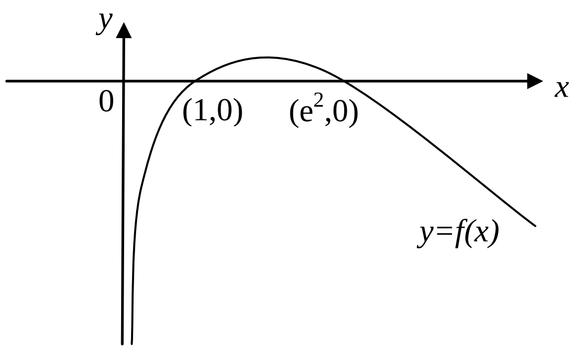
Anglo-Chinese Junior College

H2 Mathematics 9740

Qn	**Paper 1 Solution**
1(a)	$\int \frac{x}{x^2+4x+7}\,dx = \int \frac{1}{2}\frac{2x+4}{x^2+4x+7} - \frac{2}{x^2+4x+7}\,dx$ $= \frac{1}{2}\ln\left(x^2+4x+7\right) - \int \frac{2}{\left(x+2\right)^2+3}\,dx$ $= \frac{1}{2}\ln\left(x^2+4x+7\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + C$
(b)	$\int_0^a x\sqrt{a-x}\,dx = \left[-x\frac{2\left(a-x\right)^{\frac{3}{2}}}{3}\right]_0^a - \int_0^a -\frac{2\left(a-x\right)^{\frac{3}{2}}}{3}\,dx$ $= 0 + \frac{2}{3}\int_0^a \left(a-x\right)^{\frac{3}{2}}\,dx$ $= \frac{2}{3}\left[-\frac{2\left(a-x\right)^{\frac{5}{2}}}{5}\right]_0^a$ $= \frac{2}{3}\left[0 + \frac{2a^{\frac{5}{2}}}{5}\right]$ $= \frac{4}{15}a^{\frac{5}{2}}$
2	$\frac{x^2+2x}{2x^2-5x+2} \geq \frac{-3}{2x^2-5x+2}$ $\frac{x^2+2x+3}{2x^2-5x+2} \geq 0$ $\frac{\left(x+1\right)^2+2}{(2x-1)(x-2)} \geq 0$ $Since \left(x+1\right)^2+2>0,$ $(2x-1)(x-2)>0$ $x<0.5 \quad or \quad x>2$ $Since\ x \in \mathbb{R}^{+},$ $\therefore 0<x<0.5 \quad or \quad x>2$
3	When $y=0$, $x=1$ and e^2 

When $y=0$, it cuts at 2 points on the curve. Therefore it is not a one to one function.
Thus, f^{-1} does not exist.

$a=e$

$y=\ln x^2-(\ln x)^2$

$y=2\ln x-(\ln x)^2$

$-y=(\ln x-1)^2-1$

$\ln x=1\pm\sqrt{1-y}$

$x=e^{1+\sqrt{1-y}}$ (n.a) or $e^{1-\sqrt{1-y}}$

$\mathrm{f}^{-1}: x\mapsto e^{1-\sqrt{1-x}}$, $x\in(-\infty,1]$

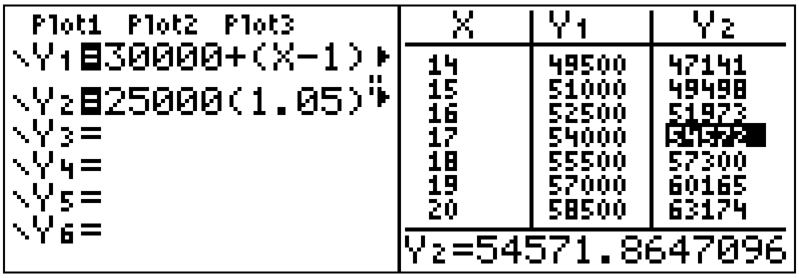
$\mathrm{R}_{gh}=(0,3e]$

4 (i) Let n be the number of years after 2013.

Tom's pay $=30000+(n-1)(1500)$

Jerry's pay $=25000(1.05)^{n-1}$

$30000+(n-1)(1500)<25000(1.05)^{n-1}$

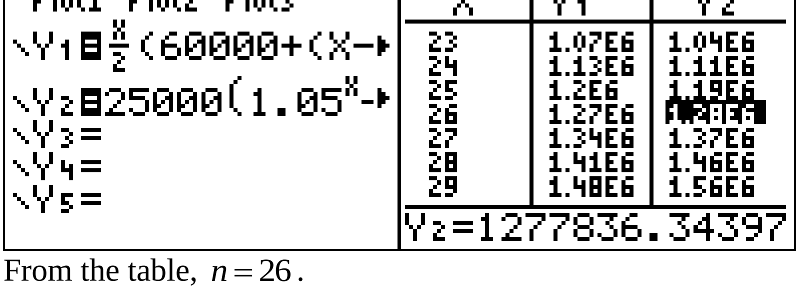


From the table, $n=17$.
Hence the first year in which Jerry's pay is higher than Tom's is 2029.

(ii) Tom's total income $=\frac{n}{2}\left[2(30000)+(n-1)(1500)\right]$

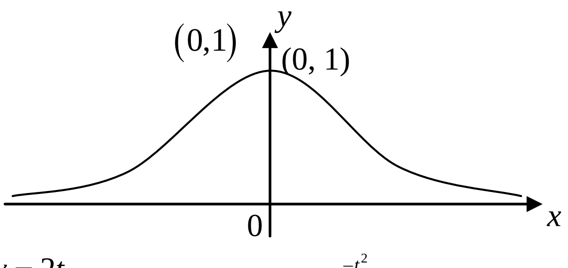
Jerry's pay $=25000\frac{1.05^n-1}{1.05-1}$

$\frac{n}{2}\left[2(30000)+(n-1)(1500)\right]<25000\frac{1.05^n-1}{1.05-1}$



From the table, $n=26$.
Hence the first year in which Jerry's pay is higher than Tom's is 2038.

5	$A = \pi r^2 B + 2\pi rhC$ $\Rightarrow h = \dfrac{A - \pi r^2 B}{2\pi rC}$ $V = \pi r^2 h$ $= \pi r^2 \left(\dfrac{A - \pi r^2 B}{2\pi rC} \right)$ $= \dfrac{r}{2C}\left(A - \pi r^2 B\right)$ $\dfrac{dV}{dr} = \dfrac{r}{2C}\left(-2\pi rB\right) + \left(A - \pi r^2 B\right)\dfrac{1}{2C} = 0$ $\dfrac{1}{2C}\left(A - 3\pi r^2 B\right) = 0$ $r^2 = \dfrac{A}{3\pi B}$ gives max V Cost of base $= \pi r^2 B$ $= \pi\left(\dfrac{A}{3\pi B}\right)B$ $= \dfrac{A}{3}$
6	$u_1 = \dfrac{\frac{3}{4}}{\frac{2}{4}+1} = \dfrac{1}{2} = \dfrac{1}{1+1}$ $u_2 = \dfrac{\frac{3}{2}}{\frac{2}{2}+1} = \dfrac{3}{4} = \dfrac{3}{1+3}$ $u_3 = \dfrac{\frac{9}{4}}{\frac{6}{4}+1} = \dfrac{9}{10} = \dfrac{9}{1+9}$ $u_4 = \dfrac{\frac{27}{10}}{\frac{18}{10}+1} = \dfrac{27}{28} = \dfrac{27}{1+27}$ $\therefore u_n = \dfrac{3^{n-1}}{1+3^{n-1}}$ Let P(n) be the statement $u_n = \dfrac{3^{n-1}}{1+3^{n-1}}$ for all $n \ge 0$. When $n = 0$, LHS = $u_0 = \dfrac{1}{4}$ RHS = $\dfrac{3^{-1}}{1+3^{-1}} = \dfrac{\frac{1}{3}}{1+\frac{1}{3}} = \dfrac{1}{4}$ $\therefore$ LHS = RHS $\therefore$ P(0) is true.

	Assume that P(k) is true for some $k \geq 0$, i.e., $u_k = \dfrac{3^{k-1}}{1+3^{k-1}}$. To prove P($k$ + 1) is true, i.e., $u_{k+1} = \dfrac{3^k}{1+3^k}$, $\text{LHS} = u_{k+1}$ $= \dfrac{3u_k}{2u_k + 1}$ $= \dfrac{3\left(\dfrac{3^{k-1}}{1+3^{k-1}}\right)}{2\left(\dfrac{3^{k-1}}{1+3^{k-1}}\right)+1}$ (by assumption) $= \dfrac{\dfrac{3^k}{1+3^{k-1}}}{\dfrac{2\cdot 3^{k-1}+1+3^{k-1}}{1+3^{k-1}}}$ $= \dfrac{3^k}{2\cdot 3^{k-1}+1+3^{k-1}}$ $= \dfrac{3^k}{1+3\cdot 3^{k-1}}$ $= \dfrac{3^k}{1+3^k} = \text{RHS}$ Since P(0) is true and P(k) is true ⇒ P(k+1) is true, by the Principle of Mathematical Induction, we conclude that P(0), P(1), P(2), P(3), … are all true. Hence P(n) is true for all integers $n \geq 0$.
7	$x = 2t, \quad y = e^{-t^2}$  $x = 2t$ $y = e^{-t^2}$ $\dfrac{dx}{dt} = 2$ $\dfrac{dy}{dt} = -2te^{-t^2}$ $\therefore \dfrac{dy}{dx} = -te^{-t^2}$ Equation of normal at $\left(2p, e^{-p^2}\right)$ is $y - e^{-p^2} = \dfrac{1}{p}e^{p^2}\left(x-2p\right)$ Equation of normal at $C\left(2, \dfrac{1}{e}\right)$ is $y - \dfrac{1}{e} = e\left(x-2\right)$ At $A\left(x,0\right)$: $x - 2 = -\dfrac{1}{e^2} \Rightarrow x = 2 - \dfrac{1}{e^2} = \dfrac{2e^2-1}{e^2}$ At $B\left(0,y\right)$: $y - \dfrac{1}{e} = -2e \Rightarrow y = \dfrac{1}{e} - 2e = \dfrac{1-2e^2}{e}$

$$\frac{|OA|}{|OB|} = \left| \frac{2e^2-1}{e^2} \Big/ \frac{1-2e^2}{e} \right| = \frac{1}{e}$$

$\therefore\ |OA| : |OB| = 1 : e$

8

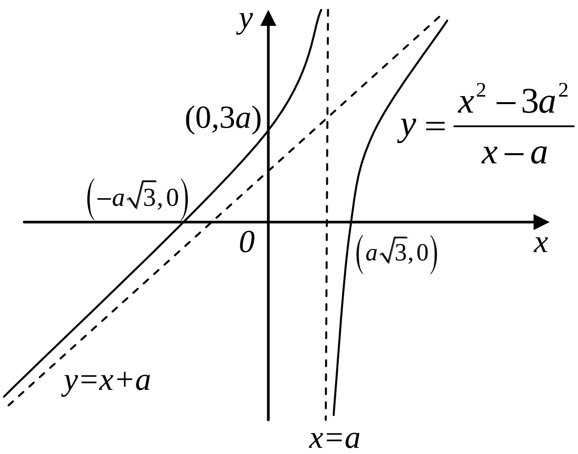
$$y = \frac{x^2-3a^2}{x-a} = x + a - \frac{2a^2}{x-a}$$

Asymptotes: $x = a, \quad y = x + a$

$$y = \frac{x^2-3a^2}{x-a} = x + a - \frac{2a^2}{x-a} \ \Rightarrow\ \frac{dy}{dx} = 1 + \frac{2a^2}{(x-a)^2}$$

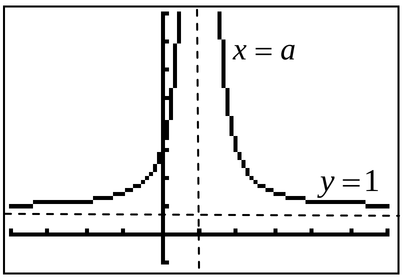
Since $2a^2 > 0$ & $(x-a)^2 \geq 0 \quad \forall x \in \mathbb{R}$, $\frac{dy}{dx} = 1 + \frac{2a^2}{(x-a)^2} > 1 \neq 0$

$y = \frac{x^2-3a^2}{x-a} = x + a - \frac{2a^2}{x-a}$ has no stationary points. (shown)



Axes intercepts: $(a\sqrt{3},0)$, $(-a\sqrt{3},0)$, $(0,3a)$

$$y = f'(x) = 1 + \frac{2a^2}{(x-a)^2}$$



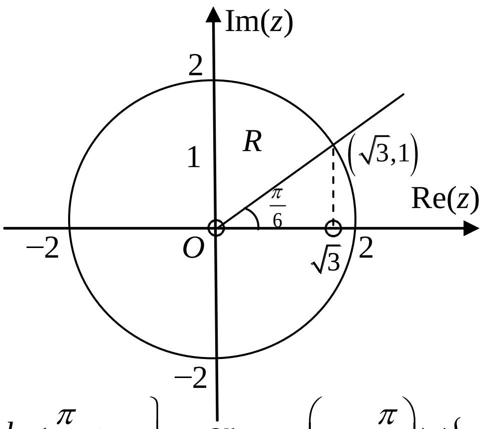
From the graph of $y = f'(x)$, f' is increasing for $x < a$

<u>OR</u>

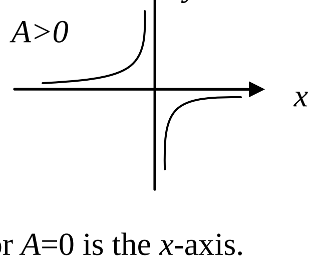
$$f''(x) = -\frac{4a^2}{(x-a)^3} \geq 0$$

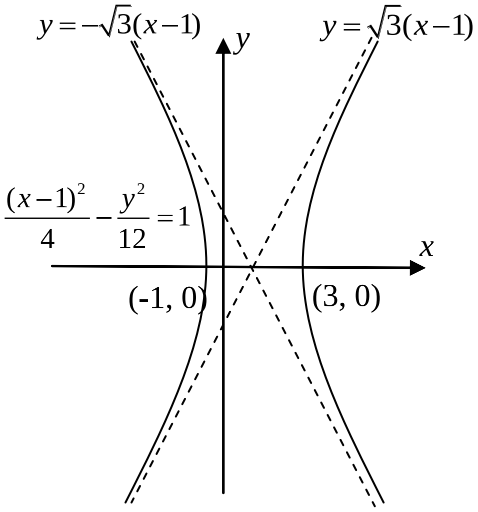
$x - a < 0$

$\therefore x < a$

9	 $\left\{k\in\mathbb{R}: -\pi<k<\frac{\pi}{2}\text{ or }\pi\right\}$ or $\left(-\pi,\frac{\pi}{2}\right)\cup\{\pi\}$
10 (i)	$\frac{1}{\sqrt{a}+\sqrt{b}}=\frac{1}{\sqrt{a}+\sqrt{b}}\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{a}-\sqrt{b}}{a-b}$
(ii)	$\sum_{r=1}^{2500}\frac{1}{\sqrt{4r-3}+\sqrt{4r+1}}$ $=\sum_{r=1}^{2500}\frac{\sqrt{4r-3}-\sqrt{4r+1}}{(4r-3)-(4r+1)}$ $=\frac{1}{4}\sum_{r=1}^{2500}\sqrt{4r+1}-\sqrt{4r-3}$ $=\frac{1}{4}\left(\begin{array}{c}\sqrt{5}-\sqrt{1}\\+\sqrt{9}-\sqrt{5}\\+\sqrt{13}-\sqrt{9}\\\vdots\\+\sqrt{9997}-\sqrt{9993}\\\sqrt{10001}-\sqrt{9997}\end{array}\right)$ $=\frac{\sqrt{10001}-1}{4}$
(iii)	$\sum_{r=0}^{2500}\frac{1}{\sqrt{4r+1}+\sqrt{4r+5}}=\sum_{r=1}^{2501}\frac{1}{\sqrt{4r-3}+\sqrt{4r+1}}$ $=\sum_{r=1}^{2500}\frac{1}{\sqrt{4r-3}+\sqrt{4r+1}}+\frac{1}{\sqrt{10001}+\sqrt{10003}}$ $=\frac{\sqrt{10001}-1}{4}+\frac{1}{\sqrt{10001}+\sqrt{10003}}$ $>\frac{\sqrt{10000}-1}{4}+\frac{1}{\sqrt{10001}+\sqrt{10003}}$ $=24.75+\frac{1}{\sqrt{10001}+\sqrt{10003}}$ >24

11 **(i)**	$\overrightarrow{OM} = \frac{\mathbf{a}+\mathbf{c}}{2}$ **Length of Projection** $= \left\| \left(\frac{\mathbf{a}+\mathbf{c}}{2}\right)\cdot\left(\frac{\mathbf{c}}{\|\mathbf{c}\|}\right) \right\|$ $= \left\| \frac{\mathbf{a}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{c}}{2\|\mathbf{c}\|} \right\|$ $= \left\| \frac{\|\mathbf{a}\|\|\mathbf{c}\|\cos 60^\circ + \|\mathbf{c}\|^2}{2\|\mathbf{c}\|} \right\|$ $= \left\| \frac{0.5\|\mathbf{a}\|\|\mathbf{c}\| + \|\mathbf{c}\|^2}{2\|\mathbf{c}\|} \right\|$ **Note:** $\|\mathbf{a}\| = \|\mathbf{c}\|$ $= \left\| \frac{0.5\|\mathbf{c}\|\|\mathbf{c}\| + \|\mathbf{c}\|^2}{2\|\mathbf{c}\|} \right\|$ $= \frac{3}{4}\|\mathbf{c}\|$ (*Shown*)
11 **(ii)**	*Area of* $\Delta OMC = \left(\frac{1}{4}\|\mathbf{a}\times\mathbf{c}\|\right)$ $= \frac{1}{4}\|\mathbf{a}\|\|\mathbf{c}\|\sin 60^\circ$ $= \frac{\sqrt{3}}{8}\|\mathbf{c}\|\|\mathbf{c}\|$ **Note:** $\|\mathbf{a}\| = \|\mathbf{c}\|$ $= \frac{\sqrt{3}}{8}\|\mathbf{c}\|^2$ $\therefore k = \frac{\sqrt{3}}{8}$
11 **(iii)**	$\overrightarrow{OD} = \frac{5}{2}\mathbf{c}$ *Shortest* $\Delta OMC = \left\| \overrightarrow{OD} \times \frac{\mathbf{a}}{\|\mathbf{a}\|} \right\|$ $= \frac{5}{2}\left\| \frac{\mathbf{c}\times\mathbf{a}}{\|\mathbf{a}\|} \right\|$ $= \frac{5}{2}\left\| \frac{\|\mathbf{c}\|\|\mathbf{a}\|\sin 60^\circ}{\|\mathbf{a}\|} \right\|$ $= \frac{5\sqrt{3}}{4}\left\| \frac{\|\mathbf{c}\|\|\mathbf{a}\|}{\|\mathbf{a}\|} \right\|$ $= \frac{5\sqrt{3}}{4}\|\mathbf{c}\|$ $\therefore t = \frac{5\sqrt{3}}{4}$

12	$w=\dfrac{y}{t^2}$ diff. w.r.t. t $\dfrac{dw}{dt}=\dfrac{t^2\dfrac{dy}{dt}-2yt}{t^4}$ $t\dfrac{dw}{dt}=w^2t^3+2wt-2w$ $t\left(\dfrac{t^2\dfrac{dy}{dt}-2yt}{t^4}\right)=\dfrac{y^2}{t^4}t^3+\dfrac{2y}{t^2}t-\dfrac{2y}{t^2}$ $t^3\dfrac{dy}{dt}-2t^2y=t^3y^2+2yt^3-2yt^2$ $\dfrac{dy}{dt}=y^2+2y$ $\int\dfrac{1}{y^2+2y}dy=\int 1\,dt$ $\int\dfrac{1}{y(y+2)}dy=\int 1\,dt$ $\int\dfrac{1}{2y}-\dfrac{1}{2(y+2)}dy=\int 1\,dt$ $\dfrac{1}{2}\ln\lvert y\rvert-\dfrac{1}{2}\ln\lvert y+2\rvert=t+c$ $\dfrac{1}{2}\ln\left\lvert\dfrac{y}{y+2}\right\rvert=t+c$ $\ln\left\lvert\dfrac{y}{y+2}\right\rvert=2t+b$ $\dfrac{y}{y+2}=Ae^{2t}$ $y\left(1-Ae^{2t}\right)=2Ae^{2t}$ $y=\dfrac{2Ae^{2t}}{1-Ae^{2t}}$ $w=\dfrac{1}{t^2}\left(\dfrac{2Ae^{2t}}{1-Ae^{2t}}\right)$  For A=0 is the x-axis.

13	Common ratio $r = \tan\theta$. For S_∞ to exist, $\lvert r\rvert < 1$, i.e. $-1 < \tan\theta < 1$ $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ For θ in this range, $1+\tan\theta+\tan^2\theta+\tan^3\theta+\cdots = \frac{1}{1-\tan\theta}$ $\frac{1}{1-\tan\theta} < \frac{3+\sqrt{3}}{2}$ $1-\tan\theta > \frac{2}{3+\sqrt{3}}$ $\tan\theta < 1-\frac{2}{3+\sqrt{3}}$ $\tan\theta < \frac{1+\sqrt{3}}{3+\sqrt{3}}$ $\tan\theta < \frac{1+\sqrt{3}}{3+\sqrt{3}}\left(\frac{3-\sqrt{3}}{3-\sqrt{3}}\right)$ $\tan\theta < \frac{2\sqrt{3}}{6}$ $\tan\theta < \frac{1}{\sqrt{3}}$ $\theta < \frac{\pi}{6}$ Hence $-\frac{\pi}{4} < \theta < \frac{\pi}{6}$.
14(a)	$\lvert z+3\rvert = 2\,\mathrm{Re}(z)$ $\sqrt{(x+3)^2+y^2} = 2x$ $(x+3)^2+y^2 = 4x^2$ $3x^2-6x-9-y^2 = 0$ $x^2-2x-3-\frac{1}{3}y^2 = 0$ $(x-1)^2-4-\frac{1}{3}y^2 = 0$ $\frac{(x-1)^2}{4}-\frac{y^2}{12} = 1$  Axes intercepts: $(-1,0)$, $(3,0)$ Asymptotes: $y = \pm\sqrt{3}(x-1)$

14(b)	$w=-2+\left(2\sqrt{3}\right)i$ $\left\|w\right\|=4$ $\arg\left(w\right)=\dfrac{2\pi}{3}$ $w^{n}=4^{n}\left[\cos\left(\dfrac{2n\pi}{3}\right)+i\sin\left(\dfrac{2n\pi}{3}\right)\right]$ w^{n} is real $\Rightarrow \sin\left(\dfrac{2n\pi}{3}\right)=0$ $\therefore\ n=\dfrac{3}{2}m,\quad m$ even, $m\in\mathbb{Z}^{+}\quad$ or $\quad n=3k,\ k\in\mathbb{Z}^{+}$ $w^{50}-\left(w^{*}\right)^{50}$ $=4^{50}\left[\cos\left(\dfrac{100\pi}{3}\right)+i\sin\left(\dfrac{100\pi}{3}\right)\right]-4^{50}\left[\cos\left(\dfrac{100\pi}{3}\right)-i\sin\left(\dfrac{100\pi}{3}\right)\right]$ $=2^{100}\left[2i\sin\left(-\dfrac{2\pi}{3}\right)\right]$ $=2^{100}\left[2i\left(-\dfrac{\sqrt{3}}{2}\right)\right]$ $=2^{100}\left(-\sqrt{3}\right)i$ $\therefore\ k=-2^{100}\left(\sqrt{3}\right)$