vectors $-\mathbf{x}$ and $\mathbf{y}$.

In fact, the line segment connecting the points with position vectors $-\mathbf{x}$ and $\mathbf{y}$ lies entirely in F as F is convex.

(b) **(i)** We first prove that any coordinate on E which undergoes a transformation can lie on any lattice point contained within the 2×2 square centred on the origin.

Define a transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2\lfloor x/2 \rfloor \\ 2\lfloor y/2 \rfloor \end{pmatrix}.$$

Suppose on the contrary that the ellipse has an area larger than 4. Then, there exist some lattice points other than the origin contained in the 2×2 square.

As $\left(2\left\lfloor \frac{x_1}{2} \right\rfloor, 2\left\lfloor \frac{y_1}{2} \right\rfloor\right)$ is a lattice point on the 2×2 square, then $\left(-2\left\lfloor \frac{x_2}{2} \right\rfloor, -2\left\lfloor \frac{y_2}{2} \right\rfloor\right)$ is also a lattice point. Suppose they have position vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively.

Using **(a)**, we establish that the point with position vector $\frac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)$ must also lie in E. That is,

$$\left(\left\lfloor \frac{x_1}{2} \right\rfloor - \left\lfloor \frac{x_2}{2} \right\rfloor, \left\lfloor \frac{y_1}{2} \right\rfloor - \left\lfloor \frac{y_2}{2} \right\rfloor\right)$$

is a lattice point in the ellipse of area larger than 4.

(ii) Note that C has an area of $4p$ units^2. Since $p\in\mathbb{Z}^+$, then the area of C must be at least 4 units^2. Consider the position vector of the required coordinate. That is,

$$\begin{pmatrix} mp-nu \\ n \end{pmatrix} = m\begin{pmatrix} p \\ 0 \end{pmatrix} - n\begin{pmatrix} u \\ 1 \end{pmatrix}.$$

This changes the basis from the standard basis vectors

$$\mathbf{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

to

$$\begin{pmatrix} p \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} u \\ 1 \end{pmatrix}$$

respectively.

So, the vector space $\mathbb{R}^2$ is now tiled by parallelograms instead of unit squares. Since every lattice point must be a vertex of one of these parallelograms, we conclude that a parallelogram lies completely inside C and the result follows.

(c) Since u^2+1 is an integer multiple of p, there exists $\lambda\in\mathbb{Z}$ such that $u^2+1=\lambda p$.

Consider $x = mp - nu$ and $y = n$. Then,

$$\begin{aligned} x^2 + y^2 &= m^2p^2 - 2mnpu + n^2u^2 + n^2 \\ &= m^2p^2 - 2mnpu + n^2\left(u^2+1\right) \\ &= m^2p^2 - 2mnpu + n^2\lambda p \\ &= p\left(m^2p + n^2\lambda - 2mnu\right) \\ &\equiv 0 \pmod p \end{aligned}$$

By considering the radius of the circle,

$$x^2 + y^2 < \left(2\sqrt{\frac{p}{\pi}}\right)^2 = \frac{4p}{\pi}.$$

As $x^2 + y^2 > 0$, then $0 < x^2 + y^2 < \dfrac{4p}{\pi}$.
Lastly, since $p < \dfrac{4p}{\pi} < 2p$, we have $x^2 + y^2 = p$.

Remark for Question 8: For **(a)**, I created an interactive simulation on Desmos. Here, we consider the general Cartesian form of a conic section which is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

and all the coefficients are real and A, B, C are all non-zero. Since the ellipse is centred on the origin, then $C = D = 0$.

An ellipse is an example of a conic section. Since the conic is non-degenerate, we have

$$ACF + \frac{1}{4}\left(BDE - AE^2 - B^2F - CD^2\right) \neq 0.$$

Also, since the conic is an ellipse, we have $4AC - B^2 > 0$.

I found a post on StackExchange which is related to **(bii)** and **(c)**. This question has some semblance to Minkowski's theorem. The convex body theorem for lattices in $\mathbb{R}^2$ is as follows. Suppose L is a lattice in $\mathbb{R}^2$ defined as $L = \{m\mathbf{v}_1 + n\mathbf{v}_2 : m, n \in \mathbb{Z}\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent vectors. That is, we cannot express $\mathbf{v}_1$ as a scalar multiple of $\mathbf{v}_2$ and vice versa. Let d be the area of a fundamental parallelogram of L. If S is a convex and origin-symmetric region with $\text{Area}(S) > 4d$, then S contains some point q, other than the origin, such that $q \in L$.

The reader can check out Blichfeldt's Theorem too.

6 2021 Paper Solutions

Question 1

(a) **(i)** Suppose $\dfrac{\ln x}{1+x^2} = 0$. Then, $\ln x = 0$, so $x = 1$.

The area of R is $-\displaystyle\int_0^1 \frac{\ln x}{1+x^2}\,dx$, whereas the area of S is $\displaystyle\int_1^\infty \frac{\ln x}{1+x^2}\,dx$.

By considering the area of S, letting $x = \dfrac{1}{t}$, we have $\dfrac{dx}{dt} = -\dfrac{1}{t^2}$. So,

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{1+x^2}\,dx &= \int_1^0 \frac{\ln\left(\frac{1}{t}\right)}{1+\left(\frac{1}{t}\right)^2}\cdot\left(-\frac{1}{t^2}\right)\,dt \\
&= \int_0^1 \frac{\ln 1 - \ln t}{1+t^2}\,dt \\
&= -\int_0^1 \frac{\ln x}{1+x^2}\,dx
\end{aligned}$$

(ii) Using the substitution $x = at$, we have $dx = a\,dt$.

The integral becomes

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{a^2+x^2}\,dx &= \int_0^\infty \frac{\ln(at)}{a^2+a^2t^2}\cdot a\,dt \\
&= \frac{1}{a}\int_0^\infty \frac{\ln a + \ln t}{1+t^2}\,dt \\
&= \frac{1}{a}\left(\ln a\int_0^\infty \frac{1}{1+t^2}\,dt + \int_0^\infty \frac{\ln t}{1+t^2}\,dt\right) \\
&= \frac{\ln a}{a}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi\ln a}{2a}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^\infty \ln\left(\frac{a^2+x^2}{x^2}\right)\,dx &= \left[x\ln\left(\frac{a^2+x^2}{x^2}\right)\right]_0^\infty + 2a^2\int_0^\infty \frac{1}{a^2+x^2}\,dx \\
&= 2a^2\int_0^\infty \frac{1}{a^2+x^2}\,dx \\
&= 2a\left[\tan^{-1}\left(\frac{x}{a}\right)\right]_0^\infty \\
&= a\pi
\end{aligned}$$

Question 2

(a) Without loss of generality, let $a \geq b \geq c > 0$. Then,

$$a^r(a-b)(a-c)+b^r(b-c)(b-a)+c^r(c-a)(c-b) = (a-b)[a^r(a-c)-b^r(b-c)]+c^r(c-a)(c-b)$$

Note that $c-a \leq 0$ and $c-b \leq 0$ so $c^r(c-a)(c-b) \geq 0$. It suffices to show that

$$a^r(a-c)-b^r(b-c) \geq 0.$$

In other words,

$$\left(\frac{a}{b}\right)^r \geq 1 \geq \frac{b-c}{a-c}.$$

$\left(\dfrac{a}{b}\right)^r \geq 1$ is merely a consequence of $a \geq b$ and $r > 0$. To see why $1 \geq \dfrac{b-c}{a-c}$, we see that the inequality is equivalent to $a-c \geq b-c$, which is true because $a \geq b$.

(b) **(i)** Note that $3abc$ can be written as $abc+abc+abc$. Suppose $a^3+abc-a^2(b+c)=a^r(a-b)(a-c)$. Then, $a(a^2+bc-ab-ac)=a^r(a-b)(a-c)$. Observe that $a^2+bc-ab-ac$ factorises as $(a-b)(a-c)$, so we can set $r=1$.

Thus, the inequality follows by setting $r=1$ in **(a)**. In particular,

$$\begin{aligned} a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) &\geq 0 \\ a(a^2-ab-ac+bc)+b(b^2-ab-bc+ac)+c(c^2-ac-bc+ab) &\geq 0 \\ a^3+b^3+c^3+3abc &\geq a^2(b+c)+b^2(c+a)+c^2(a+b) \end{aligned}$$

(ii) Consider $\dfrac{a+b+c}{a^2b^2c^2}$. Note that $\dfrac{a}{a^2b^2c^2}-\dfrac{b^2+c^2}{a^3b^2c^2}=\dfrac{a^2-b^2-c^2}{a^3b^2c^2}$.
So,

$$\begin{aligned} \frac{1}{a^5}+\frac{a}{a^2b^2c^2}-\frac{b^2+c^2}{a^3b^2c^2} &= \frac{1}{a^3}\left(\frac{1}{a^2}+\frac{a^2-b^2-c^2}{b^2c^2}\right) \\ &= \frac{(a+c)(a-c)(a+b)(a-b)}{a^5b^2c^2} \end{aligned}$$

Note that $a+c, a+b, a^5b^2c^2 \geq 0$, so it suffices to prove that

$$(a-b)(a-c)+(b-c)(b-a)+(c-a)(c-b) \geq 0.$$

This is clear by setting $r=1$ in **(a)**.

Remark for Question 2: The inequality in **(a)** is known as Schur's inequality.

Question 3

(a) Let P_n be the proposition that

$$v\frac{d^{n+2}y}{dx^{n+2}}+(n+2)\frac{dv}{dx}\frac{d^{n+1}y}{dx^{n+1}}+\binom{n+2}{2}\frac{d^2v}{dx^2}\frac{d^ny}{dx^n}=0$$

for all positive integers n, given that $y=\frac{u}{v}$.

We use the notation $u'=\dfrac{du}{dx}$, as well as $v'=\dfrac{dv}{dx}$. Note that $u=vy$. When $n=1$, we have

$$\begin{aligned} u' &= vy'+v'y \\ u'' &= vy''+2v'y'+v''y \\ u''' &= vy'''+v'y''+2v'y''+2v''y'+v'''y+v''y' \\ &= vy'''+3v'y''+3v''y' \text{ since } v \text{ is quadratic} \implies v'''=0 \end{aligned}$$

Since u is quadratic, then $u'''=0$, so it follows that

$$v\frac{d^3y}{dx^3}+3\frac{dv}{dx}\frac{d^2y}{dx^2}+3\frac{d^2v}{dx^2}\frac{dy}{dx}=0.$$

As such, P_1 is true.

Suppose P_k is true. That is to say,

$$v\frac{d^{k+2}y}{dx^{k+2}}+(k+2)\frac{dv}{dx}\frac{d^{k+1}y}{dx^{k+1}}+\binom{k+2}{2}\frac{d^2v}{dx^2}\frac{d^ky}{dx^k}=0.$$

To prove P_{k+1} is true, we need to show that

$$v\frac{d^{k+3}y}{dx^{k+3}}+(k+3)\frac{dv}{dx}\frac{d^{k+2}y}{dx^{k+2}}+\binom{k+3}{2}\frac{d^2v}{dx^2}\frac{d^{k+1}y}{dx^{k+1}}=0.$$

From P_k, we first differentiate both sides to obtain

$$vy^{(k+3)} + v'y^{(k+2)} + (k+2)v'y^{(k+2)} + (k+2)v''y^{(k+1)} + \binom{k+2}{2}v''y^{k+1} + \binom{k+2}{2}v'''y^{(k+1)} = 0$$

$$vy^{(k+3)} + (k+3)v'y^{(k+2)} + \left[k+2+\binom{k+2}{2}\right]v''y^{(k+1)} = 0 \quad \text{since } v''' = 0$$

Observe that

$$\begin{aligned} k+2+\binom{k+2}{2} &= k+2\left(\frac{2+k+1}{2}\right) \\ &= \frac{(k+3)(k+2)}{2} \\ &= \binom{k+3}{2} \end{aligned}$$

and the result follows.

(b) We first prove that z_n is an arithmetic progression.
Since $v = (\alpha - x)^2$, then $\dfrac{dv}{dx} = 2x - 2\alpha$, so $\dfrac{d^2v}{dx^2} = 2$. From **(a)**,

$$(\alpha - x)^2 \frac{d^{n+2}y}{dx^{n+2}} + 2(n+2)(x-\alpha)\frac{d^{n+1}y}{dx^{n+1}} + (n+2)(n+1)\frac{d^n y}{dx^n} = 0.$$

So,

$$\begin{aligned} (\alpha - x)^2 \frac{(n+2)!z_{n+2}}{(\alpha - x)^{n+4}} + 2(x-\alpha)\frac{(n+2)!z_{n+1}}{(\alpha - x)^{n+3}} + \frac{(n+2)!z_n}{(\alpha - x)^{n+2}} &= 0 \\ (n+2)!z_{n+2} - 2(n+2)!z_{n+1} + (n+2)!z_n &= 0 \\ z_{n+2} - z_{n+1} &= z_n - z_{n+1} \end{aligned}$$

It follows that the difference of consecutive terms is a constant.

Now, write

$$y = \frac{u}{(\alpha - x)^2} = \frac{A}{\alpha - x} + \frac{B}{(\alpha - x)^2}.$$

Then,

$$\frac{dy}{dx} = \frac{A}{(\alpha - x)^2} + \frac{2B}{(\alpha - x)^3} \text{ and } \frac{d^2y}{dx^2} = \frac{2A}{(\alpha - x)^3} + \frac{6B}{(\alpha - x)^4}.$$

As such,

$$\begin{aligned} z_2 - z_1 &= \frac{(\alpha - x)^4}{2}\frac{d^2y}{dx^2} - (\alpha - x)^3\frac{dy}{dx} \\ &= \frac{(\alpha - x)^4}{2}\left[\frac{2A}{(\alpha - x)^3} + \frac{6B}{(\alpha - x)^4}\right] - (\alpha - x)^3\left[\frac{A}{(\alpha - x)^2} + \frac{2B}{(\alpha - x)^3}\right] \\ &= A(\alpha - x) + 3B - A(\alpha - x) - 2B \\ &= B \end{aligned}$$

Recall that $u = A(\alpha - x)^2 + B$. Setting $u = B$, we have $x = \alpha$ so it follows that the common difference is $u(\alpha)$.

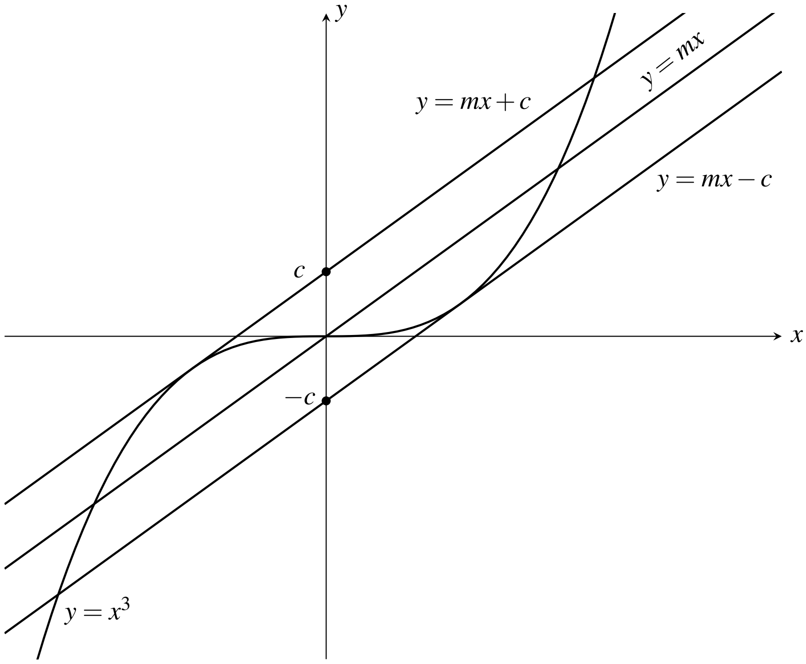
Question 4

(a) Given $y = x^3$, we have $\dfrac{dy}{dx} = 3x^2$.
Note that the curve passes through $\left(x_0, x_0^3\right)$, where x_0 is arbitrary. So, $m = 3x_0^2$.
The equation of the tangent is $y - x_0^3 = 3x_0^2(x - x_0)$. Comparing this with $y = mx + c$, we have $m = 3x_0^2$ and $c = -2x_0^3$.
As such,

$$\begin{aligned}\left(\frac{m}{3}\right)^3 &= \left(-\frac{c}{2}\right)^2 \\ \frac{m^3}{27} &= \frac{c^2}{4}\end{aligned}$$

The result follows.

(b) Consider the following sketch:



$y = mx + c$ and $y = x^3$ intersect at three points. So, $|c| < 2x_0^3$.
Squaring both sides, then multiplying by 27 yields $27c^2 < 27\left(2x_0^3\right)^2 = 108x_0^6$.
From **(a)**, since $m = 3x_0^2$, then $108x_0^6 = 108\,(m/3)^3 = 4m^3$. It follows that $27c^2 < 4m^3$.

(c) The standard equation of a circle centred at (a, b) with radius r is $(x-a)^2 + (y-b)^2 = r^2$.
Since the circle passes through the origin, then $a^2 + b^2 = r^2$.

We consider the parabola $y = x^2$. Substituting this into the equation of the circle, we have

$$\begin{aligned}(x-a)^2 + \left(x^2 - b\right)^2 &= r^2 \\ x^2 - 2ax + a^2 + x^4 - 2bx^2 + b^2 - r^2 &= 0 \\ x^4 + (1-2b)x^2 - 2ax &= 0 \quad \text{since } a^2 + b^2 = r^2\end{aligned}$$

So either $x = 0$ or $x^3 + (1-2b)x - 2a = 0$. From **(b)**, $x^3 = (2b-1)x + 2a$ has three distinct roots if $27(2a)^2 < 4(2b-1)^3$, so the coordinates of the centre of the circle (a, b) satisfy inequality

$$27a^2 < (2b-1)^3, \quad \text{where } a \neq 0.$$

The possible positions can be described by the following set:

$$\left\{(a, b) \in \mathbb{R} \backslash \{0\} \times \mathbb{R} : b > \frac{1}{2}\left(3a^{2/3} + 1\right)\right\}$$

Question 5

(a) Note that $\gcd(a+b, c+d) = z$. So, z divides $a+b$ and z also divides $c+d$. Since z also divides any linear combination of $a+b$ and $c+d$, by observing that

$$-c(a+b) + a(c+d) = ad - bc,$$

we infer that z also divides $ad - bc$. As such, there exist $\lambda, \mu \in \mathbb{N}$ such that $a+b = \lambda z$ and $c+d = \mu z$, where $\gcd(\lambda, \mu) = 1$. Note that if $\gcd(\lambda, \mu) > 1$, it would contradict the fact that $z = \gcd(a+b, c+d)$. We write $ad - bc = wz$ for some $w \in \mathbb{N}$. It follows that $w^2 = \lambda\mu$.

We consider two cases.

- **Case 1:** Suppose w divides λ. Then, $\lambda = w^2$ and $\mu = 1$. So, $a+b = w^2 z$ and $c+d = z$. Setting $x = w$ and $y = 1$, the result follows. If w divides μ instead, we can argue similarly and the result follows.
- **Case 2:** Suppose w does not divide λ and μ. Then λ and μ must be perfect squares. So, there exist $x, y \in \mathbb{N}$ such that $\lambda = x^2$ and $\mu = y^2$. The result follows.

(b) Consider

$$\alpha\left(\frac{y}{x}\right)^2 + \beta\left(\frac{y}{x}\right) + \gamma = 0,$$

where α, β, γ are constants and $\alpha \neq 0$. Multiplying both sides by x^2, we have $\alpha y^2 + \beta xy + \gamma x^2 = 0$. From **(a)**, since

$$x = \sqrt{\frac{a+b}{z}} \text{ and } y = \sqrt{\frac{c+d}{z}},$$

we have

$$\alpha\left(\frac{c+d}{z}\right) + \beta\sqrt{\left(\frac{a+b}{z}\right)\left(\frac{c+d}{z}\right)} + \gamma\left(\frac{a+b}{z}\right) = 0.$$

So, $\alpha(c+d) + \gamma(a+b) + \beta(ad - bc) = 0$. As mentioned at the start of **(a)**, we can set $\alpha = a$, $\beta = -1$ and $\gamma = -c$. So, the required quadratic equation is

$$a\left(\frac{y}{x}\right)^2 - \frac{y}{x} - c = 0.$$

As such,

$$\frac{y}{x} = \frac{1 \pm \sqrt{4ac+1}}{2a}.$$

Since $x, y \in \mathbb{N}$, then $\frac{y}{x}$ is rational. So,

$$4ac + 1 = \left(\frac{2ay}{x} - 1\right)^2,$$

where $\frac{2ay}{x} - 1$ is rational, so $4ac + 1$ is a perfect square.

Question 6

(a) Consider $2\times3\times5=30$. There are 5 ways to write express $2\times3\times5$ as a product of 3 positive integers where the order of these integers does not matter as seen below.

$$\begin{aligned}30 &= 1\times1\times30\\ &= 1\times2\times15\\ &= 1\times3\times10\\ &= 1\times6\times5\\ &= 2\times3\times5\end{aligned}$$

(b) To obtain $F(n)$, there are n cases to consider.

- **Case 1:** Suppose we have a product of $n-1$ distinct primes, so the product is given by $p_1p_2\dots p_{n-1}$. Multiply this by some other prime p_n. There are
$$F(n-1)=\binom{n-1}{n-1}F(n-1)$$
ways to do this.
- **Case 2:** Choose some prime p_i, where $1\le i\le n-1$, to be multiplied by p_n to obtain $p_1p_2\dots p_{i-1}p_{i+1}\dots p_{n-1}(p_ip_n)$. There are
$$\binom{n-1}{1}F(n-2)$$
ways to do this.
- **Case 3:** Choose two distinct primes p_i,p_j, where $1\le i<j\le n-1$ to be multiplied by p_n to obtain
$$p_1p_2\dots p_{i-1}p_{i+1}\dots p_{j-1}p_{j+1}\dots p_{n-1}(p_ip_jp_n).$$
There are
$$\binom{n-1}{2}F(n-3)$$
ways to do this.

Repeat this till the last case, where $p_1p_2\dots p_{n-1}$ is multiplied by p_n. This contributes to the $F(0)$ term in the sum.

Therefore,

$$\begin{aligned}F(n) &= \binom{n-1}{n-1}F(n-1)+\binom{n-1}{1}F(n-2)+\binom{n-3}{2}F(n-3)+\dots+F(0)\\ &= \binom{n-1}{n-1}F(n-1)+\binom{n-1}{n-2}F(n-2)+\binom{n-1}{n-3}F(n-3)+\dots+\binom{n-1}{0}F(0)\\ &= \sum_{i=0}^{n-1}\binom{n-1}{i}F(i)\end{aligned}$$

where we used the symmetry of binomial coefficients.

(c) Let $A=\alpha_1\times\alpha_2\times\dots\times\alpha_{n-2}$, which is a product of $n-2$ positive integers.
There is no duplication if the product is of the form $A\times p_{n-1}\times p_{n-1}$ (contributes to $F(n-2)$ ways) or $A\times p_{n-1}^2\times1$ (contributes to $F(n-1)$ ways). The result follows.

(d) **(i)** Note that $210=2\times3\times5\times7$ which factors into four distinct primes.
Hence, the answer is $F(4)=15$ (formula given in **(b)**.

(ii) We see that

$$\begin{aligned}150 &= 1\times 1\times 1\times 150\\ &= 1\times 1\times 2\times 75\\ &= 1\times 1\times 5\times 30\\ &= 1\times 1\times 6\times 25\\ &= 1\times 1\times 10\times 15\\ &= 1\times 2\times 3\times 25\\ &= 1\times 2\times 5\times 15\\ &= 1\times 3\times 5\times 10\\ &= 1\times 5\times 5\times 6\\ &= 2\times 3\times 5\times 5\end{aligned}$$

so there are 10 ways.

Remark for Question 6: $F(n)$ can also be thought of as the n^{th} Bell number. The Bell numbers are used to count the number of partitions of a set.

Question 7

(a) Suppose we fix x. Then, xy_k is unique and there are $p-1$ possible products for a given x. Also, xy_k is not congruent to $0 \pmod p$ as $x, y_k \in Q$. Suppose on the contrary that none of the products xy_k is congruent to $1 \pmod p$. Then, each product is congruent to either

$$2 \pmod p \quad \text{or} \quad 3 \pmod p \quad \text{or} \quad \ldots \quad \text{or} \quad p-1 \pmod p.$$

There are $p-1$ possible products and $p-2$ numbers in $[2, p-1]$.
By the pigeonhole principle, there exists $y_i, y_j \in Q$ such that $xy_i \equiv xy_j \equiv k \pmod p$ for some $2 \le k \le p-1$. So, $xy_i = xy_j$, implying that $y_i = y_j$. Thus, there exists at least one $y \in Q$ such that $xy_i \equiv 1 \pmod p$.

In fact, y is unique. Suppose there exists $y_i, y_j \in Q$ such that $xy_i \equiv xy_j \equiv 1 \pmod p$. So, $x(y_i - y_j) \pmod p$. Either x is a multiple of p or $y_i - y_j$ is a multiple of p. Since $x \in Q$, then x cannot be a multiple of p, so it forces $y_i - y_j = 0$, implying that $y_i = y_j$. This establishes the uniqueness of y such that $xy \equiv 1 \pmod p$.

(b) There are $p-1$ choices for x and $p-1$ choices for y, so there are $(p-1)^2$ choices for xy.

- **Case 1:** Suppose $xy \in Q$. Then, by **(a)**, because $xyz = (xy)z$, it follows that there are $(p-1)^2$ choices for x, y, z such that $xyz \equiv 1 \pmod p$.
- **Case 2:** Suppose $xy \notin Q$. Then, we can always reduce the equation modulo p. That is, there exists $\lambda \in Q$ such that $xy - \lambda p \in Q$. From **(a)**, there exists precisely one $z \in Q$ such that $(xy - np)z \equiv 1 \pmod p$. Since $npz \equiv 0 \pmod p$, it follows that $xyz \equiv 1 \pmod p$.
- We consider three cases.
  - **Case 1:** Suppose x, y, z are all identical. Then, it reduces to finding all $x \in Q$ such that $x^3 \equiv 1 \pmod p$. Based on the preamble, it is clear that the number of choices is N.
  - **Case 2:** Suppose 2 of the x, y, z are identical. Then, we wish to find all $x, y \in Q$ such that $x^2 y \equiv 1 \pmod p$. The number of choices is $3(p-1-N)$.
  - **Case 3:** Suppose none of the x, y, z are identical. In other words, all three of them are distinct. We wish to find an expression for the number of ways, say W, such that $xyz \equiv 1 \pmod p$.

  From **(b)**, the number of choices of integers $x, y, z \in Q$ such that $xyz \equiv 1 \pmod p$ is $(p-1)^2$. By the principle of inclusion and exclusion,

$$\begin{aligned}W &= (p-1)^2 - 3(p-1-N) - N\\ &= (p-1)(p-4) + 2N\end{aligned}$$

- From **(c)**, the number of ways to choose distinct $x, y, z \in \mathbb{Q}$ such that $xyz \equiv 1 \pmod{p}$ is divisible by 3 due to symmetry. As such,

$$\begin{aligned}(p-1)(p-4)+2N &\equiv 0 \pmod{3}\\ (p-1)(p-1)-N &\equiv 0 \pmod{3}\\ (p-1)^2 &\equiv N \pmod{3}\\ N &\equiv (p-1)^2 \pmod{3} \quad \text{by symmetry of congruence}\end{aligned}$$

- From **(d)**, $N \equiv 0 \pmod{3}$, so N is a multiple of 3. What is more important is that $N \geq 3$. So, there exists at least three distinct $x \in Q$ such that $x^3 \equiv 1 \pmod{p}$. Choose $x \in Q$, where $x \neq 1$, such that $x^3 \equiv 1 \pmod{p}$. So, $x^3 - 1 \equiv 0 \pmod{p}$. By the difference of cubes formula, $(x-1)\left(x^2+x+1\right) \equiv 0 \pmod{p}$. So, p divides $(x-1)\left(x^2+x+1\right)$. By Euclid's lemma, p divides $x-1$ or p divides x^2+x+1. But, we have chosen x such that $x-1 \neq 0$. Since p is prime, we have p divides x^2+x+1 and the result follows.

Question 8

(a) Without loss of generality, suppose $e = e_1$. By the triangle inequality, $(e_1, e_2, \ldots, e_m) \in P$ if and only if $e_2 + \ldots + e_m > e$. Adding $e_1 = e$ to both sides, the result follows.

(b) For each Q_i, set $e_i = e$, so all the Q_i's are disjoint.
As the total number of m-tuples is N^m, the result follows.

(c) We consider three cases.

- **Case 1:** Suppose $1 \leq i \leq m-1$. Since $e_i \geq 1$ (the preamble states that $e_i \in \mathbb{Z}$ and it denotes length), then $1 + x_i \geq 1$, so $x_i \geq 0$.
- **Case 2:** Suppose $i = m$. Then,

$$x_m = e_m - e_1 - e_2 - \ldots - e_{m-1}$$

Since $(e_1, e_2, \ldots, e_m) \in Q_m$, then $e_1 + e_2 + \ldots + e_{m-1} < e_m$, where we chose $e = e_m$. The result follows.
- **Case 3:** For x_{m+1}, from **(b)**, we deduced that $N \geq e_m$, so $x_{m+1} \geq 0$.

(d)

$$\begin{aligned}\sum_{i=1}^{m+1} x_i &= x_m + x_{m+1} + \sum_{i=1}^{m-1} x_i\\ &= e_m - \sum_{i=1}^{m-1} e_i + N - e_m + \sum_{i=1}^{m-1}(e_i - 1)\\ &= e_m - e_m + N + \sum_{i=1}^{m-1}(-e_i + e_i - 1)\\ &= N - m + 1\end{aligned}$$

Next, we deduce the formula for $|Q_m|$. Consider the equation

$$x_1 + x_2 + \ldots + x_{m+1} = N - m + 1.$$

The number of non-negative solutions $(x_1, \ldots, x_{m+1})$ is the number of ways to distribute $N - m + 1$ identical balls into $m+1$ distinct boxes, thus establishing a bijection.
As such,

$$|Q_m| = \binom{N+1}{m}.$$

(e) The total number of triangles that can be formed, including degenerate ones, is $10^3 = 1000$.
Setting $N = 10$, the number of 3-tuples that satisfy the triangle inequality is

$$\begin{aligned}10^3 - |Q_1| - |Q_2| - |Q_3| &= 10^3 - 3\binom{11}{3} \quad \text{by symmetry as we can choose either } e_1, e_2, e_3 \text{ to be } e\\ &= 505\end{aligned}$$

7 2022 Paper Solutions

Question 1

(a) **(i)** Number of ways is $4^{10} = 1048576$

(ii) Let E_1, E_2, E_3, E_4 denote the following events:

$$\begin{aligned}
&E_1 \text{ denotes the event when no one obtained the } A \text{ grade}\\
&E_2 \text{ denotes the event when no one obtained the } B \text{ grade}\\
&E_3 \text{ denotes the event when no one obtained the } C \text{ grade}\\
&E_4 \text{ denotes the event when no one obtained the } D \text{ grade}
\end{aligned}$$

We wish to find

$$\left|\bigcap_{i=1}^{4} E_i'\right| = 4^{10} - \left|\bigcup_{i=1}^{4} E_i\right| \quad \text{by de Morgan's law,}$$

so by the principle of inclusion and exclusion, we have

$$\begin{aligned}
\left|\bigcup_{i=1}^{4} E_i\right| &= \sum_{i=1}^{4} |E_i| - \sum_{1\le i<j\le 4} \left|E_i \cap E_j\right| + \sum_{1\le i<j<k\le 4} \left|E_i \cap E_j \cap E_k\right| - \left|\bigcap_{i=1}^{4} E_i\right| \\
&= \binom{4}{1}3^{10} - \binom{4}{2}2^{10} + \binom{4}{3}1^{10} - 0 \\
&= 230056
\end{aligned}$$

Hence, the answer is $4^{10} - 230056 = 818520$.

(iii) The Stirling numbers of the second kind account for the distribution of distinct objects into identical boxes. So we divide **(ii)**'s answer by $4!$, so $S(10,4) = 230056/4! = 34105$.

(b) **(i)** We consider two cases.

- **Case 1:** First, consider $n = k+1$. There exists a partition of X such that $k-1$ subsets each contain one object and the remaining subset, say S', contains two objects. S' can be partitioned into two subsets with one element each. So, X is the ancestor of $k+1$ partitions into $k+1$ non-empty subsets.
- **Case 2:** Next, consider $n > k+1$. There exists a partition of X such that $k-1$ subsets each contain one object and the remaining subset, say S'', contains $n-(k-1) = n-k+1$ objects. S'' can be split into two such that one subset contains one object and the other contains $n-k$ objects. There are $n-k+1$ ways to choose that one object. So, X is the ancestor of at least $n-k$ partitions into $k+1$ non-empty subsets.

The result follows.

(ii) We denote the partitions of the form X and the form Y to be

$$X_1, X_2, \ldots, X_{g(n,k)} \quad \text{and} \quad Y_1, Y_2, \ldots, Y_{g(n,k+1)} \quad \text{respectively.}$$

Also, let $d(X_i)$ and $a(Y_j)$ denote the following sets:

$$d(X_i) = \{\text{form } Y \text{ descendants that } X_i \text{ has}\} \quad \text{and} \quad a(Y_j) = \left\{\text{form } X \text{ ancestors that } Y_j \text{ has}\right\}$$

For any Y_j, its ancestors are the product of merging any 2 of its $k+1$ subsets. That is to say, for all j,

$$\left|a(Y_j)\right| = \binom{k+1}{2}.$$

We now prove the inequality.

$$\begin{aligned}
\text{RHS} &= \binom{k+1}{2} g(n,k+1) \\
&= \sum_{j=1}^{S(n,k+1)} \left| a(Y_j) \right| \\
&= \left| \left\{ \text{number of tuples } (X_i, Y_j) \text{ where } X_i \text{ is an ancestor of } Y_j \right\} \right| \\
&= \sum_{i=1}^{S(n,k)} |d(X_i)| \\
&= \sum_{i=1}^{S(n,k)} (n-k) \\
&= (n-k)S(n,k) \\
&= \text{LHS}
\end{aligned}$$

so the inequality holds.

Next, we prove that equality holds if and only if $n = k+1$.
Suppose $n = k+1$. Then, for any X_i, there will only be 1 subset with 2 elements and the rest will all have 1 element. So, X_i only has 1 descendant, implying that $|d(X_i)| = n-(n-1) = 1$ for all $1 \le i \le g(n,k)$.

Remark for Question 1: For **(bii)**, suppose $n = k+1$, one can deduce that the inequality becomes an inequality very easily by using the same argument as given to deduce that $S(k+1,k) = \binom{k+1}{2}$. However, if we are given the inequality and wish to prove that equality implies $n = k+1$, it is impossible to use the known recurrence relation for the Stirling numbers of the second kind.

Question 2

(a) Consider showing $x^2+y^2+z^2-3xyz = 0$. Using the given substitutions, we have

$$\begin{aligned}
x^2+y^2+z^2-3xyz &= a^2+(3ab-c)^2+b^2-3ab(3ab-c) \\
&= a^2+9a^2b^2-6abc+c^2+b^2-9a^2b^2+3abc \\
&= a^2+b^2+c^2-3abc \\
&= 0 \quad \text{since } (x,y,z) = (a,b,c) \text{ satisfies the equation}
\end{aligned}$$

(b) Setting $a = 1$, $b = 1$ and $c = 1$, we see that $3ab-c = 2$, so $(x,y,z) = (1,2,1)$ is another solution.
Next, set $x = 1$, $z = 2$ and $y = 3(1)(2)-1 = 5$, so $(x,y,z) = (1,5,2)$ is another solution.
Lastly, set $x = 1$, $z = 5$ and $y = 3(1)(5)-2 = 13$, so $(x,y,z) = (1,13,5)$ is another solution.

(c) Let P_n be the proposition that

$$1+F_{2n+1}^2+F_{2n-1}^2 = 3F_{2n+1}F_{2n-1}$$

for all positive integers n.

When $n = 1$, the LHS evaluates to $1+F_3^2+F_1^2 = 1+4+1 = 6$, whereas the RHS evaluates to $3F_3F_1 = 6$.
Hence, P_1 is true.
Suppose P_k is true for some positive integer k. That is, $1+F_{2k+1}^2+F_{2k-1}^2 = 3F_{2k+1}F_{2k-1}$.
To prove P_{k+1} is true, we need to show that $1+F_{2k+3}^2+F_{2k+1}^2 = 3F_{2k+3}F_{2k+1}$.
We apply **(a)** to the induction hypothesis to obtain

$$1+(3F_{2k+1}-F_{2k-1})^2+F_{2k+1}^2 = 3\,(3F_{2k+1}-F_{2k-1})\,F_{2k+1} \ (1).$$

Thus,

$$\begin{aligned}1+F_{2k+3}^2+F_{2k+1}^2-3F_{2k+3}F_{2k+1} &= 1+\left(F_{2k+2}+F_{2k+1}\right)^2+F_{2k+1}^2-3\left(F_{2k+2}+F_{2k+1}\right)F_{2k+1}\\ &= 1+\left(2F_{2k+1}+F_{2k}\right)^2+F_{2k+1}^2-3\left(2F_{2k+1}+F_{2k}\right)F_{2k+1}\\ &= 1+\left(3F_{2k+1}+F_{2k-1}\right)^2+F_{2k+1}^2-3\left(3F_{2k+1}+F_{2k-1}\right)F_{2k+1}\\ &= 0 \text{ by } (1)\end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, then P_n is true for all positive integers n by induction.

Remark for Question 2: The Diophantine equation in **(a)** is known as Markov's equation.

Question 3

(a) Write $t=an+p$, where $n\in\mathbb{Z}$ and $a>p\geq 0$. Then,

$$\begin{aligned}\int_0^a\left\lfloor\frac{x+t}{a}\right\rfloor\,dx &= \int_0^a\left\lfloor\frac{x+p}{a}+n\right\rfloor\,dx\\ &= \int_0^a\left(n+\left\lfloor\frac{x+p}{a}\right\rfloor\right)\,dx \quad \text{since } n\in\mathbb{Z}\\ &= an+\int_0^{a-p}\left\lfloor\frac{x+p}{a}\right\rfloor\,dx+\int_{a-p}^a\left\lfloor\frac{x+p}{a}\right\rfloor\,dx\\ &= an+\int_0^{a-p}0\,dx+\int_{a-p}^a 1\,dx\\ &= an+p=t\end{aligned}$$

(b) **(i)** Motivated by **(a)**, consider the substitution $x=abn'+p'$, where $n'\in\mathbb{Z}$ and $ab>p\geq 0$.
The LHS becomes

$$\begin{aligned}\left\lfloor\frac{\left\lfloor bn'+\frac{p}{a}\right\rfloor}{b}\right\rfloor &= \left\lfloor\frac{bn'+\left\lfloor\frac{p}{a}\right\rfloor}{b}\right\rfloor \quad \text{since } n'\in\mathbb{Z}\\ &= \left\lfloor n'+\frac{1}{b}\left\lfloor\frac{p}{a}\right\rfloor\right\rfloor\\ &= n' \quad \text{since } 0\leq\frac{p'}{a}<b\end{aligned}$$

We now justify that the RHS is also n'. The RHS can be written as

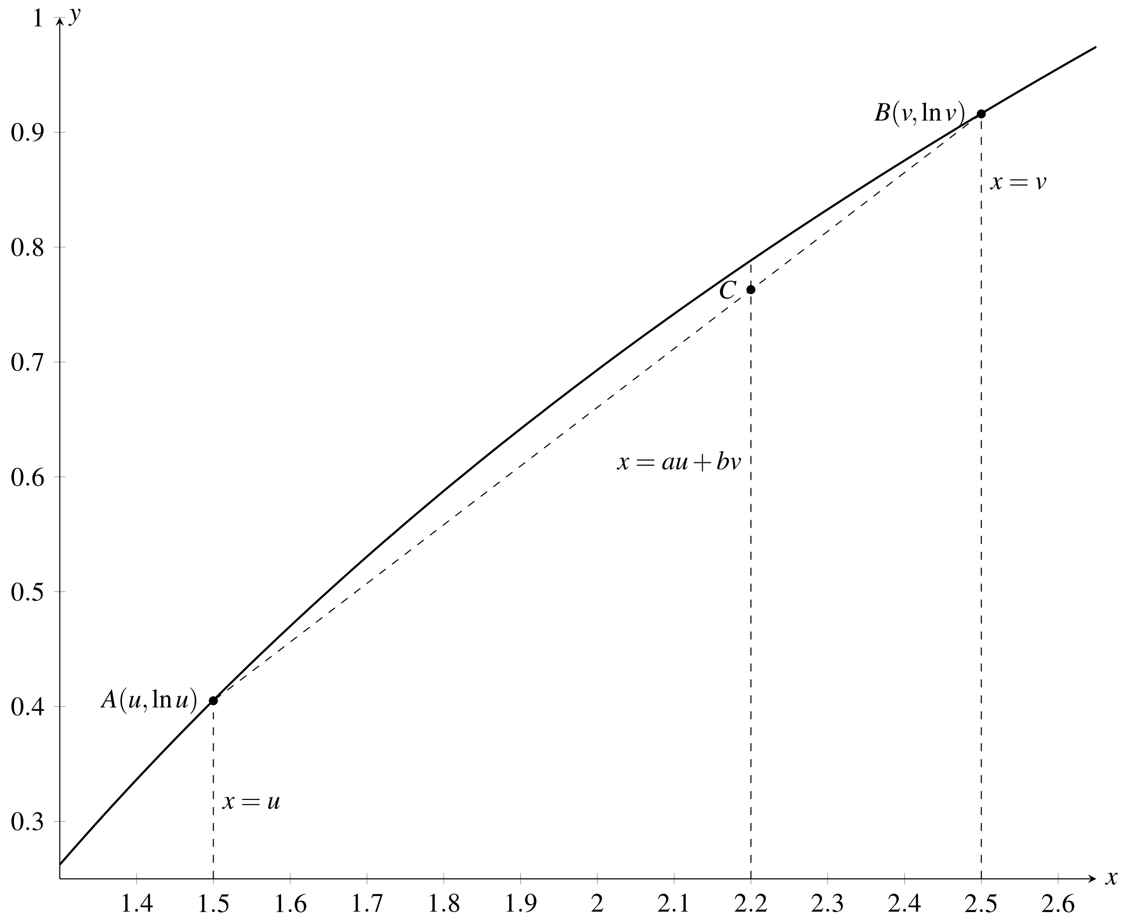
$$\begin{aligned}\left\lfloor\frac{abn'+p'}{ab}\right\rfloor &= \left\lfloor n'+\frac{p'}{ab}\right\rfloor \quad \text{since } n\in\mathbb{Z}\\ &= n'+\left\lfloor\frac{p'}{ab}\right\rfloor\\ &= n'\end{aligned}$$

(ii) Using the substitution for x in **(i)**, we have

$$\begin{aligned}\int_0^{ab}\left(fg\left(x\right)-gf\left(x\right)\right)\,dx &= \int_0^{ab}\left\lfloor\frac{\left\lfloor\frac{x+b}{a}\right\rfloor+a}{b}\right\rfloor-\left\lfloor\frac{\left\lfloor\frac{x+a}{b}\right\rfloor+b}{a}\right\rfloor\,dx\\ &= \int_0^{ab}\left\lfloor\frac{\left\lfloor\frac{x+a^2+b}{a}\right\rfloor}{b}\right\rfloor-\left\lfloor\frac{\left\lfloor\frac{x+a+b^2}{b}\right\rfloor}{a}\right\rfloor\,dx\\ &= \int_0^{ab}\left\lfloor\frac{x+a^2+b}{ab}\right\rfloor-\left\lfloor\frac{x+a+b^2}{ab}\right\rfloor\,dx \quad \text{by } \textbf{(i)}\\ &= \left(a^2+b\right)-\left(a+b^2\right) \quad \text{by } \textbf{(a)}\\ &= a^2-b^2-a+b\end{aligned}$$

Question 4

(a) We consider a sketch of the graph of $y = \ln x$. Without loss of generality, assume $u \le v$.



Since $a + b = 1$, where $a, b > 0$, $au + bv \in (u, v)$. This is because $y = \ln x$ is concave down for $x > 0$.
We first find the gradient of the line segment joining A and B. Consider the fact that $m_{AC} = m_{CB}$, so

$$\begin{aligned}
\frac{y - \ln u}{au + bv - u} &= \frac{y - \ln v}{au + bv - v} \\
auy + bvy - vy - au\ln u - bv\ln u + v\ln u &= auy + bvy - uy - au\ln v - bv\ln v + u\ln v \\
y\left(au + bv - v - au - bv + u\right) &= u\ln v - au\ln v - bv\ln v + au\ln u + bv\ln u - v\ln u \\
y &= \frac{(au + bv - v)\ln u + (u - au - bv)\ln v}{u - v} \\
&= a\ln u + b\ln v \quad \text{since } a + b = 1
\end{aligned}$$

So the y-coordinate of C is $a\ln u + b\ln v$, which is less than $\ln(au + bv)$.
As $a\ln u + b\ln v = \ln\left(u^a v^b\right)$ and $\ln x$ is an increasing function, the result follows. Equality holds if and only if $u = v$.

(b) **(i)** Let $x_n = n\left(G_n - A_n\right)$. We shall prove that $x_{n+1} \le x_n$. In other words, we can show that $x_{n+1} - x_n \le 0$.
First, note that $-(n+1)A_{n+1} + nA_n = -a_{n+1}$.

$$\begin{aligned}
x_{n+1} - x_n &= (n+1)\,G_{n+1} - (n+1)\,A_{n+1} - nG_n + nA_n \\
&= (n+1)\left(a_1 a_2 \ldots a_n a_{n+1}\right)^{\frac{1}{n+1}} - n\left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}} - a_{n+1} \quad \text{since } -(n+1)A_{n+1} + nA_n = -a_{n+1} \\
&= (n+1)\left[\left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}}\right]^{\frac{n}{n+1}} a_{n+1}^{\frac{1}{n+1}} - n\left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}} - a_{n+1} \\
&\le (n+1)\left[\frac{n}{n+1}\left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}} + \frac{1}{n+1}a_{n+1}\right] - n\left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}} - a_{n+1} \quad \text{by } \textbf{(a)} \\
&= 0
\end{aligned}$$

(ii) We can define $a_n = a_1a_2 \dots a_{n-1}$ for all $n \geq 4$. Since $a_{n-1} = a_1a_2 \dots a_{n-2}$, we have $a_n = a_{n-1}^2$.

Remark for Question 4: In **(bii)**, the sequence grows very rapidly. a_{13} has 925 digits, whereas a_{14} has 1850 digits.

Question 5

(a) **(i)** The number of ways to arrange the m married couples and s single people in a line is $(m+s)!$. We then multiply this by 2^m because within each of the m married couples, the husband and wife can swap positions.

(ii) For arrangements in a line, if the first and last persons form a married couple, then they must be seated together in the dining hall. However, this scenario is not accounted for when working with line arrangements.

(b) Define a k-vertex to be a vertex that is chosen to form our k-gon; a k^*-vertex is defined otherwise. This setup is now equivalent to distributing n vertices into k k-vertices and $n-k$ k^*-vertices, where the k-vertices are not adjacent.

We consider two cases.

- **Case 1:** Without loss of generality, suppose vertex 1 is a k-vertex, then the other two vertices are k^*-vertices. Subsequently, insert the $n-k-2$ k^*-vertices. There are now $n-k-2+1 = n-k-1$ slots between the $n-k-2$ k^*-vertices. We can insert the $k-1$ k^*-vertices such that no two k^*-vertices are adjacent in $\binom{n-k-1}{k-1}$ ways.
- **Case 2:** Again without loss of generality, suppose vertex 1 is a k^*-vertex. Then, insert the $n-k-1$ k^*-vertices so that we have $n-k-1+1 = n-k$ slots within the k^*-vertices. We then insert the k k^*-vertices such that no two k^*-vertices are adjacent in $\binom{n-k}{k}$ ways.

The total number of k-gons is $\binom{n-k-1}{k-1} + \binom{n-k}{k}$.

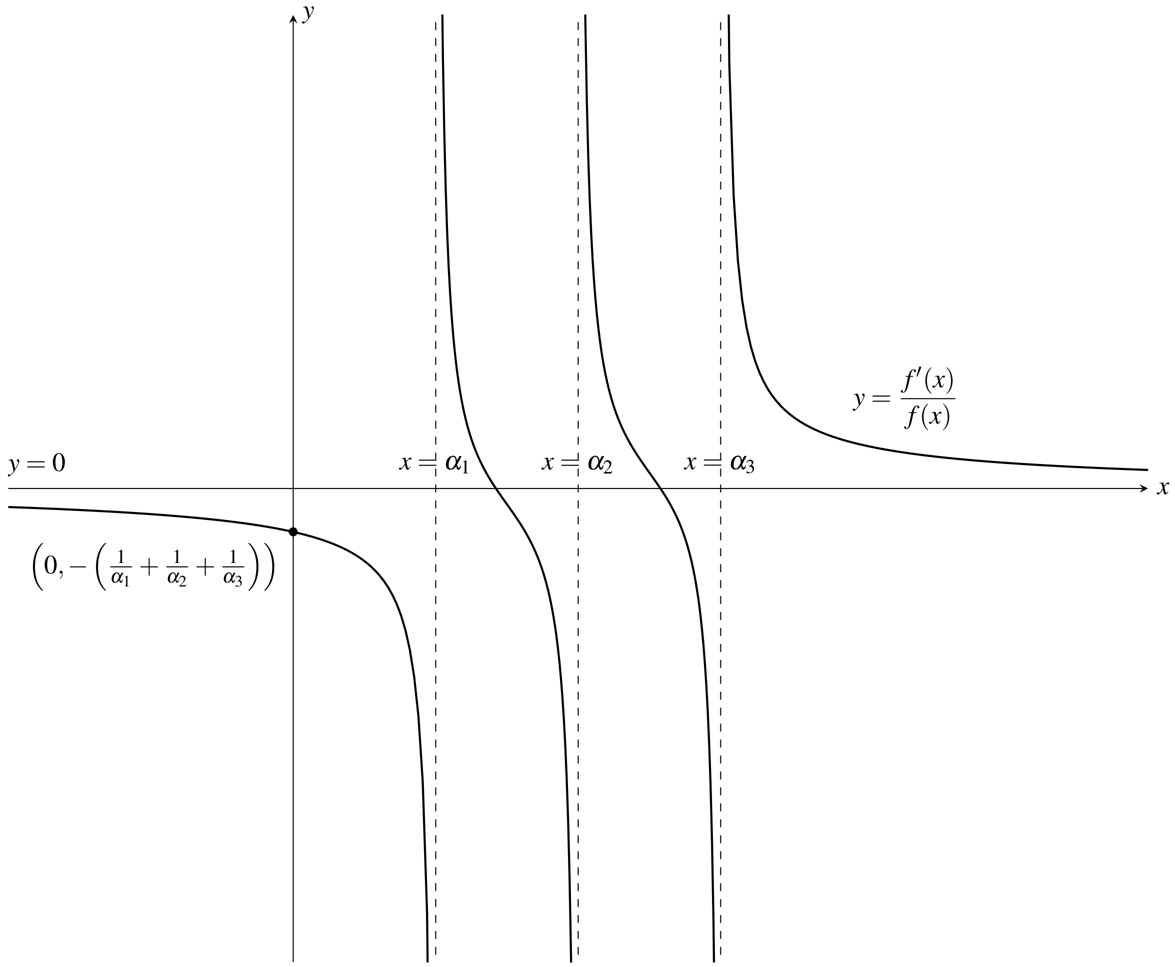
Question 6

(a) We have $f(x) = A(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$, where $A \neq 0$ is a constant. The sight of $f'(x)/f(x)$ prompts us to consider the derivative of $\ln(f(x))$.

$$\ln(f(x)) = \ln A + \sum_{i=1}^{3} \ln(x-\alpha_i)$$

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^{3} \frac{1}{x-\alpha_i} \quad \text{by differentiating both sides with respect to } x$$

(b) Consider the graph of $y = \dfrac{f'(x)}{f(x)}$. Without loss of generality, assume that $0 \leq \alpha_1 < \alpha_2 < \alpha_3$.



For the equation $f(x) - rf'(x) = 0$, we have to consider two cases.

- **Case 1:** Suppose $r = 0$, then $f(x) = 0$. Based on the preamble, $f(x) = 0$ has three distinct roots, $\alpha_1, \alpha_2, \alpha_3$, so the result follows.
- **Case 2:** Suppose $r \neq 0$. We can then rewrite the equation as $\dfrac{f'(x)}{f(x)} = \dfrac{1}{r}$. Any horizontal line $y = \dfrac{1}{r}$ intersects the graph at three distinct points, and the result follows.

(c) By **(b)**, $f(x) - \alpha_1 f'(x) = 0$ is a cubic equation with 3 distinct real roots.
Applying the result in **(b)** again, we have

$$[f(x) - \alpha_1 f'(x)] - \alpha_2[f'(x) - \alpha_1 f''(x)] = 0$$

$$f(x) - (\alpha_1+\alpha_2)f'(x) + \alpha_1\alpha_2 f''(x) = 0$$

which is a cubic equation with 3 distinct real roots.

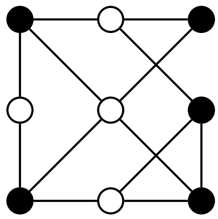
We apply **(b)** again to obtain

$$[f(x)-(\alpha_1+\alpha_2)f'(x)+\alpha\alpha_2 f''(x)]-\alpha_3[f(x)-(\alpha_1+\alpha_2)f'(x)+\alpha\alpha_2 f''(x)]=0$$
$$f(x)-(\alpha+\alpha_2+\alpha_3)f'(x)+(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)f''(x)-\alpha_1\alpha_2\alpha_3 f'''(x)=0$$
$$f(x)+af'(x)+bf''(x)+cf'''(x)=0 \quad \text{by hint (Vieta's formula)}$$

which is a cubic equation with 3 distinct real roots.

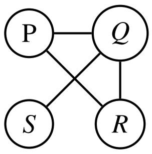
Question 7

(a) **(i)** Consider the following array which has 13 edges that link a shaded and an unshaded circle:



(ii) We first consider the case when $n=4$ as shown in the second diagram. There are $4-1$ square blocks and 3^2-3 arrowhead shapes. For some arbitrary n, there would be $n-1$ square blocks and $(n-1)^2-(n-1)=n^2-3n+2$ arrowhead shapes.

(iii) An arrowhead shape has 4 edges. Suppose on the contrary that 4 edges can link a shaded and an unshaded circle. Label the vertices as P,Q,R,S.



We consider two cases — when S is unshaded, and when S is shaded.

- **Case 1:** Suppose S is unshaded. Since S and Q share a common edge, then Q is shaded. So, P and R must be unshaded, which is a contradiction.
- **Case 2:** Suppose S is shaded. Similarly, Q is unshaded, implying that P and R are shaded, which is a contradiction as well.

As for a square block, at most 4 edges can link a shaded and an unshaded circle.

(b) **(i)** The 3×3 grid can be divided into three components which are the 4 corner squares, the 4 edge squares (but not including the corners), and the centre square. Denote the original sum by S and the final sum by S'. By symmetry, we only need to consider the cases when we shade either a corner square, an edge square or the centre square. We perform some casework.

- **Case 1:** Suppose we shade a corner square with a value of a. Then, S decreases by a, but the values of the centre square and two edge squares surrounding the corner square will increase by a total of a. So, $S'=S-a+a=S$.
- **Case 2:** Suppose we shade an edge square with a value of b, where b is the sum of the values of all the other squares. Then, S decreases by b. However, the total values in the two corner squares adjacent to it, as well as the centre squares, will increase by b, so $S'=S-b+b=S$.
- **Case 3:** Suppose we shade we shade the centre square with a value of c. Then, S decreases by c. However, S will increase by c concurrently too because each unshaded square other than the centre square will increment by some value and the total is c.

(ii) For an $n\times n$ grid,

- from **(aii)**, there are $n-1$ square blocks and from **(aiii)**, at most 4 edges in a square block can link a shaded and an unshaded circle;

- from **(aii)**, there are $n^2 - 3n + 2$ arrowhead shapes and from **(aiii)**, at most 3 edges in an arrowhead shape can link a shaded and an unshaded circle.

Thus, the maximum score is $4(n-1) + 3\left(n^2 - 3n + 2\right) = 3n^2 - 5n + 2$.

To achieve the maximum score, every square within a column must be consistently either shaded or unshaded, with adjacent columns alternating between these two states.

Question 8

(a) We see that

$$\begin{aligned}\left(r^2+s^2\right)\left(t^2+u^2\right)-(rt+su)^2 &= r^2t^2+s^2t^2+r^2u^2+s^2u^2-r^2t^2-2rstu-s^2u^2 \\ &= r^2u^2-2rstu+s^2t^2 \\ &= (ru-st)^2\end{aligned}$$

which is the square of $ru - st$.

(b) a^2 and b^2 must have opposite parities. That is to say, if a^2 is odd, then b^2 is even and vice versa.
Suppose a^2 is odd and b^2 is even. By contraposition, a is odd and b is even. So, there exists $\lambda, \mu \in \mathbb{Z}$ such that $a = 2\lambda + 1$ and $b = 2\mu$. To conclude,

$$\begin{aligned}n &= (2\lambda+1)^2+(2\mu)^2 \quad \text{since } n = a^2+b^2 \\ &= 4\lambda^2+4\lambda+4\mu^2+1\end{aligned}$$

Choosing $k = \lambda^2 + \lambda + \mu$, the result follows.

(c) There exists $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ such that $m = \alpha^2 + \beta^2$ and $n = \gamma^2 + \delta^2$. Without loss of generality, suppose α and γ are odd, and β and δ are even. So,

$$\begin{aligned}2mn &= 2\left(\alpha^2+\beta^2\right)\left(\gamma^2+\delta^2\right) \\ &= \left[(\alpha+\beta)^2+(\alpha-\beta)^2\right]\left(\gamma^2+\delta^2\right) \\ &= [(\alpha+\beta)\,\delta+(\alpha-\beta)\,\gamma]^2+[(\alpha+\beta)\,\gamma-(\alpha-\beta)\,\delta]^2 \quad \text{by } \textbf{(a)}\end{aligned}$$

which is the sum of two squares. We now show that

$$(\alpha+\beta)\,\delta+(\alpha-\beta)\,\gamma \quad \text{and} \quad (\alpha+\beta)\,\gamma-(\alpha-\beta)\,\delta$$

are odd. $\alpha + \beta$ and $\alpha - \beta$ are odd, so $(\alpha+\beta)\,\delta$ and $(\alpha-\beta)\,\delta$ are even, whereas $(\alpha-\beta)\,\gamma$ and $(\alpha+\beta)\,\gamma$ are odd. In each of the cases above, the sum of an odd integer and an even integer, so the resulting integer is even.

(d) Since the coefficients of $f(x)$ are real, by the conjugate root theorem, if $\lambda \in \mathbb{C}$ is a root of $f(x) = 0$, then λ^* is also a root of $f(x) = 0$, where λ^* is the complex conjugate of λ. We write

$$\begin{aligned}f(x) &= \text{product of all } (x-\lambda)(x-\lambda^*) \\ &= [\text{product of all } (x-\lambda)]\,[\text{product of all } (x-\lambda^*)] \\ &= [p(x)+iq(x)]\,[p(x)-iq(x)] \quad \text{where } p(x) \text{ and } q(x) \text{ are polynomials with real coefficients} \\ &= (p(x))^2+(q(x))^2\end{aligned}$$

so $f(x)$ is the sum of squares of two polynomials with real coefficients

8 2023 Paper Solutions

Question 1

(a) Recall the Cauchy-Schwarz inequality, which states that for any real numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, the inequality

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \geq \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

holds. Set $x_i = a_i$ and $y_i = 1$ for all $1 \leq i \leq n$ so the inequality becomes $n\left(a_1^2 + a_2^2 + \ldots + a_n^2\right) \geq (a_1 + a_2 + \ldots + a_n)^2$. The result follows.

(b) It suffices to show that

$$\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x} \leq \sqrt{6}\sqrt{x+y+z}.$$

Using the Cauchy-Schwarz inequality mentioned in **(i)**, setting $x_1 = \sqrt{x+y}$, $x_2 = \sqrt{y+z}$, $x_3 = \sqrt{z+x}$ and $y_i = 1$ for all $1 \leq i \leq 3$, we have

$$\left(\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x}\right)^2 \leq 3\left(x+y+y+z+z+x\right).$$

As $x+y+y+z+z+x = 2(x+y+z)$, the result follows.

(c) Think of the equation as

$$\sqrt{\frac{x+3}{x+6}} + \sqrt{\frac{x+3}{x+6}} + \sqrt{\frac{6}{x+6}} = \sqrt{6}.$$

As such, consider $x+6 = x+y+z$, $x+y = x+3$, $x+z = x+3$ and $y+z = 6$, for which this implies $y = z = 3$. So, this deals with the equality case of **(ii)**, i.e. when

$$\sqrt{\frac{x+y}{x+y+z}} = \frac{\sqrt{6}}{3}.$$

So, $9(x+3) = 6(x+6)$, which implies $x = 3$.

Question 2

(a) Using $u = \dfrac{x}{y}$, we have

$$\frac{du}{dx} = \frac{y - x\dfrac{dy}{dx}}{y^2} = \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}.$$

So,

$$\frac{du}{dx} = \frac{1}{y} - \frac{x}{y^2}\left(\frac{y}{x} - \frac{y^2}{x^3}\right) = \frac{1}{x^2}.$$

This implies that $u = -\dfrac{1}{x} + c$, where x is a constant. So, $\dfrac{x}{y} = -\dfrac{1}{x} + c$. It is easy to show that

$$y = \frac{x^2}{cx-1}.$$

This is the equation of the curve C. Since C passes through (a,b), then $b = \dfrac{a^2}{ac-1}$. So, $c = \dfrac{a^2+b}{ab}$.

(b) Using long division, the equation of the curve C can be written as

$$y = \frac{x}{c} + \frac{1}{c^2} + \frac{1}{c^2(cx-1)}.$$

For C to have two asymptotes, we must have $c^2 \neq 0$, i.e. $a^2 \neq -b$. The vertical asymptote is $x = -\dfrac{1}{c} = -\dfrac{ab}{a^2+b}$ and the oblique asymptote is $y = \dfrac{x}{c} + \dfrac{1}{c^2} = \dfrac{abx}{a^2+b} + \left(\dfrac{ab}{a^2+b}\right)^2$.

Question 3

(a) $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

(b) Starting with the RHS,

$$\sum_{j=0}^{n} \left(\binom{n}{j} \sum_{i=0}^{j} \binom{j}{i} \right) = \sum_{j=0}^{n} \binom{n}{j} 2^j \quad \text{using } \textbf{(a)}$$

Note that the binomial expansion of $(2+x)^n$ is $\sum_{j=0}^{n} \binom{n}{j} 2^j x^{n-j}$. Setting $x=1$, the result follows.

(c) **(i)** If the divisor can be factorised into r primes (which are necessarily distinct) for $1 \le r \le k$, then the number of such divisors is $\binom{k}{r}$.

So, the total number of divisors is $\sum_{r=0}^{k} \binom{k}{r} = 2^k$.

(ii) $\mu(2) = -1; \mu(3) = -1; \mu(4) = 0; \mu(6) = 1; \mu(12) = 0$

(iii) We consider two cases.

- **Case 1:** If m is prime, then its only factors are 1 and m, so

$$\sum_{d|m} \mu(d) = \mu(1) + \mu(m) = 1 + (-1) = 0.$$

- **Case 2:** If m is composite, then $m = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are primes and $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}_{\ge 0}$. So,

$$\begin{aligned}
\sum_{d|n} \mu(d) &= 1 + \sum_{r=1}^{k} \mu(p_r) + \sum_{i<j} \mu(p_i p_j) + \sum_{i<j<r} \mu(p_i p_j p_r) + \ldots + \mu(p_1 p_2 \ldots p_k) \\
&= 1 + k \cdot (-1) + \binom{k}{2} (-1)^2 + \ldots + \binom{k}{k} (-1)^k \\
&= \sum_{r=0}^{k} \binom{k}{r} (-1)^r
\end{aligned}$$

Consider $(1+x)^k = \sum_{r=0}^{k} \binom{k}{r} (-1)^r$. When we set $x=-1$, the RHS becomes the sum we wish to evaluate, while the LHS simplifies to zero.

Remark for Question 3: For **(c)**, μ is called the Möbius function.

Question 4

(a) $r_1 = 1$ as there is only one stone and that stone is coloured red; $r_2 = 0$ because if either stone is painted red, then the other cannot be painted red, otherwise, it will go against the condition that no two adjacent stones can be of the same colour.

$s_1 = 0$ as the stone cannot be painted red and not painted red concurrently; $s_2 = 3$ as there are three ways to paint the second stone, which are namely using white, yellow, or blue.

(b) Note that $r_3 = 3$ and $s_3 = 6$. So, $r_1 + s_1 = 1$, $r_2 + s_2 = 3$ and $r_3 + s_3 = 9$. So, we infer that $r_n + s_n = 3^{n-1}$.

r_{n+1} counts the number of ways to paint the stones such that the first stone is red and the $(n+1)^{\text{th}}$ stone is also red. As such, there are three choices to paint the n^{th} stone. So, the first n stones can be painted using s_n ways.

(c) Let P_n be the proposition that for all positive integers n,

$$r_n = \frac{3^{n-1} + 3(-1)^{n-1}}{4}$$

When $n=1$, we have $r_1=1$ as obtained in **(a)**. The RHS also evaluates to 1, so P_1 is true.
Assume that P_k is true for some positive integer k. That is,

$$r_k=\frac{3^{k-1}+3(-1)^{k-1}}{4}.$$

We wish to prove that P_{k+1} is true. That is,

$$r_{k+1}=\frac{3^k+3(-1)^k}{4}.$$

From **(b)**, since $r_k+s_k=3^{k-1}$ and $r_{k+1}=s_k$, then $r_k+r_{k+1}=3^{k-1}$. As such,

$$\begin{aligned}r_{k+1}&=3^{k-1}-r_k\\&=3^{k-1}-\frac{3^{k-1}+3(-1)^{k-1}}{4}\quad\text{by induction hypothesis}\\&=\frac{3\left(3^{k-1}\right)-3(-1)^{k-1}}{4}=\frac{3^k+3(-1)^k}{4}\end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, then P_n is true for all positive integers n.

(d) Suppose we colour the first stone red, then there are s_n ways to colour the remaining stones. By symmetry, the required answer is

$$\begin{aligned}4s_n&=4\left[3^{n-1}-\frac{3^{n-1}+3(-1)^{n-1}}{4}\right]\\&=3^x-3(-1)^{x-1}\end{aligned}$$

Remark for Question 4: For **(b)**, to justify that $r_n+s_n=3^{n-1}$, note that r_n+s_n counts the number of ways to place n stones on a line such that the first stone is red (and consequently, no restrictions on the last stone). Since there are $n-1$ positions to fill and there are 3 choices for each position, the result follows.

Question 5

(a) Possible remainders are 1 and 3.

(b) By Fermat's little theorem, as $z^{p-1}\equiv 1 \pmod p$, then $\left(z^2\right)^{(p-1)/2}\equiv 1 \pmod p$. So, $(-1)^{(p-1)/2}\equiv 1 \pmod p$. As such, $(p-1)/2$ must be even, so $p=4k+1$, where $k\in\mathbb{Z}$. So, $p\equiv 1 \pmod 4$. Equivalently, p is not congruent to 3 (mod 4).

(c) Possible remainders are 0, 1 and 4.

(d) **(i)** Suppose on the contrary that x is even. Then, there exists $m\in\mathbb{Z}$ such that $x=2m$, so $y^2=8m^3+7$. A perfect square is 0 or 1 mod 4, so $8m^3+7\equiv 3 \pmod 4$, which is a contradiction.

(ii) We have $y^2+1=x^3+8=(x+2)(x^2-2x+4)=(x+2)[(x-1)^2+3]$. It is clear that $(x-1)^2+3\equiv 3 \pmod 4$ and in fact, $(x-1)^2+3$ is of the form $4\alpha+3$, where $\alpha\in\mathbb{Z}$. This is because x is odd implies $x-1$ is even, so we can write $x-1=2\beta$, where $\beta\in\mathbb{Z}$. Hence, $(x-1)^2+3=4\beta^2+3$ (consequently, $\alpha=\beta^2$).

We claim that there exists a prime p such that $p\equiv 3 \pmod 4$ such that p divides y^2+1. Note that $4\alpha+3$ divides y^2+1 so there must exist some prime p of the form $4\gamma+3$ that divides $4\alpha+3$, where $\gamma\in\mathbb{Z}$. Suppose there does not exist such a prime. Then, the prime factors are of the form $4\gamma+1$. Then, the product of the prime factors will be of the form 1 mod 4, which is not 3 mod 4. Thus, we reached a contradiction.

(iii) By **(ii)**, $y^2\equiv -1 \pmod p$, where $p\equiv 3 \pmod 4$. By **(b)**, p is not congruent to 3 mod 4, which is a contradiction.

Remark for Question 5: The equation $y^2=x^3+7$ represents an elliptic curve, which has the general formula $y^2=x^3+ax+b$, where $4a^3+27b^2\neq 0$. In particular, the equation in the question belongs to a class of elliptic curves known as Mordell curves, which has the general equation $y^2=x^3+7$, where n is a non-zero integer.

Elliptic curves play an important role in abstract algebra, particularly in tackling Fermat's last theorem.

Question 6

(a)

$$\int f(x)\ dx = \int e^{mx}\sin(mx)\ dx = -\frac{1}{m}e^{mx}\cos(mx) + \int e^{mx}\cos(mx)\ dx$$
$$= -\frac{1}{m}e^{mx}\cos(mx) + \frac{1}{m}e^{mx}\sin(mx) - \int e^{mx}\sin(mx)\ dx$$

so it is clear that $\int f(x)\ dx = \frac{e^{mx}}{2m}\left[-\cos(mx)+\sin(mx)\right]+c$.

(b) Let $k = e^{-m\pi/2}$. Then,

$$\begin{aligned}\int f(x) f\left(x-\frac{\pi}{2}\right)\ dx &= k\int e^{2mx}\sin(mx)\sin\left(m\left(x-\frac{\pi}{2}\right)\right)\ dx\\ &= k\int e^{2mx}\sin(mx)\left(\sin(mx)\cos\left(\frac{m\pi}{2}\right)-\cos(mx)\sin\left(\frac{m\pi}{2}\right)\right)\ dx\\ &= \frac{k}{2}\cos\left(\frac{m\pi}{2}\right)\int e^{2mx}\ dx - \frac{k}{2}\cos\left(\frac{m\pi}{2}\right)\int e^{2mx}\cos(2mx)\ dx - \frac{k}{2}\sin\left(\frac{m\pi}{2}\right)\int e^{2mx}\sin(2mx)\ dx\\ &= \frac{ke^{2mx}}{4m}\cos\left(\frac{m\pi}{2}\right) - \frac{k}{2}\cos\left(\frac{m\pi}{2}\right)\int e^{2mx}\cos(2mx)\ dx - \frac{k}{2}\sin\left(\frac{m\pi}{2}\right)\int e^{2mx}\sin(2mx)\ dx\end{aligned}$$

Note that

$$\begin{aligned}\int e^{mx}\cos(mx)\ dx &= e^{mx}\frac{1}{m}\sin(mx) - \int e^{mx}\sin(mx)\ dx\\ &= \frac{1}{m}e^{mx}\sin(mx) + \frac{1}{2m}e^{mx}\cos(mx) - \frac{1}{2m}e^{mx}\sin(mx) \text{ using } \textbf{(a)}\\ &= \frac{e^{mx}}{2m}\left[\cos(mx)+\sin(mx)\right]+c\end{aligned}$$

so the original integral becomes

$$\begin{aligned}&\frac{ke^{2mx}}{4m}\cos\left(\frac{m\pi}{2}\right) - \frac{ke^{2mx}}{8m}\cos\left(\frac{m\pi}{2}\right)\left[\sin(2mx)+\cos(2mx)\right] - \frac{ke^{2mx}}{8m}\sin\left(\frac{m\pi}{2}\right)\left[\sin(2mx)-\cos(2mx)\right]\\ =&\frac{ke^{2mx}}{8m}\left\{\cos\left(\frac{m\pi}{2}\right)\left[2-\sin(2mx)-\cos(2mx)\right] - \sin\left(\frac{m\pi}{2}\right)\left[\sin(2mx)-\cos(2mx)\right]\right\}\end{aligned}$$

Recall the following as well:

$$\cos\left(\frac{m\pi}{2}\right) = \begin{cases}0 & \text{if } m \text{ is odd;}\\ (-1)^{m/2} & \text{if } m \text{ is even}\end{cases} \text{ and } \sin\left(\frac{m\pi}{2}\right) = \begin{cases}(-1)^{(m-1)/2} & \text{if } m \text{ is odd;}\\ 0 & \text{if } m \text{ is even}\end{cases}$$

so if m is odd, then

$$\int f(x)f\left(x-\frac{\pi}{2}\right)\ dx = \frac{(-1)^{(m+1)/2}ke^{2mx}}{8m}\left[\sin(2mx)-\cos(2mx)\right]+c,$$

and if m is even, then

$$\int f(x)f\left(x-\frac{\pi}{2}\right)\ dx = \frac{(-1)^{m/2}ke^{2mx}}{8m}\left[2-\sin(2mx)-\cos(2mx)\right]+c,$$

where c is a constant.

Question 7

(a) There are 4×3 squares of length 1 unit, 3×2 squares of length 2 units and 2×1 squares of 1 unit. The total number of squares is $12+6+2=20$.

(b) First, note that the largest square has length n units. There are $(m-1)(n-1)$ squares of length 1 unit, $(m-2)(n-2)$ squares of length 2 units and so on. So, there are $(m-k)(n-k)$ squares of length k units, where $1\le k\le n$.

The total number of squares is

$$\begin{aligned}\sum_{k=1}^{n}(m-k)(n-k) &= \sum_{k=1}^{n} mn-(m+n)\sum_{k=1}^{n} k+\sum_{k=1}^{n} k^2\\ &= mn^2-\frac{n(n+1)(m+n)}{2}+\frac{n(n+1)(2n+1)}{6}\\ &= \frac{1}{6}n[6m-3(n+1)(n+m)+(n+1)(2n+1)]\\ &= \frac{1}{6}n\left(3mn+1-n^2-3m\right)\\ &= \frac{1}{6}n(n-1)(3m-n-1)\end{aligned}$$

(c) Set

$$\frac{1}{6}n(n-1)(3m-n-1)=100.$$

Then, $n(n-1)(3m-n-1)=600$.
We must have $n(n-1)$ to divide 600. The factors of 600 up to $\left\lfloor\sqrt{600}\right\rfloor$ are listed as follows:

$$1,2,3,4,5,6,8,10,12,15,20,24$$

We shall test for $n=2,3,4,5,6$.

- **Case 1:** If $n=2$, then $m=101$
- **Case 2:** If $n=3$, then $m=34\frac{2}{3}\notin\mathbb{N}$
- **Case 3:** If $n=4$, then $m=18\frac{1}{3}\notin\mathbb{N}$
- **Case 4:** If $n=5$, then $m=12$
- **Case 5:** If $n=6$, then $m=9$.

As such, the required pairs are $(m,n)=(101,2),(12,5)$ and $(9,6)$.

Question 8

(a) Note that the prime factorisation of 2400 is $2^5\times 3\times 5^2$.
Let $A,B,C\subseteq\{1,2,\ldots,2400\}$ be the sets of integers divisible by 2, 3, and 5 respectively. We wish to find $|A'\cap B'\cap C'|$, for which by de Morgan's law, is equal to $2400-|A\cup B\cup C|$.
By the principle of inclusion and exclusion,

$$\begin{aligned}|A\cup B\cup C| &= |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|\\ &= \frac{2400}{2}+\frac{2400}{3}+\frac{2400}{5}-\frac{2400}{2\times 3}-\frac{2400}{2\times 5}-\frac{2400}{3\times 5}+\frac{2400}{2\times 3\times 5}\\ &= 1760\end{aligned}$$

Hence, the required answer is $2400-1760=640$.

(b) **(i)** We have $N(a+b)=ab$. So, $(a-N)(b-N)=ab-N(a+b)+N^2=N^2$.

(ii) Without loss of generality, assume that $a>b$. Then, we consider the following two cases:

- **Case 1:** Suppose $a-N=N$ and $b-N=N$. Then, $a=b=2N$, which is a contradiction as $a>b$.
- **Case 2:** Suppose $a-N=N^2$ and $b-N=1$. This is justified since $a=N^2+N>N+1=b$.

So, there is only one way to express $\frac{1}{N}$ as a sum of two distinct unit fractions, which is $\frac{1}{N^2+N}+\frac{1}{N+1}$.

(i) Since $f(r)=0$, then

$$r=\frac{B\pm\sqrt{B^2-4A}}{2}.$$

For r to be rational, we must have B^2-4A to be a perfect square. Suppose there exists $k \in \mathbb{Z}_{\geq 0}$ such that $B^2-4A=k^2$. Hence, $(B+k)(B-k)=4A$. So, $\frac{1}{2}(B+k)(B-k)=2A$.
Observe that

$$\begin{aligned} r &= \frac{B \pm k}{\frac{1}{2}(B+k)(B-k)} \\ &= \frac{1}{\frac{B-k}{2}} \text{ or } \frac{1}{\frac{B+k}{2}} \end{aligned}$$

If B is even, then k is even; if B is odd, then k is odd.
So, the denominators of the unit fractions, $\frac{B-k}{2}$ and $\frac{B+k}{2}$, are positive integers. The result follows.

(ii) By Vieta's formula, as the sum of roots is $\frac{7}{13}$, then $\frac{B}{A}=\frac{7}{13}$, so $7A=13B$.
Since $\gcd(7,13)=1$, there exists $k \in \mathbb{Z}$ such that $A=13k$ and $B=7k$.
We can rewrite the quadratic equation as $13kx^2-7kx+1=0$, so

$$x = \frac{7k \pm \sqrt{49k^2-52k}}{26k}.$$

Since $49k^2-52k$ is a perfect square, there exists $m \in \mathbb{Z}$ such that $49k^2-52k=m^2$. By completing the square,

$$\left(k-\frac{26}{49}\right)^2 = \frac{49m^2+26^2}{49^2},$$

which means that $49m^2+26^2$ is also a perfect square. Then, there exists $\lambda \in \mathbb{Z}$ such that $49m^2+26^2=\lambda^2$.
By the difference of squares formula, $(\lambda+7m)(\lambda-7m)=26^2$.

- **Case 1:** Suppose $\lambda+7m=169$ and $\lambda-7m=4$. Then, $14m=163$. One checks that $49m^2+26^2$ is not a perfect square.
- **Case 2:** Suppose $\lambda+7m=338$ and $\lambda-7m=2$. Then, $14m=336$, so $49m^2+26^2=170^2$.

As such, $k=4$ or $k=-\frac{144}{49}$.

For the sake of contradiction, suppose $k=-\frac{144}{49}$. Then, $49k^2-52k=576=24^2$.
However,

$$\frac{7k-\sqrt{49k^2-52k}}{26k} = \frac{7k-24}{26k} = \frac{7}{26}-\frac{12}{13k} < 0$$

which is a contradiction as $r_1, r_2 > 0$.

Thus, $k=4$, so $49k^2-52k=24^2$. This implies that

$$x = \frac{28 \pm 24}{104}.$$

Without loss of generality, set $r_1=\frac{1}{2}$ and $r_2=\frac{1}{26}$, so $A=52$ and $B=28$.

9 2024 Paper Solutions

10 2025 Specimen Paper Solutions

Question 1

(a) Since $y = x$, then $\dfrac{dy}{dx} = 1$, so the LHS of the differential equation becomes $x^2 + x^2 - x^2 - x^2$, which is zero.

(b) Letting $u = \dfrac{y}{x}$, we have $\dfrac{du}{dx} = \dfrac{1}{x^2}\left(x\dfrac{dy}{dx} - y\right)$.
The differential equation becomes

$$\frac{du}{dx} = \frac{xF(u) - y}{x^2} = \frac{F(u) - u}{x}.$$

The result follows by multiplying x on both sides of the equation.

(c) We have

$$\frac{dy}{dx} = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1.$$

Making reference to **(b)**, we see that $F\left(\dfrac{y}{x}\right) = \left(\dfrac{y}{x}\right)^2 + \dfrac{y}{x} - 1$. So, $F(u) = u^2 + u - 1$.
The differential equation becomes

$$x\frac{du}{dx} = u^2 - 1.$$

So,

$$\int \frac{1}{u^2 - 1}\, du = \int \frac{1}{x}\, dx,$$

which implies that

$$\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| = \ln|x| + c,$$

where x is an arbitrary constant. When $x = 1$ and $y = 2$, we have $u = 2$, so substituting $(x, u) = (1, 2)$ into the equation of the above equation yields $c = -\frac{1}{2}\ln 3$.

Therefore,

$$\begin{aligned}
\frac{1}{2}\ln\left|\frac{y-x}{y+x}\right| &= \ln|x| - \frac{1}{2}\ln 3 \\
\ln\left|\frac{3(y-x)}{y+x}\right| &= 2\ln|x| \\
\frac{3(y-x)}{y+x} &= x^2 \\
y &= \frac{x^3 + 3x}{3 - x^2}
\end{aligned}$$

which is the required equation of the curve.

Question 2

(a) We use integration by parts. So,

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{3}} \tan^n\theta\, d\theta \\
&= \int_0^{\frac{\pi}{3}} \tan^{n-2}\theta \tan^2\theta\, d\theta \\
&= \int_0^{\frac{\pi}{3}} \tan^{n-2}\theta \sec^2\theta\, d\theta - \int_0^{\frac{\pi}{3}} \tan^{n-2}\theta\, d\theta \quad \text{since } \tan^2\theta = \sec^2\theta - 1 \\
&= \left[\tan^{n-1}\theta\right]_0^{\frac{\pi}{3}} - (n-2)\int_0^{\frac{\pi}{3}} \tan^{n-2}\theta \sec^2\theta\, d\theta - I_{n-2} \\
&= 3^{\frac{n-1}{2}} - (n-2)\int_0^{\frac{\pi}{3}} \tan^n\theta\, d\theta - (n-2)\int_0^{\frac{\pi}{3}} \tan^{n-2}\theta\, d\theta - I_{n-2} \\
&= 3^{\frac{n-1}{2}} - (n-2)I_n - (n-2)I_{n-2} - I_{n-2} \\
(n-1)I_n &= 3^{\frac{n-1}{2}} - (n-1)I_{n-2}
\end{aligned}$$

Dividing by $n-1$ yields the result.

(b) It is clear that $I_0 = \frac{\pi}{3}$.
Also,

$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{3}} \tan\theta \, d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{\sin\theta}{\cos\theta} \, d\theta \\ &= [\ln|\cos\theta|]_0^{\frac{\pi}{3}} \\ &= \ln 2 \end{aligned}$$

So,

$$\begin{aligned} I_5 &= \frac{3^2}{4} - I_3 \\ &= \frac{3^2}{4} - \frac{3}{2} + I_1 \\ &= \frac{3^2}{4} - \frac{3}{2} + \ln 2 \\ &= \frac{3}{4} + \ln 2 \end{aligned}$$

and

$$\begin{aligned} I_6 &= \frac{9\sqrt{3}}{5} - I_4 \\ &= \frac{9\sqrt{3}}{5} - \sqrt{3} + I_2 \\ &= \frac{9\sqrt{3}}{5} - \sqrt{3} + \sqrt{3} - I_0 \\ &= \frac{9\sqrt{3}}{5} - \frac{\pi}{3} \end{aligned}$$

Question 3

(a) **(i)** By the AM-GM inequality, we have

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2\right] \geq \sqrt{\left(\frac{x}{y}\right)^2\left(\frac{y}{z}\right)^2} = \frac{x}{z}.$$

(ii) Using the Cauchy-Schwarz inequality,

$$\begin{aligned} \left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2\right]\left[\left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2 + \left(\frac{x}{y}\right)^2\right] &\geq \left[\left(\frac{x}{y}\right)\left(\frac{y}{z}\right) + \left(\frac{y}{z}\right)\left(\frac{z}{x}\right) + \left(\frac{z}{x}\right)\left(\frac{x}{y}\right)\right]^2 \\ \left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2\right]^2 &\geq \left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right)^2 \\ \left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2 &\geq \frac{x}{z} + \frac{y}{x} + \frac{z}{y} \end{aligned}$$

so we have proven the upper bound for $\frac{x}{z} + \frac{y}{x} + \frac{z}{y}$.

Next, using the AM-GM inequality, we have

$$\frac{x}{z} + \frac{y}{x} + \frac{z}{y} \geq 3\sqrt[3]{\left(\frac{x}{z}\right)\left(\frac{y}{x}\right)\left(\frac{z}{y}\right)} = 3$$

so we have proven the lower bound for $\frac{x}{z} + \frac{y}{x} + \frac{z}{y}$.

(b) **(i)** By definition of the scalar product, for any two vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

we have $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the two vectors. Since $|\cos\theta| \leq 1$, then $\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$, which implies that

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \leq \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right| \left|\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right|$$

and the result follows. For equality to hold, we must have $a_i = kb_i$ for all $1 \leq i \leq 3$ and some $k \in \mathbb{R} \backslash \{0\}$.

(ii) Using the Cauchy-Schwarz inequality,

$$\left[\left(\frac{x}{\sqrt{y+z}}\right)^2 + \left(\frac{y}{\sqrt{z+x}}\right)^2 + \left(\frac{z}{\sqrt{x+y}}\right)^2\right]\left[\left(\sqrt{y+z}\right)^2 + \left(\sqrt{z+x}\right)^2 + \left(\sqrt{x+y}\right)^2\right] \geq (x+y+z)^2$$

$$\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right)(y+z+z+x+x+y) \geq (x+y+z)^2$$

$$2\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \geq x+y+z$$

and equality holds if and only if $x = y = z$.

Remark for Question 3: Other than **(ai)**, the other three parts are the same as Question 1 of the 2017 specimen paper.

Question 4

(i) Since f is continuous on $[0, 0.4]$, $f(0) = 1 > 0$ and $f(0.4) = -0.136 < 0$, then there exists a root in $(0, 0.4)$. Next, since f is continuous on $[0.4, 2]$, $f(0.4) = -0.136 < 0$ and $f(2) = 3 > 0$, then there exists a root in $(0.4, 2)$. Lastly, since f is continuous on $[-2, 0]$, $f(-2) = -1 < 0$ and $f(0) = 1 > 0$, then there exists a root in $(-2, 0)$.

The above shows that f has at least three distinct real roots. To show that there are only three distinct real roots, consider $f'(x) = 3\left(x^2 - 1\right)$ so f is strictly increasing for $x > 1$ and strictly decreasing for $x < -1$.

(ii) Note that

$$fg(x) = f\left(\frac{1}{1-x}\right) = \left(\frac{1}{1-x}\right)^3 - 3\left(\frac{1}{1-x}\right) + 1 = -\frac{1-3x+x^3}{(1-x)^3}$$

so $g(\alpha), g(\beta)$ and $g(\gamma)$ are the roots of f. From **(i)**, we know that $\alpha \in (-2, 0)$, $\beta \in (0, 0.4)$ and $\gamma \in (0.4, 2)$. We have $g(\gamma) < 0$, which implies that $g(\gamma) = \alpha$. Suppose on the contrary that $g(\beta) = \beta$. Then,

$$\frac{1}{1-\beta} = \beta.$$

That is, $\beta^2 - \beta + 1 = 0$. However, the roots of this equation are not real, which is a contradiction. As such, $g(\beta) = \gamma$, leaving us with $g(\alpha) = \beta$.

(iii) Write $h(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Then for $x = \alpha, \beta, \gamma$, we have

$$ax^2 + bx + c = \frac{1}{1-x}$$

$$ax^2(1-x) + bx(1-x) + c(1-x) - 1 = 0$$

$$ax^2 - ax^3 + bx - bx^2 + c - cx - 1 = 0$$

$$-ax^3 + (a-b)x^2 + (b-c)x + c - 1 = 0$$

Comparing the last line with $f(x)$, we see that $a = -1$, $b = -1$ and $c = 2$. So, $h(x) = -x^2 - x + 2$.

Remark for Question 4: This question is the same as Question 6 of the 2017 specimen paper.

Question 5

(a) The derangements are

$$2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321.$$

(b) For an ordering of the numbers 1 to n, let A_n denote the event that the number i is in position i. We wish to find

$$\left|\bigcap_{i=1}^{n} A_i'\right|.$$

By de Morgan's law, the above is equal to

$$n! - \left|\bigcup_{i=1}^{n} A_i\right|.$$

By the principle of inclusion and exclusion,

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{i=1}^{n} |A_i| - \sum_{i<j} \left|A_i \cap A_j\right| + \sum_{i<j<k} \left|A_i \cap A_j \cap A_k\right| + \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n| \\ &= n\,(n-1)! - \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^{n+2}\end{aligned}$$

So,

$$\begin{aligned}D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! + \ldots + (-1)^{n+1} \\ &= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + (-1)^n \frac{1}{n!}\right)\end{aligned}$$

(c) As

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

then

$$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

So,

$$\begin{aligned}\left|D_n - \frac{n!}{e}\right| &= n!\left|\sum_{k=0}^{n} \frac{(-1)^k}{k!} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\right| \\ &= n!\left|\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - \sum_{k=0}^{n} \frac{(-1)^k}{k!}\right| \\ &= n!\left|\sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}\right| \\ &= \frac{1}{n+1} - \frac{1}{(n+1)\,(n+2)} + \frac{1}{(n+1)\,(n+2)\,(n+3)} - \frac{1}{(n+1)\,(n+2)\,(n+3)\,(n+4)} + \ldots \\ &= \frac{1}{n+1}\left[1 - \frac{1}{n+2} + \frac{1}{(n+2)\,(n+3)} - \frac{1}{(n+2)\,(n+3)\,(n+4)} + \ldots\right]\end{aligned}$$

As

$$\frac{1}{n+2} - \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \ldots \in (0,1),$$

it follows that

$$\left|D_n - \frac{n!}{e}\right| < \frac{1}{1+n}.$$

For the second part, since $n \geq 1$, then $\left|D_n - \frac{n!}{e}\right| < 1$ and the result follows.

(d) We need to show that

$$\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}.$$

This is true because

$$\lim_{n\to\infty} \frac{D_n}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{e}.$$

Remark for Question 5: For **(a)**, one sees that there are 9 derangements for the case when $n = 4$. One can verify this by using the formula given in **(b)**.

Question 6

(a) Consider the following figure. The sum of the areas of the rectangles is

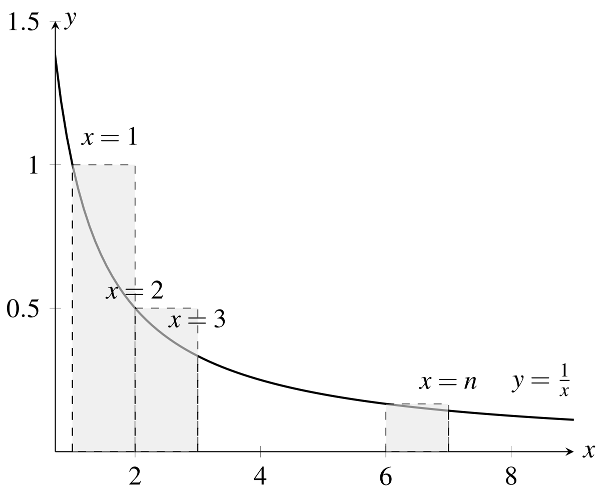
$$\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n-1} = H_{n-1},$$

whereas the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = n$ is

$$\int_1^n \frac{1}{x}\, dx = \ln n.$$

As such, $\ln n < H_{n-1}$. Adding $\frac{1}{n}$ to both sides and recognising the $H_{n-1} + \frac{1}{n} = H_n$, we obtain

$$\frac{1}{n} + \ln n < H_n.$$

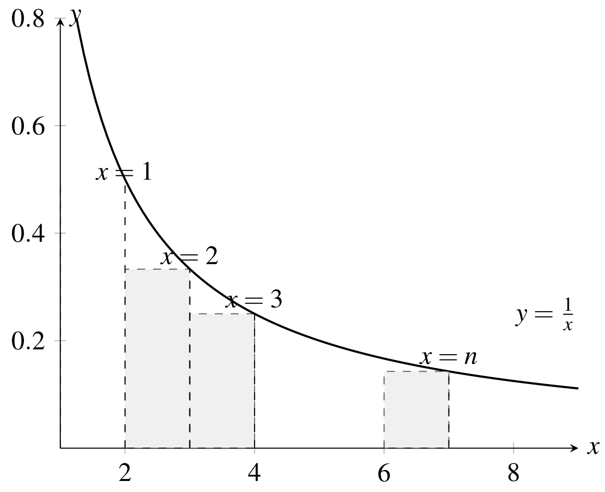


Next, consider the following figure. The sum of the areas of the rectangles is

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = H_n - 1,$$

whereas the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = n$ is $\ln n$. As such, $H_n - 1 < \ln n$. Adding 1 to both sides, it follows that

$$H_n < 1 + \ln n.$$



(b) We have

$$\begin{aligned}\lim_{n\to\infty} H_n &> \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\ln n \\ &= 1 + \lim_{n\to\infty}\ln n\end{aligned}$$

Since $\ln n$ diverges to infinity, then the harmonic series diverges too.

Next, $H_{1\ 000\ 000} < 1 + \ln(1\ 000\ 000) = 14.815 < 15$.

(c) As $p_n = n!H_n$, it suffices to show that

$$n!H_n = n(n-1)!H_{n-1} + (n-1)!.$$

Starting with the RHS, we have

$$\begin{aligned}n(n-1)!H_{n-1} + (n-1)! &= (n-1)!(nH_{n-1}+1) \\ &= n!\left(\frac{nH_{n-1}+1}{n}\right) \\ &= n!\left(H_{n-1}+\frac{1}{n}\right) \\ &= n!H_n\end{aligned}$$

(d) **(i)** $\begin{bmatrix} n \\ k \end{bmatrix}$ counts the number of ways for n distinct people to sit around 1 circular table. This is equivalent to the number of permutations of n distinct objects on a circle, which is $(n-1)!$.

(ii) Consider a person out of the n, say α. We have the following two cases:

- **Case 1:** If α is alone, then there are $\begin{bmatrix} n \\ k-1 \end{bmatrix}$ ways to distribute the remaining n people around $k-1$ tables such that no table is empty.
- **Case 2:** If α is seated with other people, we first let α sit around some arbitrary table in n ways. Then, distribute the remaining n people around k tables such that the other tables are non-empty in $\begin{bmatrix} n \\ k \end{bmatrix}$ ways.

By the addition principle, it follows that $\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + n\begin{bmatrix} n \\ k \end{bmatrix}$.

(e) From **(d)**, we have

$$\begin{aligned}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= \begin{bmatrix} n \\ 1 \end{bmatrix} + n\begin{bmatrix} n \\ 2 \end{bmatrix} \\ &= (n-1)! + n\begin{bmatrix} n \\ 2 \end{bmatrix} \\ \frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= \frac{1}{n} + \frac{1}{(n-1)!}\begin{bmatrix} n \\ 2 \end{bmatrix}\end{aligned}$$

Let $f(k) = \frac{1}{(k-1)!}\begin{bmatrix} k \\ 2 \end{bmatrix}$. Then, $f(k)$ satisfies $f(k+1) - f(k) = \frac{1}{k}$.
Summing both sides, we have

$$\sum_{k=2}^{n-1} [f(k+1) - f(k)] = \sum_{k=2}^{n-1} \frac{1}{k},$$

and it follows by the method of difference that $f(n) - f(2) = H_{n-1} - 1$. Since $f(2) = 1$, then $f(n) = H_{n-1}$.
Therefore, $f(n+1) = H_n$ and the result follows.

(f) We have,

$$\begin{aligned}2^k M(n) H_n &= \frac{2^k M(n)}{1} + \frac{2^k M(n)}{2} + \frac{2^k M(n)}{3} + \ldots + \frac{2^k M(n)}{2^k} + \ldots + \frac{2^k M(n)}{n} \\ &= 2^k \left[\frac{M(n)}{1}\right] + 2^k \left[\frac{M(n)}{2}\right] + 2^k \left[\frac{M(n)}{3}\right] + \ldots + M(n) + \ldots + 2^k \left[\frac{M(n)}{n}\right]\end{aligned}$$

- **Case 1:** Suppose n is odd. Then, $M(n) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot n$.
- **Case 2:** Suppose n is even. Then, $M(n) = 1 \cdot 3 \cdot 5 \ldots \cdot (n-1)$. Also, 2^k and n have a common factor of 2, which shows that $\frac{M(n)}{n}$ is an integer.

In each case, since $2^k \left[\frac{M(n)}{i}\right]$ is even for all i except when $i = 2^k$, the result follows.

(g) Suppose on the contrary that there exists $\beta \in \mathbb{Z}$ such that $H_n = \beta$.
Then, $2^k M(n) H_n = 2^k M(n) \beta$. From **(f)**, the LHS is odd, but the RHS is even.
This is a contradiction so no such $\beta \in \mathbb{Z}$ exists.

Remark for Question 6: In **(d)**, $\begin{bmatrix} n \\ k \end{bmatrix}$ is related to the Stirling numbers of the first kind.

11 2025 Paper Solutions

12 2026 Paper Solutions