2024 CJC JC2 H2 Maths Prelim Paper 1

1	Differentiate $\tan^{-1}\left(\dfrac{\mathrm{e}^x+2}{2\mathrm{e}^x-1}\right)$ with respect to *x*, leaving your answer in the simplest form $\dfrac{a\mathrm{e}^x}{b+c\mathrm{e}^{2x}}$ where a, b and c are integers to be determined.	[4]

2	**(a)**	Given that the real root $z_1 = -\alpha$, where $\alpha > 0$ satisfies the equation $$z^3 + \alpha z^2 + \alpha z + \alpha^2 = 0,$$ find the other roots, z_2 and z_3, in terms of α.	[4]
	(b)	Hence solve $\mathrm{i}w^3 + \alpha w^2 - \mathrm{i}\alpha w - \alpha^2 = 0$ in terms of α.	[2]

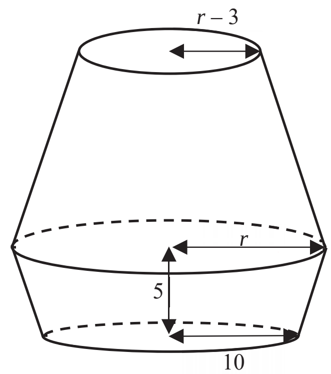
3	In an Argand diagram, the points A and B represent the complex numbers $6\mathrm{e}^{\mathrm{i}\alpha}$ and $6\mathrm{e}^{\mathrm{i}\beta}$ respectively, where $0 < \alpha < \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$.		
	(a)	Mark the points A and B on an Argand diagram. You are expected to clearly label the relevant moduli and arguments of the complex numbers represented by the points A and B, in terms of α and β.	[2]
	(b)	Show that $\left\|6\mathrm{e}^{\mathrm{i}\alpha} - 6\mathrm{e}^{\mathrm{i}\beta}\right\| = p\sin\left(\frac{\beta-\alpha}{q}\right)$, where the integers p and q are to be determined. Hence write down, in terms of α and β, the perimeter of the triangle OAB.	[4]

4	**(a)**	A sequence $u_1,\ u_2,\ u_3, \ldots$ is defined by $u_1 = -1$ and $u_r = u_{r-1} - 0.5^r + r^3$, where $r \geq 2$. By considering $\sum_{r=2}^{n}(u_r - u_{r-1})$, find an expression for u_n in terms of n. [It is given that $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.]	[5]
	The divergence test states that for a sequence $a_1,\ a_2,\ a_3,\ \ldots$, if $\lim_{n\to\infty} a_n = c$, where c is a non-zero constant or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{r=1}^{\infty} a_r$ diverges.		
	(b)	Determine whether $\sum_{n=1}^{\infty} u_n$ diverges, justifying your answer.	[2]

5 [It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]

A frustum of a cone is the portion of the cone which remains after its upper part has been cut off by a plane parallel to its base.

The diagram below shows a vessel of lava lamp. It can be modelled by two open hollow frustums with a common base radius r cm. To form this vessel, two congruent open cones with base radius r cm are cut to create two frustums which are then joined along their common base. The bottom frustum has a height of 5 cm. The radii of top and bottom surfaces of the vessel are $(r-3)$ cm and 10 cm respectively.



(a) Show that the height, in cm, of each of the original cones is $\dfrac{5r}{r-10}$. [1]

It is given that $r = 12$.

(b) Find the volume of the bottom frustum. [2]

The vessel is now mounted on a flat base and mineral oil is poured into the vessel at a rate of $10\ \text{cm}^3$ per second.

(c) **(i)** Find the depth of the mineral oil in the top frustrum when the volume of the mineral oil in the vessel is $2000\ \text{cm}^3$. [3]

(ii) Hence, using differentiation, find the rate of increase of the depth of the mineral oil at the instant when the volume of the mineral oil in the vessel is $2000\ \text{cm}^3$. [2]

6	It is given that $\ln(1+y)=\tan x$.		
	(a)	Show that $\dfrac{d^2y}{dx^2}=\dfrac{dy}{dx}\left(2\tan x+\sec^2 x\right)$.	[3]
	(b)	By further differentiation of the result in part **(a)**, find the Maclaurin series for y, up to and including the term in x^3.	[3]
	(c)	Hence find an estimate for the value of e.	[2]

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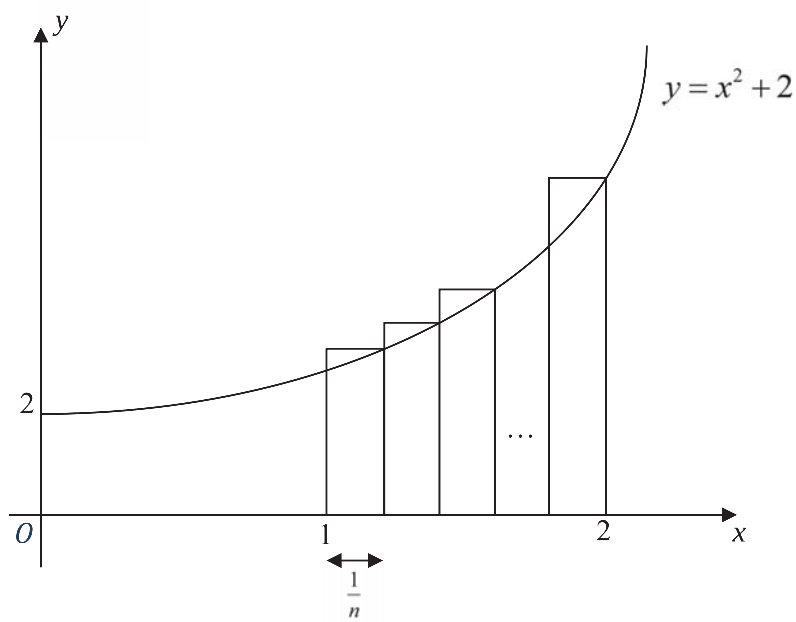
7	**(a)**	**(i)**	Find $\int \frac{x^2}{2\sqrt{1-x^2}}\,\mathrm{d}x$, using the substitution $x = \cos\theta$ where $0 < \theta < \frac{\pi}{2}$.	[5]
		(ii)	Find $\int x\sin^{-1}x\,\mathrm{d}x$.	[2]
	(b)	Find $\int \cos ax \sin bx\,\mathrm{d}x$, where a and b are real numbers and $a \neq b$.		[3]

8	A curve C has parametric equations $$x = 3 + 2\sec\theta,\ \ y = 3\tan\theta - 1,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$			
	(a)	Find the range of values of x.		[2]
	(b)	Find a cartesian equation of C.		[2]
	(c)	Hence or otherwise, sketch C, stating the coordinates of any vertices and equations of any asymptotes.		[3]
	(d)	**(i)**	Find the equation of the tangent to C at the point P where $\theta = p$, $-\frac{\pi}{2} < p < \frac{\pi}{2}$. [You do not need to simplify your answer.].	[3]
		(ii)	Show that the gradient of the tangent at P cannot lie in the interval $\left[-\frac{3}{2}, \frac{3}{2}\right]$ as p varies.	[2]

<table>
<tr><td>9</td><td>(a)</td><td>

The diagram shows part of the graph of $y = x^2 + 2$ from $x = 1$ to $x = 2$. The area under the curve in this interval may be approximated by the total area of n rectangles, A, as shown. The width of each rectangle is $\frac{1}{n}$.

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, show that $A = \frac{13}{3} + \frac{3}{2n} + \frac{1}{6n^2}$.

Deduce the exact value of $\int_1^2 \left(2 + x^2\right) \mathrm{d}x$, justifying your answer.</td><td>[6]</td></tr>
<tr><td></td><td>(b)</td><td>Region R is bounded by the curve $y = \ln x$, $x = 1$ and $y = \ln 5$.</td><td></td></tr>
<tr><td></td><td></td><td>(i) Find the area of R.</td><td>[2]</td></tr>
<tr><td></td><td></td><td>(ii) Find the exact volume of the solid generated when R is rotated through 4 right angles about the y-axis.</td><td>[4]</td></tr>
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10	Glucose is a simple carbohydrate that can be easily absorbed by the body and provides instant energy. When a patient is dehydrated or unable to take food orally, glucose is given intravenously to the patient via a glucose drip. Glucose given intravenously enters the bloodstream at a constant rate of p units per hour. It is absorbed by the body, leaving the bloodstream, at a rate proportional to the amount of glucose present in the bloodstream. G denotes the number of units of glucose in the bloodstream at time t hours after the glucose drip is administered.		
	(a)	Write down a differential equation relating G and t.	[1]
	(b)	Suppose there are 3 units of glucose in the bloodstream when $t = 0$ and that the amount of glucose in the bloodstream remains constant when $G = 8$. Show that the particular solution of the differential equation in part **(a)** is $G = 8 - 5\mathrm{e}^{-\frac{p}{8}t}$.	[6]
	(c)	Sketch the graph of G against t.	[2]
	Instead of the glucose drip, a patient who is well enough will get his glucose from the food he consumes. At the end of each meal, the rate of change in glucose in the bloodstream can be modelled as $\left(\frac{1}{2} + \sin t\right)$ units per hour where $0 \le t \le \frac{3}{2}\pi$.		
	(d)	Find the time at which the amount of glucose in the bloodstream	
		(i) increases most rapidly after the meal,	[2]
		(ii) starts to decrease.	[2]

11	Taylor is training for a triathlon, which involves swimming, running and cycling. • On Day 1, he swims 1.5 km and then swims the same distance on each subsequent day. • On Day 1, he runs 2 km and, on each subsequent day, he runs 0.5 km further than on the previous day. • On Day 1, he cycles 2.5 km and, on each subsequent day, he cycles a distance 10% further than on the previous day.		
	(a)	Find the distance that Taylor runs on Day 15.	[2]
	(b)	Find the day on which the distance that Taylor cycles first exceeds 12 km.	[3]
	(c)	On which days will the distance run exceed the distance cycled by Taylor?	[3]
	His friend, Swift, is also training for the same triathlon. Swift's training programme is similar to Taylor's except that the distance for swimming, running and cycling on Day 1 is x km.		
	(d)	Find the minimum value of x if Swift intends to cover a longer total distance in the first 30 days than the total distance covered by Taylor. Give your answer correct to 1 decimal place.	[6]

2024 CJC JC2 H2 Maths Prelim Paper 2

1	**(a)**	The points P, Q and R have position vectors $\mathbf{j}+\mathbf{k}$, $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ and $-2\mathbf{i}+\mathbf{j}+4\mathbf{k}$ respectively. Find the exact length of projection of $\overrightarrow{QR}$ on $\overrightarrow{PQ}$.	[3]
	(b)	Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\lvert\mathbf{a}\rvert=\lambda\lvert\mathbf{b}\rvert$ and $\lvert\mathbf{a}+\lambda\mathbf{b}\rvert=2\lvert\mathbf{a}-\lambda\mathbf{b}\rvert$, where λ is a positive constant. Using a suitable scalar product, show that $\cos\theta=\frac{3}{5}$, where θ is the acute angle between $\mathbf{a}$ and $\mathbf{b}$.	[3]

2	The functions f and g are defined by $$\mathrm{f} : x \mapsto \frac{3-2x}{x+2}, \text{ for } x \in \mathbb{R},\ x > -2,$$ $$\mathrm{g} : x \mapsto \lvert x \rvert - 1, \text{ for } x \in \mathbb{R}.$$		
	(a)	Find $\mathrm{f}^{-1}(x)$ and state its domain.	[3]
	(b)	Hence find $\mathrm{f}^{2024}(1)$.	[1]
	(c)	Show that fg exists and find its range.	[3]
	(d)	Solve $\mathrm{f}(x) > \mathrm{g}(x)$, giving your answer in exact form.	[4]

3	It is given that $f(x) = 6kx + x^2$, where $k > 0$.		
	(a)	Sketch the graph of $y = f(x)$, indicating clearly the coordinates of the points where the curve crosses the axes and of the turning points, if any.	[2]
	(b)	Describe the transformation that maps the graph of $y = f(x)$ onto the graph of $y = x^2 - 9k^2$.	[2]
	It is given that $$g(x) = \begin{cases} 6kx + x^2 & \text{for } -3k \le x < k \\ 7k^2 & \text{for } k \le x < 2k \end{cases} \quad \text{where } k > 0$$ and that $g(x) = g(x + 5k)$ for all real values of x.		
	(c)	Sketch the graph of $y = g(x)$ for $-3k \le x < 5k$.	[4]
	(d)	Sketch the graph of $y = \dfrac{1}{g(x)}$ for $-3k \le x < 2k$.	[3]

4	The plane p has equation $\mathbf{r}=\begin{pmatrix}1\\3\\4\end{pmatrix}+s\begin{pmatrix}2\\0\\1\end{pmatrix}+t\begin{pmatrix}0\\1\\-1\end{pmatrix}$, and the line l has the equation $\mathbf{r}=\begin{pmatrix}1\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}$, where λ, s and t are parameters.		
	(a)	Show that l is perpendicular to p and find the coordinates of the point where l and p intersect.	[5]
	The line m meets l at the point $A(1,1,3)$ and it meets p at the point $B(1,3,4)$.		
	(b)	The line n is the reflection of m in p. Show that a vector equation of n is $\mathbf{r}=\begin{pmatrix}1\\3\\4\end{pmatrix}+\mu\begin{pmatrix}-4\\2\\5\end{pmatrix}$, $\mu\in\mathbb{R}$.	[2]
	(c)	Find the acute angle between n and p.	[2]
	(d)	Hence or otherwise, find the area bounded by the lines l, m and n.	[3]

5	Andrea participates in a game show. In each round of the game, Andrea responds to a series of questions until she answers a question wrongly or she has answered five questions correctly. She is awarded one point for every correct answer and is awarded an additional 3 bonus points if she answers all five questions correctly. It is given that she answers each question independently and the probability she answers each question correctly is $\frac{4}{5}$.		
	(a)	Show that the probability that Andrea obtains 2 points in one round of the game is $\frac{16}{125}$.	[1]
	(b)	Find the probability distribution of the points that Andrea obtains in one round of the game.	[2]
	(c)	Find the expectation and variance of the points that Andrea obtains in one round of the game.	[2]
	(d)	Find the probability that Andrea obtains less than 3 points after two rounds of the game.	[3]

6 An investigation on the number of hours, t, spent on mobile games per day, of 5 students and the marks, m, they obtained in an examination out of 100 marks was carried out.

t	1.1	2.2	3.4	3.8	4.2
m	72	62	54	48	42

(a) Sketch a scatter diagram of the data. [1]

(b) Use your calculator to find the least squares regression line of t on m, and state the product moment correlation coefficient between t and m. [3]

(c) Use the equation of the regression line found in part **(b)** to estimate the number of hours spent on playing mobile games per day if a student obtains 30 marks. Explain whether you would expect this value to be reliable. [2]

(d) Without any further calculations, explain if the product moment correlation between t and $\frac{m}{100}$ would be different from the value obtained in part **(b)**. [1]

7	Aran is a telemarketer working in a large city. Each day, he speaks to 50 potential customers and on average, he successfully makes a sale 6% of the time. The number of successful sales each day is denoted by the random variable T.		
	(a)	State, in context, two assumptions needed for T to be well modelled by a binomial distribution.	[2]
	Assume now that T has a binomial distribution.		
	(b)	Find the probability that Aran makes at least 5 successful sales on a randomly chosen day.	[2]
	(c)	Aran receives a bonus if he makes at least 5 successful sales a day. Find the probability that he receives a bonus on 2 days out of a randomly chosen 5-day work week.	[2]
	Aran is part of a team of 100 similar telemarketers.		
	(d)	Estimate the probability that on a randomly chosen day, the team makes more than 3.5 successful sales on average.	[3]

8	A giant panda's daily diet consists almost entirely of leaves, stems and shoots of various bamboo species. A panda foundation claims that giant pandas consume on average at most 25 kg of bamboo every day to meet their energy needs. A random sample of 40 pandas is taken and the mass, x kilograms, of bamboo consumed per day by this group of pandas is summarised below. $$\sum(x-25)=80.4,\ \sum(x-25)^2=1893.5$$		
	(a)	Calculate the unbiased estimates of the population mean and variance for the mass of bamboo consumed per day by pandas.	[2]
	(b)	State the hypotheses that can be used to test whether the panda foundation's claim is valid. By finding the p-value, carry out the test at 2% level of significance, giving your conclusion in the context of the question.	[4]
	(c)	State, in context, the meaning of the p-value found in part **(b).**	[1]
	Another random sample of n pandas is selected by zookeepers in a certain country and the sample mean was found to be 26.9 kg per day. A test was carried out at 2% level of significance.		
	(d)	By using the unbiased estimate of the population variance found in **(a)**, find the least possible value of n such that this new sample achieves a different conclusion from that in **(b)**.	[4]

9	A student takes the bus to school every morning. During peak hours, the traffic is either moderate or heavy. The time taken, in minutes, for a randomly chosen bus journey through moderate or heavy traffic follow independent normal distributions with means and standard deviations as shown in the table.		

	Mean	Standard deviation
Moderate Traffic	14	σ
Heavy Traffic	27	2.5

	(a)	Given that 20% of the student's bus journeys through moderate traffic exceed 15 minutes, show that $\sigma = 1.19$ correct to 3 significant figures.	[2]
	(b)	The probability of a randomly chosen bus journey through heavy traffic taking more than a minutes is larger than 0.9. Find the range of values of a.	[2]
	(c)	The student is late for school if his bus journey takes more than 32 minutes, assuming he leaves home at the same time as usual. He gets a demerit point if he is late on any two consecutive mornings. For three consecutive mornings, the student's bus journeys encountered heavy traffic. Find the probability that he gets a demerit point.	[3]
	(d)	Find the probability that the time taken for two randomly chosen bus journeys through moderate traffic differs by less than 0.5 minutes.	[2]
	(e)	Find the probability that the total time taken for two randomly chosen bus journeys through heavy traffic is greater than four times the time taken for a randomly chosen bus journey through moderate traffic.	[3]

10	**(a)**	In a certain university, there are 110 Co-Curricular Activities (CCAs) clustered into 4 categories. There are 22 Arts and Culture CCAs, 16 Community Service CCAs, 38 Physical Sports CCAs and 34 Special Interest CCAs.		
		(i)	Albert wishes to find out about approaches to training of the Physical Sports CCAs, so he sends a questionnaire to 22 Physical Sports CCAs. Explain whether these 22 Physical Sports CCAs form a sample or a population.	[1]
		(ii)	Benedict wishes to investigate the level of student engagement in CCAs, but does not want to obtain the detailed information necessary from all 110 CCAs. Explain how he should carry out his investigation, and why he should do the investigation in this way.	[2]
		(iii)	Find the number of different possible samples of 16 CCAs, with 4 CCAs chosen from each category.	[2]
	(b)	The probability of events A, B and C occurring are given by $\mathrm{P}(A)=\frac{8}{25}$, $\mathrm{P}(B)=\frac{61}{100}$ and $\mathrm{P}(C)=\frac{7}{20}$ respectively. It is also known that $\mathrm{P}(A\cap C)=\frac{11}{100}$, $\mathrm{P}(B\mid C)=\frac{13}{35}$ and $\mathrm{P}(A\cap B\cap C)=\frac{7}{100}$.		
		(i)	Find $\mathrm{P}(B\cap C)$.	[2]
		(ii)	If events A and B are independent, find $\mathrm{P}(A\cap B\cap C')$.	[2]
		(iii)	If events A and B are **not** known to be independent, find the greatest and least possible values of $\mathrm{P}(A\cap B\cap C')$.	[4]

2024 CJC JC2 H2 Maths Prelim Paper 1

1	Differentiate $\tan^{-1}\left(\dfrac{e^x+2}{2e^x-1}\right)$ with respect to x, leaving your answer in the simplest form $\dfrac{ae^x}{b+ce^{2x}}$ where a, b and c are integers to be determined.	[4]

Solution:

$$\frac{d}{dx}\tan^{-1}\left(\frac{e^x+2}{2e^x-1}\right)$$

$$=\frac{1}{1+\left(\dfrac{e^x+2}{2e^x-1}\right)^2}\times\frac{\left(2e^x-1\right)(e^x)-\left(e^x+2\right)(2e^x)}{\left(2e^x-1\right)^2}$$

$$=\frac{\left(2e^x-1\right)^2}{\left(2e^x-1\right)^2+\left(e^x+2\right)^2}\times\frac{2e^{2x}-e^x-2e^{2x}-4e^x}{\left(2e^x-1\right)^2}$$

$$=\frac{-5e^x}{4e^{2x}-4e^x+1+e^{2x}+4e^x+4}$$

$$=\frac{-5e^x}{5+5e^{2x}}$$

$$=\frac{-e^x}{1+e^{2x}}$$

Alternative Method

Let $y=\tan^{-1}\left(\dfrac{e^x+2}{2e^x-1}\right)$

$$\tan y=\left(\frac{e^x+2}{2e^x-1}\right)$$

Differentiating with respect to x:

$$\sec^2 y\frac{dy}{dx}=\frac{\left(2e^x-1\right)(e^x)-\left(e^x+2\right)(2e^x)}{\left(2e^x-1\right)^2}$$

$$\left(1+\tan^2 y\right)\frac{dy}{dx}=\frac{\left(2e^x-1\right)(e^x)-\left(e^x+2\right)(2e^x)}{\left(2e^x-1\right)^2}$$

$$\frac{dy}{dx}=\frac{1}{1+\tan^2 y}\times\frac{\left(2e^x-1\right)(e^x)-\left(e^x+2\right)(2e^x)}{\left(2e^x-1\right)^2}$$

$$\frac{dy}{dx}=\frac{1}{1+\left(\dfrac{e^x+2}{2e^x-1}\right)^2}\times\frac{\left(2e^x-1\right)(e^x)-\left(e^x+2\right)(2e^x)}{\left(2e^x-1\right)^2}$$

$$= \frac{\left(2e^x - 1\right)^2}{\left(2e^x - 1\right)^2 + \left(e^x + 2\right)^2} \times \frac{2e^{2x} - e^x - 2e^{2x} - 4e^x}{\left(2e^x - 1\right)^2}$$

$$= \frac{-5e^x}{4e^{2x} - 4e^x + 1 + e^{2x} + 4e^x + 4}$$

$$= \frac{-5e^x}{5 + 5e^{2x}}$$

$$= \frac{-e^x}{1 + e^{2x}}$$

Examiner's Report:

This question was successful for 70% of cohort.

Main problems were:

- forgetting to use chain rule on the expression
- made mistakes in apply quotient rule
- made algebraic mistakes when simplifying the differentiated expression

Almost none used the Alternative Method.

2	**(a)**	Given that the real root $z_1 = -\alpha$, where $\alpha > 0$ satisfies the equation $$z^3 + \alpha z^2 + \alpha z + \alpha^2 = 0,$$ find the other roots, z_2 and z_3, in terms of α.	[4]
	(b)	Hence solve $\mathrm{i}w^3 + \alpha w^2 - \mathrm{i}\alpha w - \alpha^2 = 0$ in terms of α.	[2]

Solution:

(a)

Since $z_1 = -\alpha$ is a real root, so $(z_1 + \alpha)$ is a factor.

Let $z^3 + \alpha z^2 + \alpha z + \alpha^2 = (z + \alpha)(z^2 + Az + B)$ where A, B are real constants.

Compare coefficient of z^2: $\alpha = A + \alpha \Rightarrow A = 0$

Compare coefficient of constant: $\alpha^2 = B\alpha \Rightarrow B = \alpha$

Hence, $z^3 + \alpha z^2 + \alpha z + \alpha^2 = (z + \alpha)(z^2 + \alpha)$

Thus $(z^2 + \alpha) = 0$

$$z = \pm\sqrt{-\alpha} = \pm\sqrt{\alpha}\,\mathrm{i}$$

Thus $z_2 = \sqrt{\alpha}\,\mathrm{i}$, $z_3 = -\sqrt{\alpha}\,\mathrm{i}$

Method 2

Do long-division on $z^3 + \alpha z^2 + \alpha z + \alpha^2$ using $(z + \alpha)$

$$\begin{array}{rl}
 & z^2 + \alpha \\
z + \alpha \,\big) & z^3 + \alpha z^2 + \alpha z + \alpha^2 \\
(-) & z^3 + \alpha z^2 \\
 & \qquad\qquad \alpha z + \alpha^2 \\
(-) & \qquad\qquad \alpha z + \alpha^2 \\
 & \qquad\qquad\quad 0
\end{array}$$

Thus $(z^2 + \alpha) = 0 \Rightarrow z = \pm\sqrt{-\alpha} = \pm\sqrt{\alpha}\,\mathrm{i}$

Thus $z_2 = \sqrt{\alpha}\,\mathrm{i}$, $z_3 = -\sqrt{\alpha}\,\mathrm{i}$

Method 3

Since all coefficients are real and $z_1 = -\alpha$ is a real root,

the other two roots are complex conjugates.

Let the complex roots be $z = x + y\mathrm{i}$ and $z = x - y\mathrm{i}$

$$z^3 + \alpha z^2 + \alpha z + \alpha^2 = (z+\alpha)(z - x + y\mathrm{i})(z - x - y\mathrm{i})$$
$$= (z+\alpha)\left(z^2 - 2xz + x^2 + y^2\right)$$
$$= z^3 + (\alpha - 2x)z^2 + \left(x^2 + y^2 - 2\alpha x\right)z + \alpha\left(x^2+y^2\right)s$$

Equating coeff of z^2: $\alpha - 2x = \alpha \Rightarrow x = 0$

Equating constants: $\alpha\left(x^2 + y^2\right) = \alpha^2 \Rightarrow x^2 + y^2 = \alpha$ (since $\alpha > 0$)

$\Rightarrow y^2 = \alpha$ (since $x = 0$)

$\Rightarrow y = \pm\sqrt{\alpha}$

Thus $z_2 = \sqrt{\alpha}\ \mathrm{i}$, $z_3 = -\sqrt{\alpha}\ \mathrm{i}$

Method 4

Let the other roots be $z = A$ and $z = B$

$$z^3 + \alpha z^2 + \alpha z + \alpha^2 = (z+\alpha)(z - A)(z - B)$$
$$= (z+\alpha)\left(z^2 - (A+B)z + AB\right)$$
$$= z^3 + \left(\alpha - (A+B)\right)z^2 + \left(AB - \alpha(A+B)\right)z + \alpha AB s$$

Equating coeff of z^2: $\alpha - (A+B) = \alpha \Rightarrow A + B = 0 \Rightarrow A = -B$

Equating constants: $\alpha AB = \alpha^2 \Rightarrow AB = \alpha$ (since $\alpha > 0$)

$\Rightarrow -B^2 = \alpha$ (since $A = -B$)

$\Rightarrow B = \pm\sqrt{-\alpha} = \pm\sqrt{\alpha}\ \mathrm{i}$

Thus, the other roots are $\sqrt{\alpha}\ \mathrm{i}$ and $-\sqrt{\alpha}\ \mathrm{i}$

(b)

Let $z = \mathrm{i}w$

$$(\mathrm{i}w)^3 + \alpha(\mathrm{i}w)^2 + \alpha(\mathrm{i}w) + \alpha^2 = 0$$

$$-\mathrm{i}w^3 - \alpha w^2 + \alpha\mathrm{i}w + \alpha^2 = 0$$

$$\mathrm{i}w^3 + \alpha w^2 - \alpha\mathrm{i}w - \alpha^2 = 0$$

Thus $\mathrm{i}w = -\alpha \Rightarrow w = \mathrm{i}\alpha$

$\mathrm{i}w = \sqrt{\alpha}\mathrm{i} \Rightarrow w = \sqrt{\alpha}$,

$\mathrm{i}w = -\sqrt{\alpha}\mathrm{i} \Rightarrow w = -\sqrt{\alpha}$

Examiner's Report:

(a)

This part was successful for 50% of the cohort.

Main problems were:

- stated that since all coefficients are real, so the 'conjugate' $z = \alpha$ is also a root, without realising that α is real and so does not have a complex conjugate.
- those who stated that $z = \alpha$ is also a root did not subst into the equation to check and see that it is not.
- some students realised that the other roots are complex conjugates but were unable to use comparing coefficients to find the roots.
- some students were confused about what the question wants and tried to solve for the value of α
- many students tried Methods 3 and 4 but messed up during the expansion or did not know what to do with the equations.

(b)

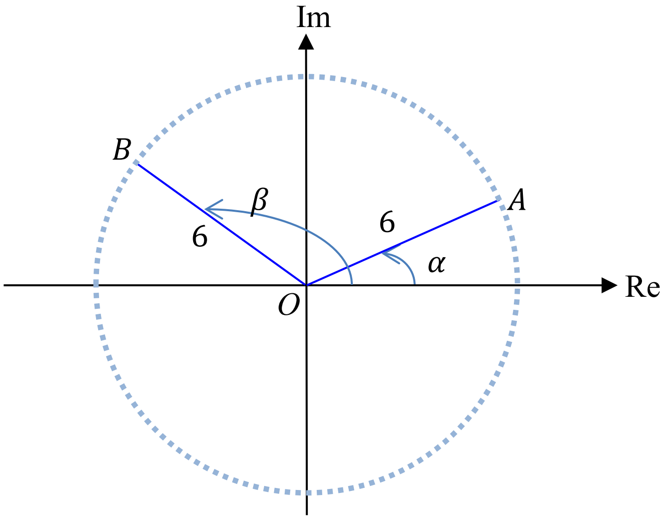
Majority of students did not manage to find the correct substitution.

For those who did, some did not manage to simplify the roots for *w*.

3	In an Argand diagram, the points A and B represent the complex numbers $6e^{i\alpha}$ and $6e^{i\beta}$ respectively, where $0<\alpha<\frac{\pi}{2}$ and $\frac{\pi}{2}<\beta<\pi$.		
	(a)	Mark the points A and B on an Argand diagram. You are expected to clearly label the relevant moduli and arguments of the complex numbers represented by the points A and B, in terms of α and β.	[2]
	(b)	Show that $\left\|6e^{i\alpha}-6e^{i\beta}\right\| = p\sin\left(\frac{\beta-\alpha}{q}\right)$, where the integers p and q are to be determined. Hence write down, in terms of α and β, the perimeter of the triangle OAB.	[4]

Solution:

(a)



(b) **Method 1**

$$\left|6e^{i\alpha}-6e^{i\beta}\right|$$

$$=6\left|e^{i\alpha}-e^{i\beta}\right|$$

$$=6\left|e^{i\left(\frac{\beta+\alpha}{2}\right)}\left(e^{-i\left(\frac{\beta-\alpha}{2}\right)}-e^{i\left(\frac{\beta-\alpha}{2}\right)}\right)\right|$$

$$=6\left|e^{i\left(\frac{\beta+\alpha}{2}\right)}\right|\left|-\left[e^{i\left(\frac{\beta-\alpha}{2}\right)}-e^{-i\left(\frac{\beta-\alpha}{2}\right)}\right]\right|$$

$$=6(1)\left|e^{i\left(\frac{\beta-\alpha}{2}\right)}-e^{-i\left(\frac{\beta-\alpha}{2}\right)}\right|$$

$$=6(1)\left|2i\sin\left(\frac{\beta-\alpha}{2}\right)\right|$$

$$=12\sin\left(\frac{\beta-\alpha}{2}\right) \quad \text{where } p=12, q=2$$

Method 2

$$\left|6e^{i\alpha}-6e^{i\beta}\right|$$
$$=AB$$
$$=2AC \text{ where } C \text{ is the midpoint of } AB$$
$$=2\left(6\sin\angle AOC\right)$$
$$=12\sin\left(\frac{\beta-\alpha}{2}\right) \quad \text{where } p=12, q=2$$

Method 3

$$\left|6e^{i\alpha}-6e^{i\beta}\right|$$
$$=AB$$
$$=\sqrt{6^2+6^2-2(6)(6)\cos(\beta-\alpha)}$$
$$=\sqrt{72-72\cos(\beta-\alpha)}$$
$$=\sqrt{72\left[1-\cos(\beta-\alpha)\right]}$$
$$=\sqrt{72\left[2\sin^2\left(\frac{\beta-\alpha}{2}\right)\right]}$$
$$=12\sin\left(\frac{\beta-\alpha}{2}\right) \quad \text{where } p=12, q=2$$

Method 4

$$\frac{\left|6e^{i\alpha}-6e^{i\beta}\right|}{\sin(\beta-\alpha)}=\frac{6}{\sin\left(\frac{\pi-(\beta-\alpha)}{2}\right)}$$
$$\left|6e^{i\alpha}-6e^{i\beta}\right|=\frac{6\sin(\beta-\alpha)}{\sin\left(\frac{\pi}{2}-\frac{\beta-\alpha}{2}\right)}$$
$$=\frac{6\left[2\sin\left(\frac{\beta-\alpha}{2}\right)\cos\left(\frac{\beta-\alpha}{2}\right)\right]}{\cos\left(\frac{\beta-\alpha}{2}\right)}$$
$$=12\sin\left(\frac{\beta-\alpha}{2}\right) \quad \text{where } p=12, q=2$$

Perimeter of the triangle OAB

$$=6+6+12\sin\left(\frac{\beta-\alpha}{2}\right)$$
$$=12\left[1+\sin\left(\frac{\beta-\alpha}{2}\right)\right] \text{ units}$$

Examiner's Report:

3 **(a)** This part was generally well done.
Students were able to label the points A and B in the correct quadrants.
However, some students labelled the $\arg\left(6\mathrm{e}^{\mathrm{i}\beta}\right)=\beta$ wrongly. The argument should be measured from <u>the positive Re axis</u> to the vector OB.
Some students also failed to mark the moduli of both complex numbers.

Students should take note that $\mathrm{Im}\left(6\mathrm{e}^{\mathrm{i}\alpha}\right)\neq\mathrm{Im}\left(6\mathrm{e}^{\mathrm{i}\beta}\right)$ (i.e. the points A and B are not symmetrical about Im axis). The question did not give a specific relationship between α and β.

(b) This part was not well attempted.

- Only a small percentage of students attempted factorization in the exponential form. However, most of them were not able to arrive at the final answer.
- Most students converted the exponential form to the trigonometric form and applied Factor Formulae.

In both methods, a significant percentage of students were unable to proceed after the first step. Students did not understand the significance of the modulus sign (which is the magnitude of a complex number) and often removed the modulus sign without consequence.

There was a small percentage of students who recognized the geometrical meaning of $\left|6\mathrm{e}^{\mathrm{i}\alpha}-6\mathrm{e}^{\mathrm{i}\beta}\right|$ as the length AB in the triangle OAB. They were able to proceed by applying the Cosine/Sine Rule but they did not know how to simplify to the required form.

4	**(a)**	A sequence u_1, u_2, u_3,... is defined by $u_1 = -1$ and $u_r = u_{r-1} - 0.5^r + r^3$, where $r \geq 2$. By considering $\sum_{r=2}^{n}(u_r - u_{r-1})$, find an expression for u_n in terms of n. [It is given that $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.]	[5]
	The divergence test states that for a sequence a_1, a_2, a_3, ... , if $\lim_{n\to\infty} a_n = c$, where c is a non-zero constant or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{r=1}^{\infty} a_r$ diverges.		
	(b)	Determine whether $\sum_{n=1}^{\infty} u_n$ diverges, justifying your answer.	[2]

Solution:

(a)

$u_r = u_{r-1} - 0.5^r + r^3$

$u_r - u_{r-1} = r^3 - 0.5^r$

$$\sum_{r=2}^{n}(u_r - u_{r-1}) = \sum_{r=2}^{n}(r^3 - 0.5^r)$$

$$\sum_{r=2}^{n}(r^3 - 0.5^r) = \frac{n^2(n+1)^2}{4} - 1 - \left(\frac{0.5^2(1-0.5^{n-1})}{1-0.5}\right)$$

$$= \frac{n^2(n+1)^2}{4} - 1 - 0.5(1-0.5^{n-1})$$

$$= \frac{n^2(n+1)^2}{4} - 1.5 + 0.5^n \text{ for } n \geq 2$$

$$\sum_{r=2}^{n}(u_r - u_{r-1}) = u_n - u_1 = u_n + 1$$

So $u_n = \frac{n^2(n+1)^2}{4} - 1.5 + 0.5^n - 1 = \frac{n^2(n+1)^2}{4} - 2.5 + 0.5^n$ for $n \geq 2$

When $n = 1$, $u_1 = \frac{n^2(n+1)^2}{4} - 2.5 + 0.5^n = \frac{1^2(1+1)^2}{4} - 2.5 + 0.5 = -1$

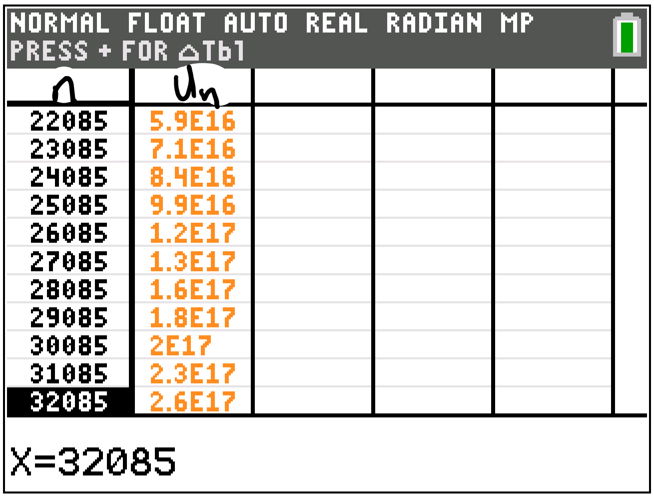
Thus, $u_n = \frac{n^2(n+1)^2}{4} - 2.5 + 0.5^n$ for $n \geq 1$

(b) $u_n = \dfrac{n^2(n+1)^2}{4} - 2.5 + 0.5^n$ for $n \geq 1$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\dfrac{n^2(n+1)^2}{4} - 2.5 + 0.5^n\right)$ does not exist. Since as $n \to \infty$, $\dfrac{n^2(n+1)^2}{4} - 2.5 \to \infty$, $0.5^n \to 0$.

Thus $\sum_{n=1}^{\infty} u_n$ diverges.

Or graph

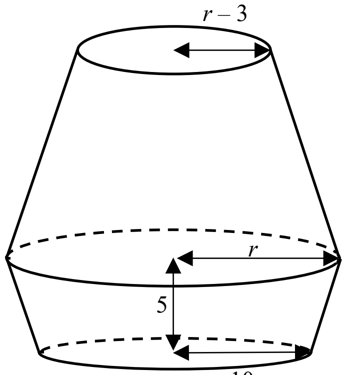


As $n \to \infty$, $\lim_{n\to\infty} u_n \to \infty$, thus $\sum_{n=1}^{\infty} u_n$ diverges.

<u>Examiner's Report:</u>

(a) This part is poorly done generally. Many students did not realise that they need to apply method of difference to sum $\sum_{r=2}^{n}(u_r - u_{r-1})$ and mistakenly linked it to $S_n - S_{n-1}$. Secondly, while summing $\sum_{r=2}^{n}(-0.5^r + r^3)$, students mistakenly wrote -0.5^ras $(-0,5)^r$, but they are not the same. Those who rewrite into $\sum_{r=2}^{n}(r^3 - 0.5^r)$ are more successful in proceeding further correctly. Thirdly, many students split the sum wrongly as $\sum_{r=2}^{n}(r^3) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{2} r^3$, showing that they are still not familiar with the rule for splitting partial sum. Next, a sizeable number of students could not sum the geometric series $\sum_{r=2}^{n}(0.5^r)$successfully, often identifying the wrong first term and the number of terms in the series, showing clearly that they have not revised or understood the concepts well. Among the handful who successfully find u_n, they forgot to check that u_n is indeed -1.

(b) This part has a mixed response depending on the expression the students obtained in part (a). Some students clearly misread "diverges" as "converges". Students who merely stated that the infinite series diverges without clear reference to the expression they obtained from part (a) will receive no credit. The key reasons must refer to the behaviour of $\dfrac{n^2(n+1)^2}{4}$ and 0.5^n for large values of n.

5	[It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.] A frustum of a cone is the portion of the cone which remains after its upper part has been cut off by a plane parallel to its base. The diagram below shows a vessel of lava lamp. It can be modelled by two open hollow frustums with a common base radius r cm. To form this vessel, two congruent open cones with base radius r cm are cut to create two frustums which are then joined along their common base. The bottom frustum has a height of 5 cm. The radii of top and bottom surfaces of the vessel are $(r-3)$ cm and 10 cm respectively.  		
	(a) Show that the height, in cm, of each of the original cones is $\frac{5r}{r-10}$.		[1]
	It is given that $r=12$.		
	(b) Find the volume of the bottom frustum.		[2]
	The vessel is now mounted on a flat base and mineral oil is poured into the vessel at a rate of 10 cm^3 per second.		
	(c) **(i)** Find the depth of the mineral oil in the top frustrum when the volume of the mineral oil in the vessel is 2000 cm^3.		[3]
	(ii) Hence, using differentiation, find the rate of increase of the depth of the mineral oil at the instant when the volume of the mineral oil in the vessel is 2000 cm^3.		[2]

<u>Solution:</u>

5 **(a)** Let the height of each original cone be h cm

Using similar cones (original cone and bottom removed cone),

$$\frac{h}{h-5}=\frac{r}{10}$$

$$10h=rh-5r$$

$$h(r-10)=5r$$

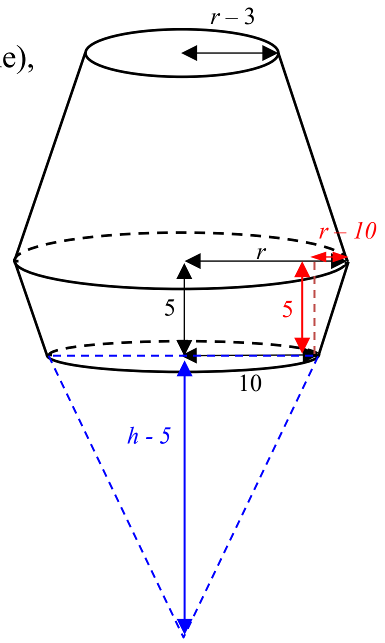
$$h=\frac{5r}{r-10}$$

OR

Using similar cones,

$$\frac{h}{r}=\frac{5}{r-10}$$

$$h=\frac{5r}{r-10}$$



(b) $r=12\Rightarrow h=30$

Volume of bottom frustum

$$=\frac{1}{3}\pi(12)^2(30)-\frac{1}{3}\pi(10)^2(25)$$

$$=\frac{1820}{3}\pi$$

$$=1905.899543$$

$$=1910\text{ cm}^3\text{ (3 sf)}$$

(c)(i) Since volume of oil = 2000 cm^3 > 1910 cm^3, the bottom frustum is completely filled.

Let top radius, depth and volume of mineral oil in the top frustum at time t be x cm, y cm and V cm^3 respectively

Using similar triangles (original cone and top removed cone),

$$\frac{30-y}{30}=\frac{x}{12}$$

$$x=\frac{2}{5}(30-y)$$

$$V=\frac{1}{3}\pi r^2h-\frac{1}{3}\pi x^2(30-y)$$

$$=\frac{1}{3}\pi(12)^2(30)-\frac{1}{3}\pi\left[\frac{2}{5}(30-y)\right]^2(30-y)$$

$$=1440\pi-\frac{4\pi}{75}(30-y)^3$$

Using G.C.,

When $V = 2000 - 1905.899543 = 94.100457$

$y = 0.20946682$

The depth of the mineral oil is 0.209 cm.

(c)(ii) $\dfrac{\mathrm{d}V}{\mathrm{d}y} = \dfrac{4\pi}{25}(30-y)^2$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}t}$$

$$10 = \frac{4\pi}{25}(30-0.2094668)^2 \times \frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.0224 \text{ cm/s}$$

<u>Examiner's Report:</u>

5 **(a)** A significant percentage of students did not recall how to use the ratio of lengths for similar objects and left the part blank.

(b) Some students did not understand the definition of frustum in the question.
A significant percentage of students failed to use the correct radii/heights in the calculation. The original cone and the removed cone have different radii and heights.

(c) This part was badly attempted.
Common mistakes included:

- Students did not realize that they needed to establish a new relationship between the height and the radius of unoccupied volume above the oil (as a cone). They re-used the relationship in part **(a)**.
- Students equated the volume of oil in the top frustum to the volume of a cone.

(d) This part was badly attempted.
Common mistakes included:

- Students treated the radius as constant and attempted differentiation. They failed to realise that both the radius and height were variables.
- Students substituted of the radius to find the volume in terms of h. They failed to realise that the height became a constant the moment they did that.

6	It is given that $\ln(1+y)=\tan x$.		
	(a)	Show that $\dfrac{d^2y}{dx^2}=\dfrac{dy}{dx}\left(2\tan x+\sec^2 x\right)$.	[3]
	(b)	By further differentiation of the result in part **(a)**, find the Maclaurin series for y, up to and including the term in x^3.	[3]
	(c)	Hence find an estimate for the value of e.	[2]

Solution:

(a)

$\ln(1+y)=\tan x$

$1+y=e^{\tan x}$

Diff wrt x:

$\dfrac{dy}{dx}=e^{\tan x}\sec^2 x$

Diff wrt x:

$$\frac{d^2y}{dx^2}=e^{\tan x}(2\sec x)(\sec x\tan x)+\sec^2 x\left(e^{\tan x}\sec^2 x\right)$$

$$\frac{d^2y}{dx^2}=e^{\tan x}\left(\sec^2 x\right)\left(2\tan x+\sec^2 x\right)$$

$$=\frac{dy}{dx}\left(2\tan x+\sec^2 x\right)$$

(b)

Diff wrt x:

$$\frac{d^3y}{dx^3}=\frac{d^2y}{dx^2}\left(2\tan x+\sec^2 x\right)+\frac{dy}{dx}\left(2\sec^2 x+2\sec x\left(\sec x\tan x\right)\right)$$

When $x=0$, $y=0,\dfrac{dy}{dx}=1,\dfrac{d^2y}{dx^2}=1,\dfrac{d^3y}{dx^3}=3$

$$y=x+\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots$$

(c)

When $x=\frac{\pi}{4}$, $y+1=e^{\tan\frac{\pi}{4}}=e$

$$y=\frac{\pi}{4}+\frac{1}{2}\left(\frac{\pi}{4}\right)^2+\frac{1}{2}\left(\frac{\pi}{4}\right)^3+\ldots=1.3361$$

$e=y+1=2.34$ (corr to 3sf)

Examiner's Report:

(a) There were many approaches to this part and generally, majority were successful in getting $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Most were also able to manipulate their obtained $\frac{d^2y}{dx^2}$ to prove the required.

(b) This part is generally well done with many correctly differentiating the required form to obtain the 3rd derivative, and thereafter, the Maclaurin series for *y*, up to and including the term in x^3. A handful made algebraic slip in obtaining the coeff of x^3.

(c) This part is poorly done. Students should note the word, hence and work to obtain *y* as $y = e^{\tan x} - 1$. The estimate of *e* is then obtained by letting $x = \frac{\pi}{4}$ accordingly.

7	(a)	(i)	Find $\int \dfrac{x^2}{2\sqrt{1-x^2}}\,\mathrm{d}x$, using the substitution $x=\cos\theta$ where $0<\theta<\dfrac{\pi}{2}$.	[5]
		(ii)	Find $\int x\sin^{-1}x\,\mathrm{d}x$.	[2]
	(b)	Find $\int \cos ax\sin bx\,\mathrm{d}x$, where a and b are real numbers and $a\neq b$.		[3]

Solution:

(ai)

$$\begin{aligned}\int \frac{x^2}{2\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{(\cos\theta)^2}{2\sqrt{1-(\cos\theta)^2}}\,(-\sin\theta)\,\mathrm{d}\theta\\
&= \int \frac{(\cos\theta)^2}{2\sqrt{(\sin\theta)^2}}\,(-\sin\theta)\,\mathrm{d}\theta\\
&= \int \frac{(\cos\theta)^2}{2\sin\theta}\,(-\sin\theta)\,\mathrm{d}\theta\\
&= -\int \frac{1}{2}(\cos\theta)^2\mathrm{d}\theta\\
&= -\frac{1}{2}\int \frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta\\
&= -\frac{1}{4}\int 1+\cos 2\theta\,\mathrm{d}\theta\\
&= -\frac{1}{4}\left[\theta+\frac{\sin 2\theta}{2}\right]+C\\
&= -\frac{1}{4}\left[\cos^{-1}x+\sin\theta\cos\theta\right]+C\\
&= -\frac{1}{4}\left[\cos^{-1}x+x\sqrt{1-x^2}\right]+C\end{aligned}$$

(aii)

$$\begin{aligned}\int x\sin^{-1}x\,\mathrm{d}x &= \sin^{-1}x\cdot\frac{x^2}{2}-\int \frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x\\
&= \frac{x^2\sin^{-1}x}{2}-\int \frac{x^2}{2\sqrt{1-x^2}}\,\mathrm{d}x\\
&= \frac{x^2\sin^{-1}x}{2}+\frac{1}{4}\left[\cos^{-1}x+x\sqrt{1-x^2}\right]+C\end{aligned}$$

(b)

$$\begin{aligned}\int \cos ax\sin bx\,\mathrm{d}x &= \frac{1}{2}\int 2\cos ax\sin bx\,\mathrm{d}x\\
&= \frac{1}{2}\int \sin(ax+bx)-\sin(ax-bx)\,\mathrm{d}x\\
&= \frac{1}{2}\left[-\frac{\cos(ax+bx)}{a+b}\right]-\frac{1}{2}\left[-\frac{\cos(ax-bx)}{a-b}\right]+C\\
&= \frac{1}{2}\left[\frac{\cos(ax-bx)}{a-b}\right]-\frac{1}{2}\left[\frac{\cos(ax+bx)}{a+b}\right]+C\end{aligned}$$

Examiner's Report:

(a) Many students were able to replace the terms correctly, except for a handful that differentiated $\cos\theta$ wrongly. However, quite a number of students were not able to apply the double angle formula correctly as well as carry out the integration of the $\cos 2\theta$ expression correctly. Many students were not able to get the final answer mark as they either did not give the final answer in terms of *x* or tried to do it without simplifying the $\sin 2\theta$ to $\sin\theta\cos\theta$.

(b) Some students could identify the need to perform integration by parts. However, they either chose the wrong terms for *u* and $\frac{dv}{dx}$, or carried out the respective differentiation and integration wrongly. Again many students did not see the link between (a)(i) and (a)(ii) and tried to solve $\int \frac{x^2}{2\sqrt{1-x^2}}\,\mathrm{d}x$ instead of using the answer from (a)(i)

(c) Many students were not able to see that they should use factor formula here. Instead, they tried to solve the integration using by parts. Some tried to apply the factor formula but were not able to obtain the correct expressions for the angle, that is, $\sin(ax+bx)-\sin(ax-bx)$. There were also quite a few who did not integrate $\sin(ax+bx)-\sin(ax-bx)$ correctly.

8	A curve C has parametric equations $$x=3+2\sec\theta,\quad y=3\tan\theta-1,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$			
	(a)	Find the range of values of x.		[2]
	(b)	Find a cartesian equation of C.		[2]
	(c)	Hence or otherwise, sketch C, stating the coordinates of any vertices and equations of any asymptotes.		[3]
	(d)	**(i)**	Find the equation of the tangent to C at the point P where $\theta=p$, $-\frac{\pi}{2}<p<\frac{\pi}{2}$. [You do not need to simplify your answer.].	[3]
		(ii)	Show that the gradient of the tangent at P cannot lie in the interval $\left[-\frac{3}{2},\frac{3}{2}\right]$ as p varies.	[2]

Solution:

(a)

For $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$,

$$0<\cos\theta\le 1$$

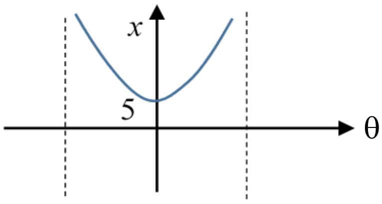
$$\sec\theta\ge 1$$

$$3+2\sec\theta\ge 5$$

$$\therefore x\ge 5$$

Alternative Method

Sketch the graph of $x=3+2\sec\theta$



From the graph, $x\ge 5$

(b)

$x=3+2\sec\theta \Rightarrow \sec\theta=\frac{x-3}{2}$

$y=3\tan\theta-1 \Rightarrow \tan\theta=\frac{y+1}{3}$

Substituting into $1+\tan^2\theta=\sec^2\theta$,

$$1+\left(\frac{y+1}{3}\right)^2=\left(\frac{x-3}{2}\right)^2$$

$$\frac{(x-3)^2}{2^2}-\frac{(y+1)^2}{3^2}=1,\quad x\ge 5.$$

(c)

<u>Method 1 (using the cartesian equation)</u>

From (b), $\dfrac{(x-3)^2}{2^2}-\dfrac{(y+1)^2}{3^2}=1,\quad x\geq 5$

So, C is a hyperbola for $x\geq 5$, with centre $(3,-1)$

To find the oblique asymptotes:

$$\left(\frac{x-3}{2}\right)^2=\left(\frac{y+1}{3}\right)^2$$

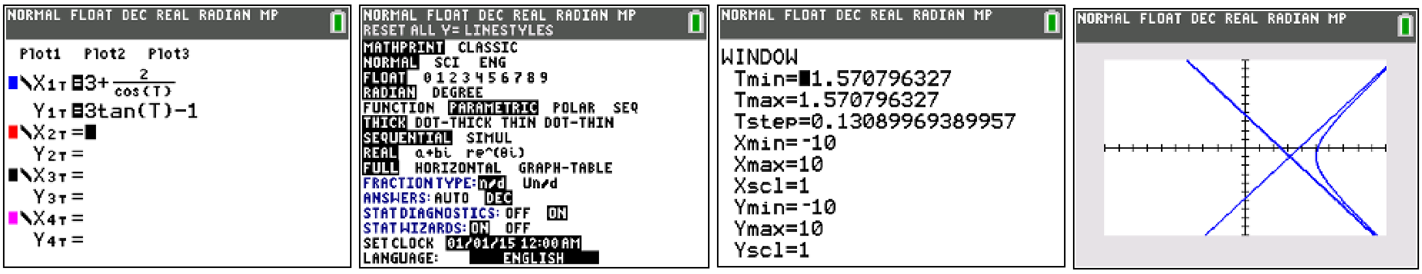
$$\left(\frac{y+1}{3}\right)=\pm\left(\frac{x-3}{2}\right)$$

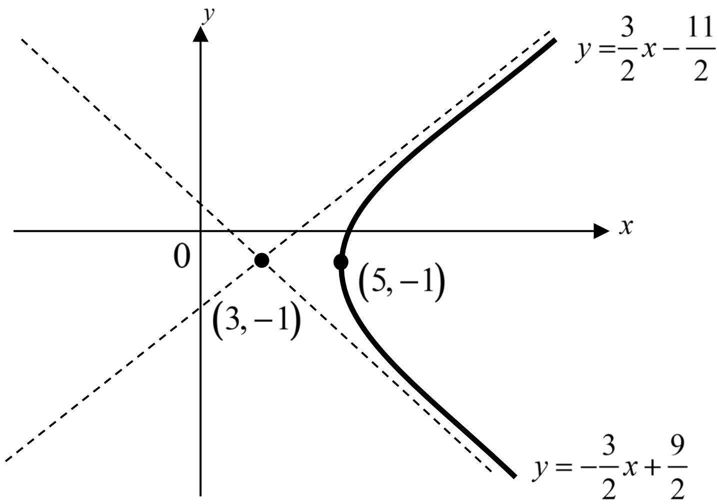
$$y=-1\pm 3\left(\frac{x-3}{2}\right)$$

$$y=-1+3\left(\frac{x-3}{2}\right)=-1+\frac{3}{2}x-\frac{9}{2}=\frac{3}{2}x-\frac{11}{2}$$

or $$y=-1-3\left(\frac{x-3}{2}\right)=-1-\frac{3}{2}x+\frac{9}{2}=-\frac{3}{2}x+\frac{7}{2}$$

<u>Method 2 (using the parametric equations)</u>





(d)(i)

$x = 3 + 2\sec\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec\theta\tan\theta$

$y = 3\tan\theta - 1 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 3\sec^2\theta$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3\sec^2\theta}{2\sec\theta\tan\theta} = \dfrac{3\sec\theta}{2\tan\theta} = \dfrac{3}{2\sin\theta}$

At $P = \left(3 + 2\sec p, 3\tan p - 1\right)$,

At P, gradient of tangent is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{2\sin p}$

Equation of tangent at P is

$y - \left(3\tan p - 1\right) = \dfrac{3}{2\sin p}\left(x - 3 - 2\sec p\right)$

(d)(ii)

Since $-\dfrac{\pi}{2} < p < \dfrac{\pi}{2}$,

So $\quad -1 < \sin p < 1$

Hence, $\dfrac{3}{2\sin p} < -\dfrac{3}{2}$ or $\dfrac{3}{2\sin p} > \dfrac{3}{2}$

Thus the gradient of the tangent at P cannot lie in the interval $\left[-\dfrac{3}{2}, \dfrac{3}{2}\right]$.

[Fun Fact: These are the gradients of the oblique asymptotes of the hyperbola.]

<u>Alternative Method</u>

Observe that the gradient of the oblique asymptotes are $-\dfrac{3}{2}$ and $\dfrac{3}{2}$.

As the tangents of the curve cannot lie between the two oblique asymptotes,

the gradient of the tangent at P cannot lie in the interval $\left[-\dfrac{3}{2}, \dfrac{3}{2}\right]$.

<u>**Examiner's Report:**</u>

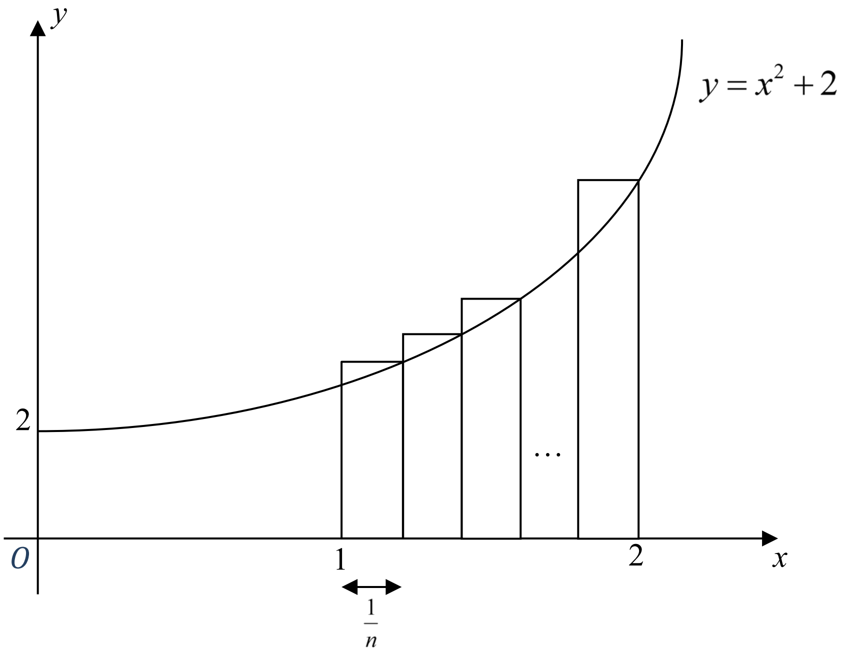
(a) Students performed poorly in this part. They failed to consider the range of possible values of $\cos\theta$ for the theta values given i.e. $\left(0,1\right]$. Instead, substituted the upper and lower limits of theta in the equation of x, erroneously assuming that will give the corresponding upper and lower limits of x. Students need to understand this method only works provided the equation of x is an increasing function.

(b) Students who remember using trigonometry identities as a strategy to eliminate theta generally obtained full credit for this part. However, about half of them did not. A significant number performed parametric differentiation, which as irrelevant to this part. Students who considered right angle triangles and basic trigonometry may get partial credit for obtaining partial answers such as $y = 9\sqrt{\dfrac{\left(x-3\right)^2}{4} - 1}$, which only formed the top half of the graph.

(c) Again, students performed badly in this part. Majority of them failed to relate the graph to a hyperbola. For those who did, they either failed to exclude the portion where x <5, or include equations of asymptotes, or both.

(di) More students were able to get full credit for this part. Some of them lost 1 mark for algebraic manipulation errors. Other than that, majority of students know the steps to obtain tangent of parametric curve.

(dii) Students performed poorly in this part. Like part (a), they failed to consider the range of possible values of $\sin\theta$ and subsequently, $\frac{3}{2\sin\theta}$. Instead, many tried proof by contradiction. These students let gradients be $-\frac{3}{2}$, $\frac{3}{2}$ and explained why it could not be so. However, this method only shows gradient cannot be 2 values, instead of a whole range of values.

9	**(a)**	 The diagram shows part of the graph of $y=x^2+2$ from $x=1$ to $x=2$. The area under the curve in this interval may be approximated by the total area of n rectangles, A, as shown. The width of each rectangle is $\frac{1}{n}$. Given that $\sum_{r=1}^{n} r^2=\frac{n}{6}(n+1)(2n+1)$, show that $A=\frac{13}{3}+\frac{3}{2n}+\frac{1}{6n^2}$. Deduce the exact value of $\int_1^2 \left(2+x^2\right)\,\mathrm{d}x$, justifying your answer.	[6]
	(b)	Region R is bounded by the curve $y = \ln x$, $x = 1$ and $y = \ln 5$.	
		(i) Find the area of R.	[2]
		(ii) Find the exact volume of the solid generated when R is rotated through 4 right angles about the y-axis.	[4]

Solution:

(a) Width of 1 rectangle = $\frac{1}{n}$

$x_1=1+\frac{1}{n}$: Height of 1st rectangle $=\left(1+\frac{1}{n}\right)^2+2=\left(\frac{n+1}{n}\right)^2+2$

$x_2=1+\frac{2}{n}$: Height of 2nd rectangle $=\left(1+\frac{2}{n}\right)^2+2=\left(\frac{n+2}{n}\right)^2+2$

$x_3=1+\frac{3}{n}$: Height of 3rd rectangle $=\left(1+\frac{3}{n}\right)^2+2=\left(\frac{n+3}{n}\right)^2+2$

…… ……

$x_n=1+\frac{n}{n}=2$: Height of nth rectangle $=\left(1+\frac{n}{n}\right)^2+2=\left(\frac{n+n}{n}\right)^2+2$

Total area of n rectangles, A

$$= \frac{1}{n}\left[\left(1+\frac{1}{n}\right)^2+2\right]+\frac{1}{n}\left[\left(1+\frac{2}{n}\right)^2+2\right]+\frac{1}{n}\left[\left(1+\frac{3}{n}\right)^2+2\right]+\ldots+\frac{1}{n}\left[\left(1+\frac{n}{n}\right)^2+2\right]$$

$$= \frac{1}{n}\sum_{r=1}^{n}\left[\left(1+\frac{r}{n}\right)^2+2\right]$$

$$= \frac{1}{n}\sum_{r=1}^{n}\left(3+\frac{2r}{n}+\frac{r^2}{n^2}\right)$$

$$= 3+\frac{2}{n^2}\sum_{r=1}^{n}r+\frac{1}{n^3}\sum_{r=1}^{n}r^2$$

$$= 3+\frac{2}{n^2}\left[\frac{n}{2}(n+1)\right]+\frac{1}{n^3}\left[\frac{n}{6}(n+1)(2n+1)\right]$$

$$= 3+\frac{1}{n}(n+1)+\frac{1}{6n^2}\left(2n^2+3n+1\right)$$

$$= 3+\left(1+\frac{1}{n}\right)+\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)$$

$$= \frac{13}{3}+\frac{3}{2n}+\frac{1}{6n^2} \quad \text{(Shown)}$$

Or,

Total area of n rectangles, A

$$= \frac{1}{n}\left[\left(\frac{n+1}{n}\right)^2+2\right]+\frac{1}{n}\left[\left(\frac{n+2}{n}\right)^2+2\right]+\frac{1}{n}\left[\left(\frac{n+3}{n}\right)^2+2\right]+\ldots+\frac{1}{n}\left[\left(\frac{n+n}{n}\right)^2+2\right]$$

$$= \frac{1}{n}\sum_{r=1}^{n}\left[\left(\frac{n+r}{n}\right)^2+2\right]$$

$$= \frac{1}{n^3}\sum_{r=n+1}^{2n}r^2+2$$

$$= \frac{1}{n^3}\left(\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2\right)+2$$

$$= \frac{1}{n^3}\left[\frac{2n}{6}(2n+1)(4n+1)-\frac{n}{6}(n+1)(2n+1)\right]+2$$

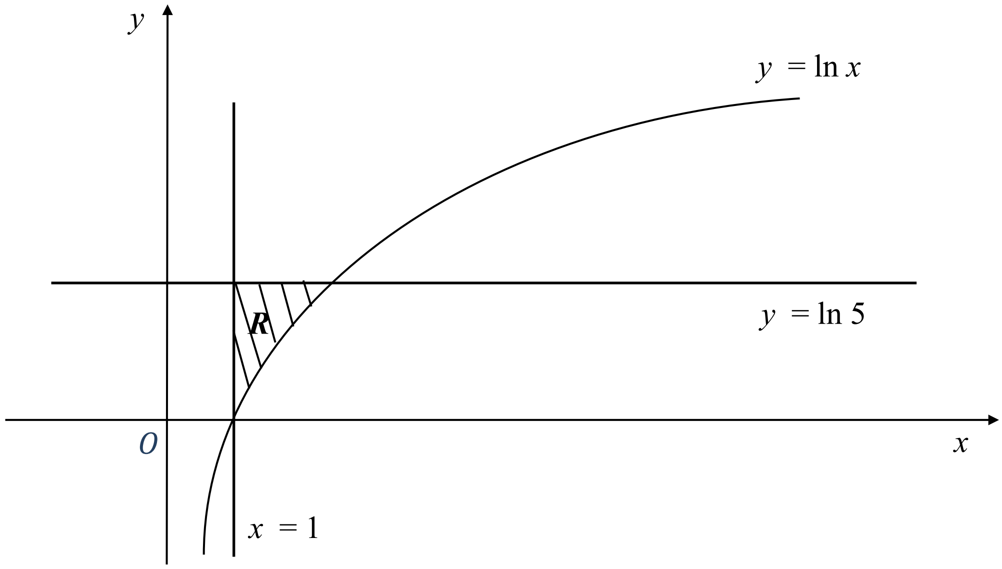
$$= \frac{1}{n^3}\left[\frac{n}{6}(2n+1)(7n+1)\right]+2$$

$$= \frac{1}{6n^2}\left[14n^2+9n+1\right]+2$$

$$= \frac{13}{3}+\frac{3}{2n}+\frac{1}{6n^2} \quad \text{(Shown)}$$

$$\int_1^2\left(2+x^2\right)\,\mathrm{d}x=\lim_{n\to\infty}A=\lim_{n\to\infty}\left(\frac{13}{3}+\frac{3}{2n}+\frac{1}{6n^2}\right)=\frac{13}{3} \quad \text{units}^2$$

(b)



(bi)

Area of $R = \int_1^5 (\ln 5 - \ln x)\,\mathrm{d}x = 2.39$ units^2 (to 3 s.f.)

Or,

Area of $R = (5-1)(\ln 5) - \int_1^5 \ln x\,\mathrm{d}x = 2.39$ units^2 (to 3 s.f.)

Or,

Area of $R = \int_0^{\ln 5} \left(\mathrm{e}^y - 1\right)\mathrm{d}x = 2.39$ units^2 (to 3 s.f.)

Or,

Area of $R = \left(\int_0^{\ln 5} \mathrm{e}^y\,\mathrm{d}y\right) - (1-0)(\ln 5) = 2.39$ units^2 (to 3 s.f.)

(bii)

Volume of solid generated when R is rotated about y-axis

$$= \pi\int_0^{\ln 5}\left(\mathrm{e}^{2y} - 1\right)\mathrm{d}y$$

$$= \pi\left[\frac{1}{2}\mathrm{e}^{2y} - y\right]_0^{\ln 5}$$

$$= \pi\left[\left(\frac{1}{2}\mathrm{e}^{2\ln 5} - \ln 5\right) - \left(\frac{1}{2} - 0\right)\right]$$

$$= \pi\left[\frac{1}{2}\left(5^2\right) - \frac{1}{2} - \ln 5\right]$$

$$= \pi\left(12 - \ln 5\right) \quad \text{units}^2$$

Or,

Volume of solid generated when R is rotated about y-axis

$$= \pi\int_0^{\ln 5} e^{2y}\,dy - \pi(1)^2(\ln 5)$$
$$= \pi\left[\frac{1}{2}e^{2y}\right]_0^{\ln 5} - \pi\ln 5$$
$$= \frac{1}{2}\pi\left[e^{2\ln 5} - 1\right] - \pi\ln 5$$
$$= \frac{1}{2}\pi\left[\left(5^2\right) - 1\right] - \pi\ln 5$$
$$= \pi(12 - \ln 5) \quad \text{units}^2$$

Examiner's Report:

Poorly attempted question.
Many students left it blank too.

(a) Students fail to use the correct x-value for getting the height of each rectangle, and some students even got the first/last rectangle wrong. For those who managed to get the correct dimensions of the rectangles, many students have difficulties writing the series in summation notation and/or evaluating the sum.

For the last part in getting the exact value of the definite integral, many students used integration instead of finding the limit of A as n approaches infinity.

(b)(i) Students fail to identify the correct region R, which is bounded by 2 graphs (a straight line $y = \ln 5$ and a curve $y = \ln x$ between $x = 1$ and $x = 5$, or $x = e^y$ and $x = 1$ between $y = 0$ and $y = \ln 5$). Many students also integrated $\ln x$ wrongly instead of using GC to evaluate the definite integral as the question did not ask for an exact area.

(b)(ii) Students fail to notice that the region R is rotated about the y-axis, which meant that the definite integral should be an integration with respect to y. Many students also used the wrong formula for finding volume of solid generated by a region bounded by 2 graphs $V = \pi\int_c^d \left(\left(g_1(y)\right)^2 - \left(g_2(y)\right)^2\right)dy$, and could not simplify $e^{2\ln 5}$ as $e^{\ln 5^2} = 5^2 = 25$ to get the exact volume.

10	Glucose is a simple carbohydrate that can be easily absorbed by the body and provides instant energy. When a patient is dehydrated or unable to take food orally, glucose is given intravenously to the patient via a glucose drip. Glucose given intravenously enters the bloodstream at a constant rate of p units per hour. It is absorbed by the body, leaving the bloodstream, at a rate proportional to the amount of glucose present in the bloodstream. G denotes the number of units of glucose in the bloodstream at time t hours after the glucose drip is administered.		
	(a)	Write down a differential equation relating G and t.	[1]
	(b)	Suppose there are 3 units of glucose in the bloodstream when $t = 0$ and that the amount of glucose in the bloodstream remains constant when $G = 8$. Show that the particular solution of the differential equation in part **(a)** is $G = 8 - 5\mathrm{e}^{-\frac{p}{8}t}$.	[6]
	(c)	Sketch the graph of G against t.	[2]
	Instead of the glucose drip, a patient who is well enough will get his glucose from the food he consumes. At the end of each meal, the rate of change in glucose in the bloodstream can be modelled as $\left(\frac{1}{2} + \sin t\right)$ units per hour where $0 \le t \le \frac{3}{2}\pi$.		
	(d)	Find the time at which the amount of glucose in the bloodstream	
		(i) increases most rapidly after the meal,	[2]
		(ii) starts to decrease.	[2]

Solution:

(a)

$$\frac{\mathrm{d}G}{\mathrm{d}t} = p - kG, \quad k>0$$

(b)

$$\frac{1}{p-kG}\frac{\mathrm{d}G}{\mathrm{d}t} = 1$$

$$\int \frac{1}{p-kG}\,\mathrm{d}G = \int 1\,\mathrm{d}t$$

$$-\frac{1}{k}\ln\left|p-kG\right| = t + c$$

$$\left|p-kG\right| = e^{-kt-kc}$$

$$p - kG = \pm e^{-kt-kc} = Ae^{-kt}$$

$$G = \frac{1}{k}\left(p - Ae^{-kt}\right)$$

Amount of glucose in the bloodstream remains constant when $G = 8$:

$$0 = p - k(8) \Rightarrow k = \frac{p}{8}$$

When $t = 0$, $G = 3$

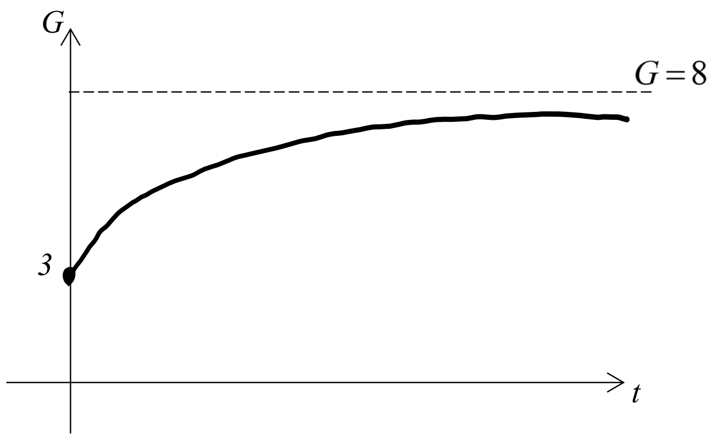
$$3 = \frac{1}{k}\left(p - Ae^{-0}\right) = \frac{1}{k}\left(p - A\right)$$

$$\Rightarrow\ p - 3k = A$$

Subst $k = \frac{p}{8}$ $\Rightarrow$ $A = p - 3\left(\frac{p}{8}\right) = \frac{5p}{8}$

Thus $G = \frac{1}{\frac{p}{8}}\left(p - \frac{5p}{8}e^{-\frac{p}{8}t}\right) = 8 - 5e^{-\frac{p}{8}t}$ (shown)

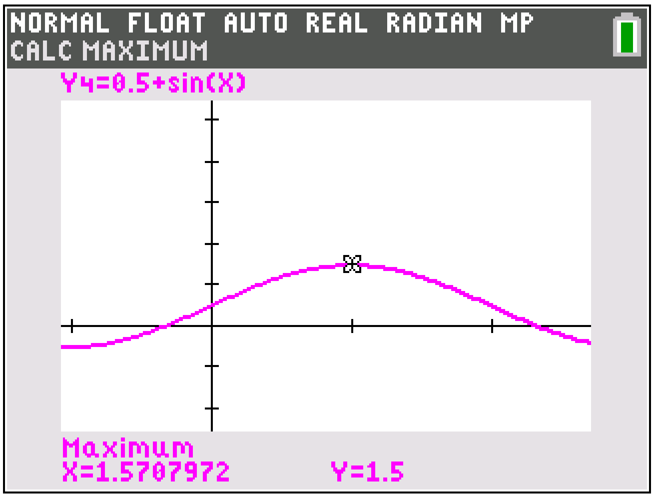
(c)



(di)

$$\frac{dG}{dt} = \left(\frac{1}{2} + \sin t\right)$$

For G to be increasing mostly rapidly, $\frac{dG}{dt}$ must be at its maximum



$\rightarrow$ $t = \frac{\pi}{2} \approx 1.57$ hr after the meal

OR

$$\frac{\mathrm{d}G}{\mathrm{d}t}=\left(\frac{1}{2}+\sin t\right)$$

$$\frac{\mathrm{d}^2G}{\mathrm{d}t^2}=\cos t=0$$

$$t=\frac{\pi}{2},\frac{3\pi}{2}$$

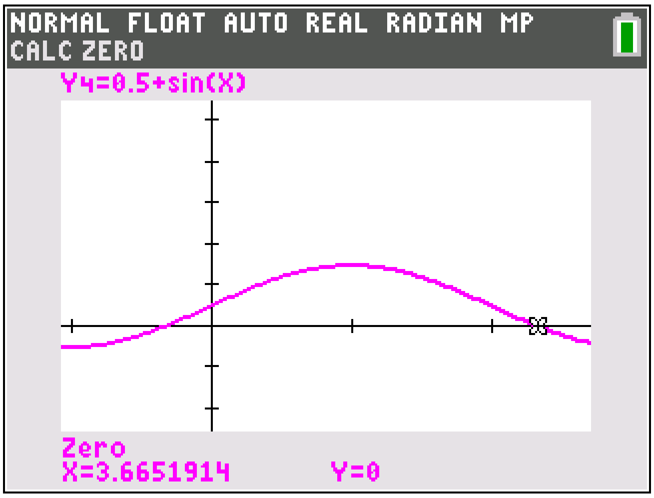
When $t=\frac{\pi}{2}$, $\left.\frac{\mathrm{d}^3G}{\mathrm{d}t^3}\right|_{t=\frac{\pi}{2}}=-\sin t<0$

When $t=\frac{3\pi}{2}$, $\left.\frac{\mathrm{d}^3G}{\mathrm{d}t^3}\right|_{t=\frac{3\pi}{2}}=-\sin t>0$

Thus $t=\frac{\pi}{2}$ gives max $\frac{\mathrm{d}G}{\mathrm{d}t}$.

(dii)

For G to decrease, $\frac{\mathrm{d}G}{\mathrm{d}t}<0$



→ $t=\frac{7\pi}{6}\approx 3.6652$ or t = 3.67 hr after the meal

Examiner's Report:

(a) This part is poorly done. Students need to understand that a differential equation relating G and t is that of $\frac{\mathrm{d}G}{\mathrm{d}t}$, most students gave an equation in, and not one involving its derivative. A way to form the required DE is to look at the net change in G *as* $\frac{\mathrm{d}G}{\mathrm{d}t}=\frac{\mathrm{d}G_{enters}}{\mathrm{d}t}-\frac{\mathrm{d}G_{leaves}}{\mathrm{d}t}$.

(b) This part is often left incomplete. In cases where students attempted the question, students were generally able to perform variable separable, though some made error in integrating $\int \frac{1}{p-kG}\,\mathrm{d}G = \int 1\,\mathrm{d}t \Rightarrow \ln|p-kG| = t+c$, erroneously missing out the $-\frac{1}{k}$. The correct integration should have been $\int \frac{1}{p-kG}\,\mathrm{d}G = \int 1\,\mathrm{d}t \Rightarrow -\frac{1}{k}\ln|p-kG| = t+c$.

(c) Students are advised to attempt this part even if they are not able to do parts (a) and (b) since the equation of G is given. There were many incomplete and inaccurate sketches, for example, sketching for $t < 0$ when clearly $t \geq 0$. Also, the initial condition of 3 units of glucose in the bloodstream when $t = 0$ and that the amount of glucose in the bloodstream remains constant when $G = 8$ meant (0,3) and HA at G =8 respectively.

(d) Many students left (i) and (ii) attempted, where there were attempts to solve (i), many interpret increases most rapidly after the meal as finding the value of t when G is a max. This is not the case. For this question, we are finding the value of t when $\frac{\mathrm{d}G}{\mathrm{d}t}$ is at its max. For cases where students recognise the need to look for max $\frac{\mathrm{d}G}{\mathrm{d}t}$, they tend to do so via differentiation (set $\frac{\mathrm{d}^2G}{\mathrm{d}t^2} = 0$) than a sketch of $\frac{\mathrm{d}G}{\mathrm{d}t}$. While both methods work, students who had used the derivative method, will need to show that $\frac{\mathrm{d}^3G}{\mathrm{d}t^3} < 0$ at $t = \frac{\pi}{2}$ for max value. For part (ii), G starts to decrease implies $\frac{\mathrm{d}G}{\mathrm{d}t} < 0$, which can be read off from the sketch of $\frac{\mathrm{d}G}{\mathrm{d}t}$ when $t < 0$. There were quite a few students who gave a negative value of t without realizing that cannot be the case.

11	Taylor is training for a triathlon, which involves swimming, running and cycling. • On Day 1, he swims 1.5 km and then swims the same distance on each subsequent day. • On Day 1, he runs 2 km and, on each subsequent day, he runs 0.5 km further than on the previous day. • On Day 1, he cycles 2.5 km and, on each subsequent day, he cycles a distance 10% further than on the previous day.		
	(a)	Find the distance that Taylor runs on Day 15.	[2]
	(b)	Find the day on which the distance that Taylor cycles first exceeds 12 km.	[3]
	(c)	On which days will the distance run exceed the distance cycled by Taylor?	[3]
	His friend, Swift, is also training for the same triathlon. Swift's training programme is similar to Taylor's except that the distance for swimming, running and cycling on Day 1 is x km.		
	(d)	Find the minimum value of x if Swift intends to cover a longer total distance in the first 30 days than the total distance covered by Taylor. Give your answer correct to 1 decimal place.	[6]

Solution:

Let r_n and c_n be the distance covered on the day n in running and cycling respectively.

(a) $r_{15} = 2 + 14(0.5) = 9km$

(b) Consider $c_n = 2.5(1.1)^{n-1} > 12$

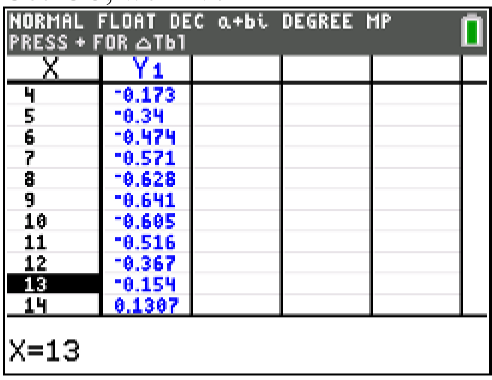
Taking ln on both sides, we have $n - 1 > \dfrac{\ln\left(\dfrac{12}{2.5}\right)}{\ln(1.1)}$

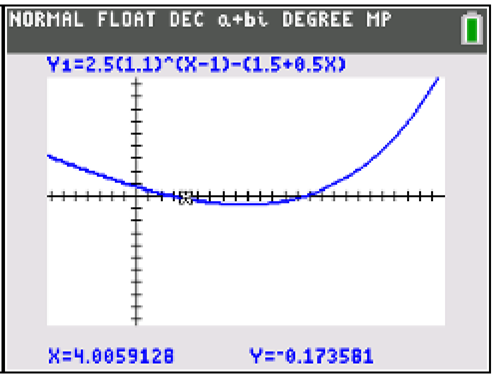
On Day 18.

Alternative method: Using GC

(c) Consider $2.5(1.1)^{n-1} - \left[2 + (n-1)0.5\right] > 0$

Use GC, we have





Thus, distance cycled is less than distance run between Day 4 and Day 13 inclusive.

(d) Let a be the distance covered by Swift on Day 1.

Consider $30a+\frac{30}{2}\left[2a+\frac{29}{2}\right]+a\left(\frac{1.1^{30}-1}{1.1-1}\right)>1.5(30)+\frac{30}{2}\left[2(2)+\frac{29}{2}\right]+2.5\left(\frac{1.1^{30}-1}{1.1-1}\right)$

Using GC, we have

a	$30a+\frac{30}{2}\left[2a+\frac{29}{2}\right]+a\left(\frac{1.1^{30}-1}{1.1-1}\right)$
2.2	711.39
2.3	733.84
2.4	756.29

The starting distance is at least 2.3km.

Examiner's Report:

(a) This part is well done, except those students who misunderstood the question and used sum of first n terms instead.
(b) This part is generally well done, students used algebraic or GC to answer the question.
(c) Several students failed to find all the values of n that satisfy the given condition as they only looked at the first few values from the GC table and concluded. The more careful students scrolled down the table to check further. Students who used graphical method are more successful.
(d) This part is well done for those who took time to solve it. One common mistake is that students who used GC table to solve for the minimum value of x did not set the interval of n to 0.1 unit. Many used the default '1' unit interval and failed to locate the value of x to 1 decimal place.

2024 CJC JC2 H2 Maths Prelim Paper 2

1	**(a)**	The points P, Q and R have position vectors $\mathbf{j}+\mathbf{k}$, $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ and $-2\mathbf{i}+\mathbf{j}+4\mathbf{k}$ respectively. Find the exact length of projection of $\overrightarrow{QR}$ on $\overrightarrow{PQ}$.	[3]
	(b)	Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\lvert\mathbf{a}\rvert=\lambda\lvert\mathbf{b}\rvert$ and $\lvert\mathbf{a}+\lambda\mathbf{b}\rvert=2\lvert\mathbf{a}-\lambda\mathbf{b}\rvert$, where λ is a positive constant. Using a suitable scalar product, show that $\cos\theta=\frac{3}{5}$, where θ is the acute angle between $\mathbf{a}$ and $\mathbf{b}$.	[3]

Solution:

(a) $$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}1\\-2\\3\end{pmatrix}-\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-3\\2\end{pmatrix}$$

$$\overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}=\begin{pmatrix}-2\\1\\4\end{pmatrix}-\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}-3\\3\\1\end{pmatrix}$$

Length of projection of $\overrightarrow{QR}$ on $\overrightarrow{PQ}$

$$=\left|\overrightarrow{QR}\bullet\widehat{\overrightarrow{PQ}}\right|=\frac{\left|\overrightarrow{QR}\bullet\overrightarrow{PQ}\right|}{\left|\overrightarrow{PQ}\right|}=\frac{\left|\begin{pmatrix}-3\\3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-3\\2\end{pmatrix}\right|}{\sqrt{14}}$$

$$=\frac{10}{\sqrt{14}}=\frac{5}{7}\sqrt{14}\text{ units}$$

(b) $$\lvert\mathbf{a}+\lambda\mathbf{b}\rvert=2\lvert\mathbf{a}-\lambda\mathbf{b}\rvert$$

$$\lvert\mathbf{a}+\lambda\mathbf{b}\rvert^2=\left(2\lvert\mathbf{a}-\lambda\mathbf{b}\rvert\right)^2$$

$$(\mathbf{a}+\lambda\mathbf{b})\bullet(\mathbf{a}+\lambda\mathbf{b})=4(\mathbf{a}-\lambda\mathbf{b})\bullet(\mathbf{a}-\lambda\mathbf{b})$$

$$\mathbf{a}\bullet\mathbf{a}+2\lambda\mathbf{a}\bullet\mathbf{b}+\lambda^2\mathbf{b}\bullet\mathbf{b}=4\left(\mathbf{a}\bullet\mathbf{a}-2\lambda\mathbf{a}\bullet\mathbf{b}+\lambda^2\mathbf{b}\bullet\mathbf{b}\right)$$

$$3\lvert\mathbf{a}\rvert^2+3\lambda^2\lvert\mathbf{b}\rvert^2=10\lambda\left(\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta\right)$$

$$\cos\theta=\frac{6\lambda^2\lvert\mathbf{b}\rvert^2}{10\lambda^2\lvert\mathbf{b}\rvert^2}\quad\text{since }\lvert\mathbf{a}\rvert=\lambda\lvert\mathbf{b}\rvert$$

$$\therefore\cos\theta=\frac{3}{5}\quad\text{(Shown)}$$

<u>Examiner's Report:</u>

(a) This part was well-done in general. Common mistakes were:

- Mis-quoting the formula for length of projection as $\overrightarrow{QR} \cdot \widehat{\overrightarrow{PQ}}$ (without the modulus sign) or $\left|\overrightarrow{QR} \times \widehat{\overrightarrow{PQ}}\right|$ (used vector product instead of scalar product).
- Computing scalar product in a wrong manner. E.g. $\dfrac{\left|\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\right|}{\sqrt{14}} = \dfrac{\left|\begin{pmatrix} 9 \\ 7 \\ 6 \end{pmatrix}\right|}{\sqrt{14}}$, not realizing that the result of a scalar product is a scalar, and not a vector.

(b) This part was badly done. A large proportion of students did not or hardly attempted the question. Common mistakes were:

- Thinking that modulus signs are distributive over addition and subtraction i.e. and then using quadratic expansion thereafter. For example,
  $$\begin{aligned} |\mathbf{a} + \lambda \mathbf{b}|^2 &= \left(|\mathbf{a}| + \lambda |\mathbf{b}|\right)^2 \\ &= |\mathbf{a}|^2 + 2\lambda |\mathbf{a}||\mathbf{b}| + \lambda^2 |\mathbf{b}|^2 . \end{aligned}$$
- Many failed to recognize that $|\mathbf{a} + \lambda \mathbf{b}|^2 = (\mathbf{a} + \lambda \mathbf{b}) \cdot (\mathbf{a} + \lambda \mathbf{b})$ and therefore could not get much going.
- Erroneously denoting $\mathbf{a} \cdot \mathbf{a}$ as $\mathbf{a}^2$ instead of $|\mathbf{a}|^2$
- Writing $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \cos\theta$ instead of $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

2	The functions f and g are defined by $$\mathrm{f}: x \mapsto \frac{3-2x}{x+2},\ \text{for } x \in \mathbb{R},\ x > -2,$$ $$\mathrm{g}: x \mapsto \lvert x \rvert - 1,\ \text{for } x \in \mathbb{R}.$$		
	(a)	Find $\mathrm{f}^{-1}(x)$ and state its domain.	[3]
	(b)	Hence find $\mathrm{f}^{2024}(1)$.	[1]
	(c)	Show that fg exists and find its range.	[3]
	(d)	Solve $\mathrm{f}(x) > \mathrm{g}(x)$, giving your answer in exact form.	[4]

Solution:

(a) Let $y = \dfrac{3-2x}{x+2}$

$$xy + 2y = 3 - 2x$$
$$xy + 2x = 3 - 2y$$
$$x = \frac{3-2y}{y+2}$$

Hence $\mathrm{f}^{-1}(x) = \dfrac{3-2x}{x+2}$

$\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}}$

$= x \in \mathbb{R},\ x > -2$

(b) Since f is self-inverse, $\mathrm{f}^2(x) = x$

$\mathrm{f}^{2024}(x) = x$

$\mathrm{f}^{2024}(1) = 1$

(c) $\mathrm{R}_{\mathrm{g}} = [-1, \infty)$ and $\mathrm{D}_{\mathrm{f}} = (-2, \infty)$

Since $\mathrm{R}_{\mathrm{g}} \subseteq \mathrm{D}_{\mathrm{f}}$, fg exists.

$\mathrm{R}_{\mathrm{fg}} = (-2, 5]$

(d) At point of intersection, $x - 1 = \dfrac{3-2x}{x+2}$

Simplifying to get $x^2 + 3x - 5 = 0$

Solving, we have $x = \dfrac{-3 \pm \sqrt{29}}{2}$.

Solution is $-2 < x < \dfrac{\sqrt{29}-3}{2}$

Alternative method(not recommended)

$\frac{3-2x}{x+2} > |x|-1 \Rightarrow |x| < \frac{3-2x}{x+2}+1 = \frac{5-x}{x+2}$

$\Rightarrow -\frac{5-x}{x+2} < x < \frac{5-x}{x+2}$

$-\frac{5-x}{x+2} < x$ and $x < \frac{5-x}{x+2}$

For $x < \frac{5-x}{x+2}$, we have $x - \frac{5-x}{x+2} < 0 \Rightarrow \frac{x^2+3x-5}{x+2} < 0$

Using test-point method gives $x < \frac{-3-\sqrt{29}}{2}$ or $-2 < x < \frac{\sqrt{29}-3}{2}$

For $x > -\frac{5-x}{x+2}$, we have $x + \frac{5-x}{x+2} > 0 \Rightarrow \frac{x^2+x+5}{x+2} > 0$

Since $x^2+x+5>0$ (either complete the square or use of discriminant), $x > -2$.

Combining, we have $-2 < x < \frac{\sqrt{29}-3}{2}$

Examiner's Report:

(a) This part is generally well-done although there were some carelessness/incomplete understanding, misrepresenting $D_{f^{-1}} = (\infty,-2)$ when it should have been $D_{f^{-1}} = (-2,\infty)$.

(b) About half obtained the correct answer, recognizing $f^{2024}(x) = x$ and so $f^{2024}(1) = 1$. The other part simply assume $f^{2024}(1) = \frac{3-2(1)}{1+2} = \frac{1}{3}$, demonstrating no understanding of part (a) to (b). Students are advised to see the f^{-1} as defined in (a) is the same as the given function f in attempting to show this part.

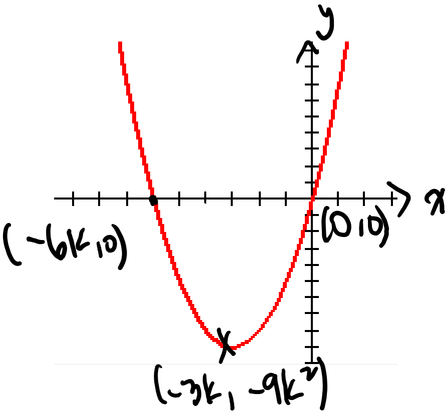
(c) There was a significant proportion of students who could not recall the condition for composite function to exist, this knowledge is very much needed. In cases where condition is correctly stated, there were instances of not stating R_g correctly. There was varied degree of success in finding R_{fg}, some approached it by finding fg and sketching the graph accordingly, in such cases, they generally missed the HA. Others used the mapping approach and, in this case, some made the error of writing the $R_{fg} = [5,-2)$ when it should have been $R_{fg} = (-2,5]$.

(d) Varied response to this part, for those who approached the question via a sketch of f(x) and g(x), they generally were more successful. A sketch would have been a good starting point as at intersection, $g(x) = x+1$. Students could then move on to find the exact roots, then the region of $f(x) > g(x)$ via the sketch. In cases where students attempted the test point method, the splitting of $f(x) > g(x) \Rightarrow |x| < \frac{5-x}{x+2}$ to $-\frac{5-x}{x+2} < x < \frac{5-x}{x+2}$ is poorly understood. Also, there were many cross multiplication when attempting to solve $x < \frac{5-x}{x+2} \Rightarrow x(x+2) < 5-x$. This is incorrect when there was no explanation to say $x > -2$. Even in cases where no cross multiplication was done, the critical points in solving the inequalities were wrongly identified. Finally, the solution set is the intersection of both inequalities, which was also often not done.

3	It is given that $\mathrm{f}(x) = 6kx + x^2$, where $k > 0$.			
	(a)	Sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the coordinates of the points where the curve crosses the axes and of the turning points, if any.	[2]	
	(b)	Describe the transformation that maps the graph of $y = \mathrm{f}(x)$ onto the graph of $y = x^2 - 9k^2$.	[2]	
	It is given that $$\mathrm{g}(x) = \begin{cases} 6kx + x^2 & \text{for } -3k \le x < k \\ 7k^2 & \text{for } k \le x < 2k \end{cases} \quad \text{where } k > 0$$ and that $\mathrm{g}(x) = \mathrm{g}(x + 5k)$ for all real values of x.			
	(c)	Sketch the graph of $y = \mathrm{g}(x)$ for $-3k \le x < 5k$.	[4]	
	(d)	Sketch the graph of $y = \dfrac{1}{\mathrm{g}(x)}$ for $-3k \le x < 2k$.	[3]	

<u>Solution:</u>

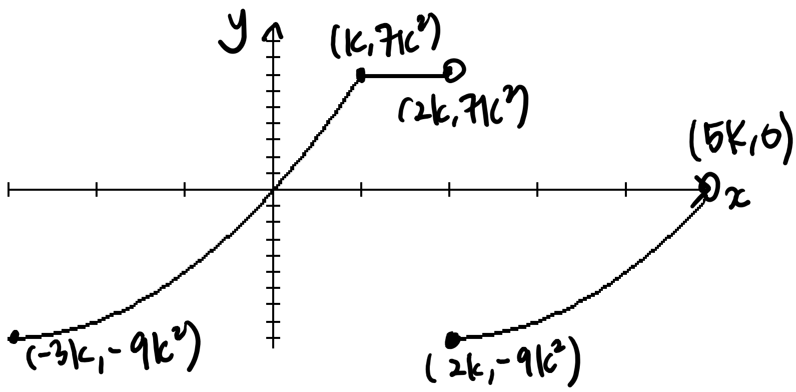
(a) $\mathrm{f}(x) = 6kx + x^2 = x(6k + x) = (x + 3k)^2 - 9k^2$



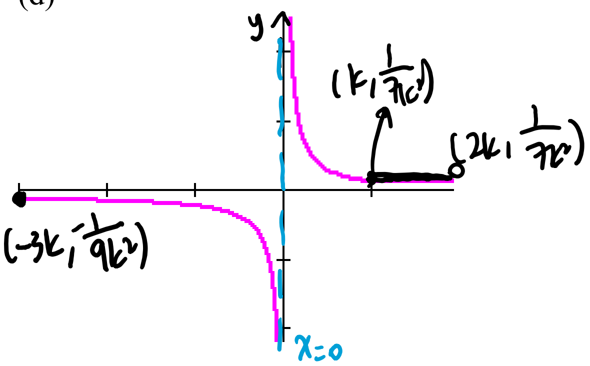
(b) $\mathrm{f}(x) = 6kx + x^2 = (x + 3k)^2 - 9k^2$

Comparing with $y = x^2 - 9k^2$, the transformation is a translation of $3k$ units in the positive x-axis direction.

(c) $$\mathrm{g}(x) = \begin{cases} \mathrm{f}(x) & -3k \le x < k \\ 7k^2 & k \le x < 2k \end{cases}$$



(d)



Examiner's Report:

(a) This part is quite well done. The common mistakes are that students tend to substitute k with a number say, 1 so that they can use GC to sketch the graph but forgot to replace by k when they label the features of the graph, thus not getting full credit.

(b) This part is poorly done as students could not see that they need to complete the square so that they can compare the two expressions to see the transformation needed. Students who started with $y = x^2 - 9k^2 = (x-3k)(x+3k)$ are also equally successful to identify the transformation. Another common mistake made by students who did not pay attention to the use of precise description of transformation include using words like "shift to the right" instead of translation of 3k units in the positive x-axis direction.

(c) About 50% of students could do this part. Common mistakes include not labelling the start and endpoints in coordinates clearly, not indicating open and close circles clearly and not paying attention to the domain required. Another common mistake is to misread $7k^2$ as $7x^2$.

(d) Another poorly done part since it relates to part ©. Quite a number of students assume that the domain of part (d) is the same as that of part ©. Students need to read questions carefully.

4	The plane p has equation $\mathbf{r}=\begin{pmatrix}1\\3\\4\end{pmatrix}+s\begin{pmatrix}2\\0\\1\end{pmatrix}+t\begin{pmatrix}0\\1\\-1\end{pmatrix}$, and the line l has the equation $\mathbf{r}=\begin{pmatrix}1\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}$, where λ, s and t are parameters.		
	(a)	Show that l is perpendicular to p and find the coordinates of the point where l and p intersect.	[5]
	The line m meets l at the point $A(1,1,3)$ and it meets p at the point $B(1,3,4)$.		
	(b)	The line n is the reflection of m in p. Show that a vector equation of n is $\mathbf{r}=\begin{pmatrix}1\\3\\4\end{pmatrix}+\mu\begin{pmatrix}-4\\2\\5\end{pmatrix}$, $\mu\in\mathbb{R}$.	[2]
	(c)	Find the acute angle between n and p.	[2]
	(d)	Hence or otherwise, find the area bounded by the lines l, m and n.	[3]

<u>Solution:</u>

(a) $\begin{pmatrix}2\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$

Since direction of l = normal of p, l is perpendicular to p.

Let F be the point of intersection between l and p.

$$p:\ \mathbf{r}\bullet\begin{pmatrix}-1\\2\\2\end{pmatrix}=\begin{pmatrix}1\\3\\4\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\2\end{pmatrix}=13$$

Since F lies on l, $\overrightarrow{OF}=\begin{pmatrix}1\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}$ for some λ

Since F lies on p, $\overrightarrow{OF}\bullet\begin{pmatrix}-1\\2\\2\end{pmatrix}=13$

$$\left[\begin{pmatrix}1\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}\right]\bullet\begin{pmatrix}-1\\2\\2\end{pmatrix}=13$$

$$7+9\lambda=13$$

$$\lambda=\frac{2}{3}$$

Hence $F\left(\frac{1}{3},\frac{7}{3},\frac{13}{3}\right)$

(b) Let the reflection of A be A'.

Using mid-point theorem, $\overrightarrow{OF} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$

$$\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA}$$

$$= 2\begin{pmatrix} \frac{1}{3} \\ \frac{7}{3} \\ \frac{13}{3} \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} \\ \frac{11}{3} \\ \frac{17}{3} \end{pmatrix}$$

$$\overrightarrow{BA'} = \begin{pmatrix} -\frac{1}{3} \\ \frac{11}{3} \\ \frac{17}{3} \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$$

Hence, $n : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix},\ \mu \in \mathbb{R}$

(c) $$\theta = \sin^{-1} \frac{\left| \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix} \right|}{\sqrt{9}\sqrt{45}}$$

$$= 1.11 \text{ rad or } 63.4^\circ$$

(d) $$\left|\overrightarrow{BA'}\right| = \left| \frac{1}{3}\begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix} \right|$$

$$= \frac{1}{3}\sqrt{45}$$

$$= \sqrt{5}$$

Hence area $= \left(\dfrac{1}{2}\right)\left(\sqrt{5}\right)^2 \sin\left(2 \times 1.107148718\right)$

$$= 2$$

Alternatively,
Area

$$=\left(\frac{1}{2}\right)\left|\overrightarrow{AA'}\right|\left|\overrightarrow{BF}\right|$$

$$=\left(\frac{1}{2}\right)\left|\begin{pmatrix}-\frac{1}{3}\\ \frac{11}{3}\\ \frac{17}{3}\end{pmatrix}-\begin{pmatrix}1\\1\\3\end{pmatrix}\right|\left|\begin{pmatrix}\frac{1}{3}\\ \frac{7}{3}\\ \frac{13}{3}\end{pmatrix}-\begin{pmatrix}1\\3\\4\end{pmatrix}\right|$$

$$=\left(\frac{1}{2}\right)(4)(1)$$

$$=2$$

<u>Examiner's Report:</u>

(a) Students performed well for this part.

(b) About half of the students know to apply midpoint theorem in order to find the reflection of A about p. However, a significant number made mistakes such as:
- using point B as the midpoint of A and A', when the correct point should be F.
- for students who used B as reference point instead of origin O, they formed the equation

$\overrightarrow{FB}=\frac{\overrightarrow{BA}+\overrightarrow{BA'}}{2}$ when it should be $\overrightarrow{BF}=\frac{\overrightarrow{BA}+\overrightarrow{BA'}}{2}$.

(c) Majority of students got this part correct. Most common mistake is using $\cos^{-1}\frac{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\5\end{pmatrix}\right|}{\sqrt{9}\sqrt{45}}$, without realizing this is acute angle between line and **<u>normal</u>** of plane, not the plane itself.

(d) Students who attempted this part generally did well.

5	Andrea participates in a game show. In each round of the game, Andrea responds to a series of questions until she answers a question wrongly or she has answered five questions correctly. She is awarded one point for every correct answer and is awarded an additional 3 bonus points if she answers all five questions correctly. It is given that she answers each question independently and the probability she answers each question correctly is $\frac{4}{5}$.	
(a)	Show that the probability that Andrea obtains 2 points in one round of the game is $\frac{16}{125}$.	[1]
(b)	Find the probability distribution of the points that Andrea obtains in one round of the game.	[2]
(c)	Find the expectation and variance of the points that Andrea obtains in one round of the game.	[2]
(d)	Find the probability that Andrea obtains less than 3 points after two rounds of the game.	[3]

Solution:

5 **(a)** Probability that Andrea obtains 2 points in a round of the game

$$=\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)$$

$$=\frac{16}{125}$$

(b) Let X be the random variable denoting the number of points obtained in a round.

x	0	1	2	3	4	8
$\mathrm{P}(X=x)$	$\frac{1}{5}$	$\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)=\frac{4}{25}$	$\frac{16}{125}$	$\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)=\frac{64}{625}$	$\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)=\frac{256}{3125}$	$\left(\frac{4}{5}\right)^5=\frac{1024}{3125}$

(c) **Method 1**

Using G.C.,

Expectation $=3.67232$ (exact)

Variance $=3.22334^2$

$=10.4$ (3 s.f.)

Method 2

$$\mathrm{E}(X)=0\left(\frac{1}{5}\right)+1\left(\frac{4}{25}\right)+2\left(\frac{16}{125}\right)+3\left(\frac{64}{625}\right)+4\left(\frac{256}{3125}\right)+8\left(\frac{1024}{3125}\right)$$

$$=3.67232$$

$$\mathrm{E}(X^2)=0^2\left(\frac{1}{5}\right)+1^2\left(\frac{4}{25}\right)+2^2\left(\frac{16}{125}\right)+3^2\left(\frac{64}{625}\right)+4^2\left(\frac{256}{3125}\right)+8^2\left(\frac{1024}{3125}\right)$$

$$=23.87584$$

$$\operatorname{Var}(X)=\mathrm{E}\left(X^2\right)-\left[\mathrm{E}(X)\right]^2$$
$$=23.87584-(3.67232)^2$$
$$=10.4 \text{ (to 3 s.f.)}$$

(d) Probability that Andrea obtains less than 3 points after two rounds

$$=\mathrm{P}\left(X_1+X_2<3\right)$$
$$=\mathrm{P}\left(X_1+X_2=0\right)+\mathrm{P}\left(X_1+X_2=1\right)+\mathrm{P}\left(X_1+X_2=2\right)$$
$$=\mathrm{P}\left(X_1=0\cap X_2=0\right)+2\mathrm{P}\left(X_1=0\cap X_2=1\right)+2\mathrm{P}\left(X_1=0\cap X_2=2\right)+\mathrm{P}\left(X_1=1\cap X_2=1\right)$$
$$=\left(\frac{1}{5}\right)^2+2\left(\frac{1}{5}\right)\left(\frac{4}{25}\right)+2\left(\frac{1}{5}\right)\left(\frac{16}{125}\right)+\left(\frac{4}{25}\right)^2$$
$$=\frac{113}{625} \text{ or } 0.1808 \text{ (exact)}$$

Examiner's Report:

5 **(a)** This part was well attempted. Some students did not understand the problem and this affected part **(b)**.

(b) This part was well attempted. The common mistake was to omit the outcome of 0.

(c) Even though the students who did not get the complete probability distribution in part **(b)**, they were often able to obtain full credit in this part as the outcome 0 did not affect the expectation and variance.

Calculation errors were often seen, for example, some students transferred the values wrongly from part **(b)** to part **(c)**, or some students pressed the calculator wrongly.

Some students still did not understand that they were not expected to round off an exact answer (a decimal that terminates or recurs, i.e., a value that can be expressed as a ratio of 2 integers (or in other words, a fraction)).

(d) This part was not well done.

Some students did not know how to start the question. Some students even brought in the binomial or normal distribution which were both not suitable in the context.

Some students listed cases for only two points. No credit was given in this case as it could not be inferred these students interpreted the question correctly (when the number of points obtained for 2 rounds is <u>less than three</u> points).

There were some students who tried to order the two identical outcomes e.g. 0 and 0 or 1 and 1. The two outcomes (e.g. 0 and 0) are indistinguishable from each other. Hence there is no need to multiply by two.

6 An investigation on the number of hours, t, spent on mobile games per day, of 5 students and the marks, m, they obtained in an examination out of 100 marks was carried out.

t	1.1	2.2	3.4	3.8	4.2
m	72	62	54	48	42

(a) Sketch a scatter diagram of the data. [1]

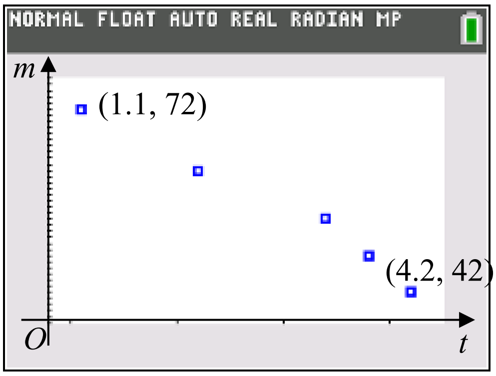
(b) Use your calculator to find the least squares regression line of t on m, and state the product moment correlation coefficient between t and m. [3]

(c) Use the equation of the regression line found in part **(b)** to estimate the number of hours spent on playing mobile games per day if a student obtains 30 marks. Explain whether you would expect this value to be reliable. [2]

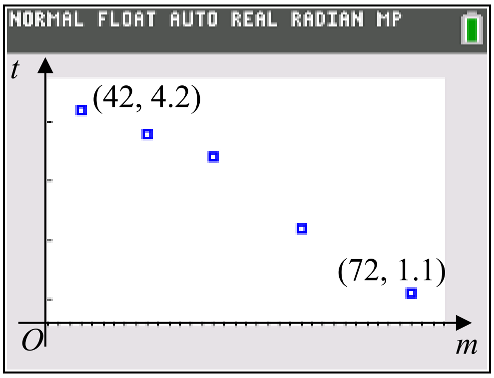
(d) Without any further calculations, explain if the product moment correlation between t and $\frac{m}{100}$ would be different from the value obtained in part **(b)**. [1]

<u>Solution:</u>

(a)



or



(b) Equation of regression line of t on m is $t = -0.107m + 8.88$.
$r = -0.990$ (3 s.f.)

(c) When $m = 30$, estimated time spent $= -0.10684(30) + 8.8805 = 5.68$ hours (3 s.f.)

As $m = 30$ lies outside the data range $42 \le m \le 72$, extrapolation is carried out for estimation. Therefore, the estimate might be unreliable.

(e) There is no change in the product moment correlation coefficient as it is independent of the scale of measurement.

Examiner's Report:

6 **(a)** This part was generally well done. Students could sketch either m against t or t against m. There were no clear independent and dependent variables here.

Some common mistakes:
- Failed to label axes
- Failed to label the minimum and maximum values of t and m
- The relative positions of the data points were wrong (e.g. some points were colinear when they were not)
- Wrong number of data points

(b) Students generally were able to find the correlation coefficient. There was a small percentage of students who found $|r|$, not r itself. There was also a small percentage of students who used the formula to find r – this method took a longer time.

A significant percentage of students were not able to find the equation of the regression line of <u>t on m</u> (in other words, to find an equation of the form $\underline{t = Am + B}$).

A small percentage of students found the equation of the regression line of m on t and made t the subject. This equation is not the same the equation of the regression line of t on m.

(c) For students who had obtained the wrong equation in part **(b)**, they lost the mark for finding t.

For the part on reliability, a small percentage of students used the term 'reliability' interchangeably with the term 'accuracy'. These two are not equivalent.

Common mistakes:
- Some students were not clear about the extrapolation process.
  - Some students said that the value of m was extrapolated.
  - Some students said that the estimate was outside the data range of t.
- Some students thought that the estimated value would be reliable as long as the correlation coefficient was close to one.
- Some students were not precise with their language, for example, the value is not reliable as <u>**it**</u> (refers to t-value, not m-value) is not within the data range $42 \le m \le 72$.

(d) This part was well attempted. A small percentage of students failed to <u>provide a reason</u> (explain) for their answer – they simply stated no.

r is independent of any linear transformation (i.e. translation and scaling). The strength and direction of the linear relationship between d and m is not affected by these linear transformations. The degree of scatter remains unchanged.

Common mistakes:
- The residuals were a constant.
- The equation of the regression line remained unchanged.
- The gradient of the regression line remained unchanged.
- Only the gradient of the regression line changed.

7	Aran is a telemarketer working in a large city. Each day, he speaks to 50 potential customers and on average, he successfully makes a sale 6% of the time. The number of successful sales each day is denoted by the random variable T.		
	(a)	State, in context, two assumptions needed for T to be well modelled by a binomial distribution.	[2]
	Assume now that T has a binomial distribution.		
	(b)	Find the probability that Aran makes at least 5 successful sales on a randomly chosen day.	[2]
	(c)	Aran receives a bonus if he makes at least 5 successful sales a day. Find the probability that he receives a bonus on 2 days out of a randomly chosen 5-day work week.	[2]
	Aran is part of a team of 100 similar telemarketers.		
	(d)	Estimate the probability that on a randomly chosen day, the team makes more than 3.5 successful sales on average.	[3]

Solution:

(a) The probability of successfully making a sale is constant at 0.06 for every customer.
The success of making a sale with one customer is independent of the next customer.

(b)
$$T \sim \mathrm{B}(50, 0.06)$$
$$P(T \geq 5) = 1 - P(T \leq 4)$$
$$= 0.179$$

(c) Let X be the random variable denoting the number of days Aran receives a bonus, out of 5.
$$X \sim \mathrm{B}(5, 0.17940)$$
$$P(X = 2) = 0.178$$

(d)
$$\mathrm{E}(T) = 50 \times 0.06$$
$$= 3$$

$$\mathrm{Var}(T) = 50 \times 0.06 \times (1 - 0.06)$$
$$= 2.82$$

Since $n = 100$ is sufficiently large, by Central Limit Theorem, $\overline{T} \sim \mathrm{N}\left(3, \frac{2.82}{100}\right)$ approximately.

$$\mathrm{P}(\overline{T} > 3.5) = 0.00145$$

Examiner's Report:

(a) Examples of wrong assumptions pertain mostly to independence (problematic parts are underlined):

- "The <u>probability</u> that a sales attempt is successful is <u>independent</u> of other sales attempts."
  Problem: Events can be independent of each other but not probabilities.
- "The <u>number of</u> successful sales <u>each day</u> is independent of <u>other days</u>."
  Problem: It is not about the *number of* successful sales but a sale being successful that matters.
- "Whether a sales attempt is successful on <u>one day</u> is independent of <u>other days</u>.
  Problem: The idea of independence pertains to customers, not days per se.
- "Aran's sales attempt with a customer is either successful or unsuccessful."
  Problem: Even if we were to assume that a follow-up call might be made after the first call, it does not change the fact that the first call must still have ended either with a successful sale or without a successful sale.

Conceptually dubious assumptions (but marks not deducted out of leniency). Kindly highlight to students to avoid such phrasing.

- "The success of making a sale with one customer is independent of <u>the success of making a sale with other customers</u>." The correct version should be, "The success of making a sale with one customer is independent of <u>other customers</u>."

- "The probability of successfully making a sale is <u>equal</u> for every customer". The word 'equal' does not necessarily mean 'constant'. The correct version should be, "The probability of successfully making a sale is <u>constant</u> for every customer."

(b) This part was well done. Among the small number of them who got it wrong, almost all of them mistook binomcdf to be applicable for $\mathrm{P}(T<t)$ rather than $\mathrm{P}(T \le t)$.

(c) This part was also quite well done. It is noteworthy that a sizeable number of students correctly solved it like a typical probability question e.g. $\binom{5}{2}(0.17940)^2(1-0.17940)^3$ without making an explicit reference to binomial pdf concept. Unfortunately, quite a number of them forgot to include $\binom{5}{2}$.

(d) This question was not done well. Only about 1/10 of the cohort got it right. Many students solved this question without using the idea of approximation (ie. without using Central Limit Theorem). Below is the erroneous answer.

Let T denote the number of successful sales made in a day by a randomly chosen telemarketer.

$$T \sim \mathrm{B}(50, 0.06)$$

$$T_1 + T_2 + T_3 + \ldots T_{100} \sim \mathrm{B}(5000, 0.06)$$

$$\begin{aligned} P(\bar{T} > 3.5) &= P\left(\frac{T_1 + T_2 + T_3 + \ldots T_{100}}{100} > 3.5\right) \\ &= P(T_1 + T_2 + T_3 + \ldots T_{100} > 350) \\ &= 1 - P(T_1 + T_2 + T_3 + \ldots T_{100} \le 350) \\ &= 0.00163 \text{ (3s.f)} \end{aligned}$$

Kindly highlight to students that the above method would have been right if the question had asked for exact answers. Unfortunately, it is wrong in the context of this question because the instruction was clearly to "estimate" the probability.

8	A giant panda's daily diet consists almost entirely of leaves, stems and shoots of various bamboo species. A panda foundation claims that giant pandas consume on average at most 25 kg of bamboo every day to meet their energy needs. A random sample of 40 pandas is taken and the mass, x kilograms, of bamboo consumed per day by this group of pandas is summarised below. $$\sum(x-25)=80.4,\ \sum(x-25)^2=1893.5$$		
	(a)	Calculate the unbiased estimates of the population mean and variance for the mass of bamboo consumed per day by pandas.	[2]
	(b)	State the hypotheses that can be used to test whether the panda foundation's claim is valid. By finding the p-value, carry out the test at 2% level of significance, giving your conclusion in the context of the question.	[4]
	(c)	State, in context, the meaning of the p-value found in part **(b).**	[1]
	Another random sample of n pandas is selected by zookeepers in a certain country and the sample mean was found to be 26.9 kg per day. A test was carried out at 2% level of significance.		
	(d)	By using the unbiased estimate of the population variance found in **(a)**, find the least possible value of n such that this new sample achieves a different conclusion from that in **(b)**.	[4]

Solution:

(a) unbiased estimate of the population mean of X

$$=\frac{\sum(x-25)}{40}+25$$

$$=\frac{80.4}{40}+25$$

$$=27.01 \text{ (exact)}$$

unbiased estimate of the population variance of X

$$=\frac{1}{40-1}\left[\sum(x-25)^2-\frac{\left[\sum(x-25)\right]^2}{40}\right]$$

$$=\frac{1}{39}\left[1893.5-\frac{80.4^2}{40}\right]$$

$$=44.408 \text{ (5 s.f.)} = 44.4 \text{ (3 s.f.)}$$

(b) Let X be the random variable denoting the mass (in kg) of bamboo consumed per day by a panda.

$H_0 : \mu = 25$

$H_1 : \mu > 25$

Under H_0, since $n = 40$ is sufficiently large, by Central Limit Theorem,

$\overline{X} \sim N\left(25, \frac{44.408}{40}\right)$ approximately.

$z_{test} = \frac{27.01 - 25}{\sqrt{\frac{44.408}{40}}} = 1.9076$ (5 s.f.)

p-value = 0.028219 (5 s.f.) = 0.0282 (3 s.f.)

Since p-value = 0.0282 > 0.02, we do not reject H_0 and conclude that we have insufficient evidence at 2% significance level that the mean mass (in kg) of bamboo consumed per day by a panda is more than 25 kg.

(c)

The probability of getting a sample of 40 giant pandas whose mean mass of bamboo consumed per day is more than or equal to 27.01 kg when the population mean is 25 kg is 0.0282.

(d) $H_0 : \mu = 25$

$H_1 : \mu > 25$

Under H_0, since n is sufficiently large, by Central Limit Theorem,

$\overline{X} \sim N\left(25, \frac{44.408}{n}\right)$ approximately.

$z_{test} = \frac{26.9 - 25}{\sqrt{\frac{44.408}{n}}}$

Different conclusion from (c) so we reject H_0.

$z_{test} \geq z_{critical}$

$\frac{26.9 - 25}{\sqrt{\frac{44.408}{n}}} \geq 2.0537$ (5 s.f.)

$n \geq 51.883$ (5 s.f.)

$n \geq 52$ since $n \in \mathbb{Z}^+$

Least possible value of n is 52.

Alternative method

$$\frac{26.9-25}{\sqrt{\frac{44.408}{n}}} \geq 2.0537 \text{ (5 s.f.)}$$

Using G.C.,

n	$z_{test}=\frac{26.9-25}{\sqrt{\frac{44.408}{n}}}$	
51	2.0361	< 2.0537
52	2.0560	> 2.0537
53	2.0757	> 2.0537

Least possible value of n is 52.

Examiner's Report:

Generally not well-attempted.

(a) Most students could only do this part using the correct formula although a few had poor notations and rounded off the sample mean to 3 s.f. when it is an exact value for this question. Some students only found $\frac{\sum(x-25)}{40}=2.01$, or misinterpreted the mean to be 25 and used $\frac{\sum(x-25)^2}{40-1}$.

(b) Many did not state the correct hypothesis and used a 2-tailed test. Some students also did not state the distribution of $\overline{X}$ under H_0 correctly, and/or calculated the p-value wrongly with their GC. A number of students were confused with the conclusion and made the following mistakes:

- Since $p > 0.02$, we reject H_0 (we should only reject H_0 when $p \leq 0.02$).
- Since $p > 0.02$, we do not reject H_0 and conclude that the foundation's claim is valid at 2% level of significance (the foundation's claim is stated in H_0 for this question which is rejected based on the p-value, and we should only state H_1 in our conclusion i.e. mean mass of bamboo consumed per day by a panda is more than 25 kg or that the foundation's claim is invalid).

(c) Only a few students were able to define the p-value correctly. Many wrote the definition for the level of significance. Also accepted a few students who wrote: p-value is the smallest level of significance at which H_0 is rejected under the assumption that H_0 is true where H_0 claims that the population mean mass is 25 kg.

(d) Marked this part based on their response to part (b) and did not penalize their hypothesis statements and distribution of $\overline{X}$ under H_0 again. Some students wrote the correct inequality but did not solve it correctly to state the least value of n. Some students also misinterpreted the question and used a different pair of H_0 and H_1 instead of using a different conclusion from their part (b).

9 A student takes the bus to school every morning. During peak hours, the traffic is either moderate or heavy. The time taken, in minutes, for a randomly chosen bus journey through moderate or heavy traffic follow independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard deviation
Moderate Traffic	14	σ
Heavy Traffic	27	2.5

(a) Given that 20% of the student's bus journeys through moderate traffic exceed 15 minutes, show that $\sigma = 1.19$ correct to 3 significant figures. [2]

(b) The probability of a randomly chosen bus journey through heavy traffic taking more than a minutes is larger than 0.9. Find the range of values of a. [2]

(c) The student is late for school if his bus journey takes more than 32 minutes, assuming he leaves home at the same time as usual. He gets a demerit point if he is late on any two consecutive mornings. For three consecutive mornings, the student's bus journeys encountered heavy traffic.
Find the probability that he gets a demerit point. [3]

(d) Find the probability that the time taken for two randomly chosen bus journeys through moderate traffic differs by less than 0.5 minutes. [2]

(e) Find the probability that the total time taken for two randomly chosen bus journeys through heavy traffic is greater than four times the time taken for a randomly chosen bus journey through moderate traffic. [3]

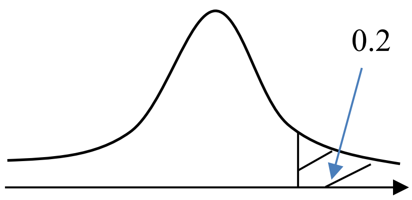
Solution:

(a)

Let M and H be random variables denoting the time taken, in minutes, of a randomly chosen bus journey during moderate and heavy traffic respectively.

$\therefore M \sim \mathrm{N}\left(14,\ \sigma^2\right)$

$\mathrm{P}(M > 15) = 0.2$



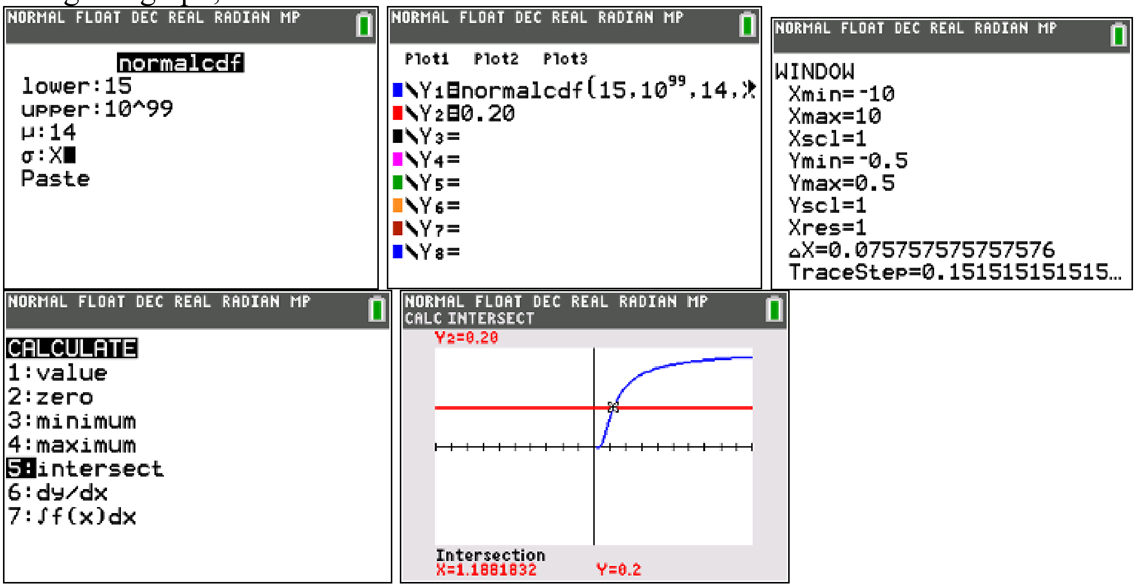
$\mathrm{P}(Z > \frac{15-14}{\sigma}) = 0.2$

$\frac{15-14}{\sigma} = 0.84162$

$\sigma = 1.1882 = 1.19$ (3 s.f.) (Shown)

Method 2

Using GC graph,



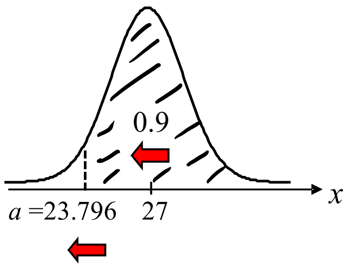
The point of intersection is (1.1881832, 0.2)

Hence, $\sigma = 1.1882 = 1.19$ (3 s.f.) (Shown)

(b)

$H \sim \mathrm{N}\left(27,\ 2.5^2\right)$

$\mathrm{P}(H > a) > 0.9$



Let $\mathrm{P}(H > a) = 0.9$

From GC, $a = 23.796$

Since $\mathrm{P}(H > a) > 0.9$,

$0 < a < 23.796$

$\therefore 0 < a < 23.7$ (3 s.f.)

Note: $\mathrm{P}(H > 23.7) = 0.90658$

$\mathrm{P}(H > 23.8) = 0.89973$]

Method 2

$H \sim \mathrm{N}\left(27,\ 2.5^2\right)$

$\mathrm{P}(H > a) > 0.9$

$$\mathrm{P}\left(Z > \frac{a-27}{2.5}\right) > 0.9$$

$$\frac{a-27}{2.5} < -1.28155$$

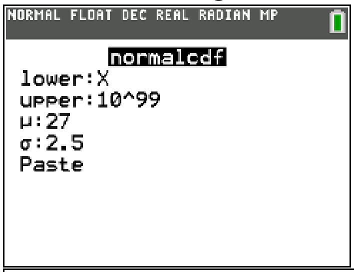
$0 < a < 23.796$

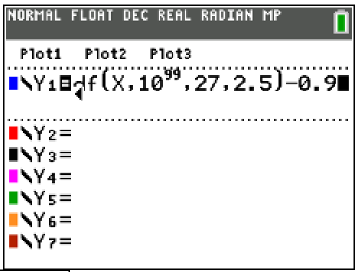
$\therefore 0 < a < 23.7$ (3 s.f.)

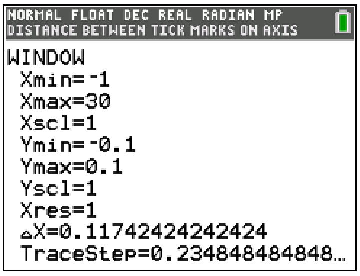
<u>Method 3</u>

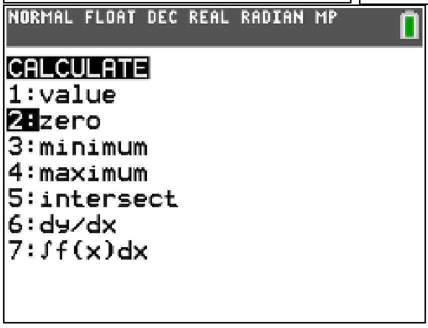
Using GC, sketch the graph $y = \mathrm{P}(H > a) - 0.9$

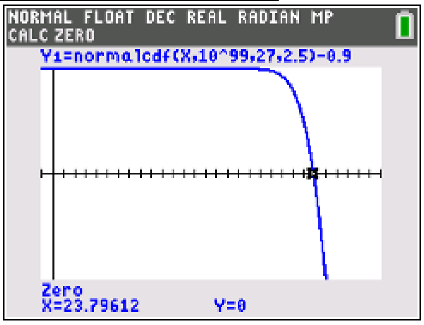
Then look for region > 0











$\therefore 0 < a < 23.7$ (3 s.f.)

(c)

P(student gets a demerit point)

=P(consecutive bus journeys take more than 32 minutes for 3 days)

=P(1st day not late, 2nd day late, 3$^{\text{rd}}$ day late)

+ P(1st day late, 2nd day late, 3$^{\text{rd}}$ day not late)

+ P(all 3 days late)

$=\mathrm{P}(H < 32, H > 32, H > 32) + \mathrm{P}(H > 32, H > 32, H < 32) + \mathrm{P}(H > 32, H > 32, H > 32)$

$= \mathrm{P}\left(H < 32\right)\times\left[\mathrm{P}\left(H > 32\right)\right]^2 + \left[\mathrm{P}\left(H > 32\right)\right]^2 \times \mathrm{P}\left(H < 32\right) + \left[\mathrm{P}\left(H > 32\right)\right]^3$

$= (0.97725)(0.022750)^2 \times 2 + (0.022750)^3$

$= 0.0010234$

≈ 0.00102 (3 s.f.)

<u>Alternative Method</u>

P(student gets a demerit point)

=P(consecutive bus journeys take more than 32 minutes for 3 days)

=P(1st day not late, 2nd day late, 3$^{\text{rd}}$ day late)

+ P(1st day late, 2nd day late)

$=\mathrm{P}(H < 32, H > 32, H > 32) + \mathrm{P}(H > 32, H > 32)$

$= \mathrm{P}\left(H < 32\right)\times\left[\mathrm{P}\left(H > 32\right)\right]^2 + \left[\mathrm{P}\left(H > 32\right)\right]^2$

≈ 0.00102 (3 s.f.)

(d)

$M_1 - M_2 \sim \left(14-14,\, 1.19^2 + 1.19^2\right)$

$M_1 - M_2 \sim \mathrm{N}(0,\ 2.8322)$

$\mathrm{P}\left(\left|M_1 - M_2\right| < 0.5\right)$

$= \mathrm{P}\left(-0.5 < M_1 - M_2 < 0.5\right)$

$= 0.23361$

≈ 0.234 (3 s.f.)

OR

$M_1 - M_2 \sim \left(14-14,\, 1.1882^2 + 1.1882^2\right)$

$M_1 - M_2 \sim \mathrm{N}(0,\ 2.8236)$

$\mathrm{P}\left(\left|M_1 - M_2\right| < 0.5\right)$

$= \mathrm{P}\left(-0.5 < M_1 - M_2 < 0.5\right)$

$= 0.23396$

≈ 0.234 (3 s.f.)

(e)

$H_1 + H_2 - 4M \sim \mathrm{N}\left(2\times 27 - 4\times 14,\ 2\times 2.5^2 + 4^2 \times 1.19^2\right)$

$H_1 + H_2 - 4M \sim \mathrm{N}(-2,\ 35.1576)$

$\mathrm{P}\left(H_1 + H_2 > 4M\right)$

$= \mathrm{P}\left(H_1 + H_2 - 4M > 0\right)$

$= 0.36794$

≈ 0.368 (3 s.f.)

OR

$H_1 + H_2 - 4M \sim \mathrm{N}\left(2\times 27 - 4\times 14,\ 2\times 2.5^2 + 4^2 \times 1.1882^2\right)$

$H_1 + H_2 - 4M \sim \mathrm{N}(-2,\ 35.089)$

$\mathrm{P}\left(H_1 + H_2 > 4M\right)$

$= \mathrm{P}\left(H_1 + H_2 - 4M > 0\right)$

$= 0.36782$

≈ 0.368 (3 s.f.)

<u>Examiner's Report:</u>

(a)

This part was generally well-done except for those who did not study this topic.

Students using GC to get the answer must show the working or at least first give the value to 5 s.f. since this is a "show" question.

(b)
This part is not well-done.
Main problems:
- most did not sketch the bell-shaped graph and blindly gave the answer as $a > 23.7$
- most did not consider the context of the question and left out the lower bound zero
- most forgot that invNorm can be used for other distributions besides Z~N(0,1) and hence used the longer standardization method
- some used the GC table method without considering that *a* may not be an integer and hence did not change the step-size to 2 d.p. and gave an integer answer. Students should have taken the cue from the variance that minutes in this context are not integer values.

(c)
This part was not well-done as students did not think clearly about the possible cases.
Main problems:
- some used Binomial Distribution B(3, prob of late) without considering that the case L L' L does not result in a demerit point.
- some careless mistakes in writing the prob of being late as 0.2275 instead of 0.002275
- some considered the prob of two consecutive late days as P(X>32) + P(X>32) instead of multiply

(d)
Most students could do this part:
Main problems:
- did not understand the question clearly and did not consider modulus of the difference in variables
- made careless mistakes by using 1.1882 instead of 1.1882^2 when calculating the new variance
- a small minority added instead of subtract the random variables
- a small minority did not know how to break the modulus and wrote it as
$\mathrm{P}\left(\left|M_1 - M_2\right| < 0.5\right) = \mathrm{P}\left(M_1 - M_2 < 0.5\right) + \mathrm{P}\left(M_1 - M_2 < -0.5\right)$ or a variation of this.
Student should draw a number line to help themselves visualize this.
- did not do their working to 5 s.f. although this is not penalized

(e)
Most students could do this part. Most were correct in calculating the new mean but the new variance had the usual variety of errors.
Main problems:
- some wrote $2H$ instead of $H + H$, leading in errors in calculating the new variance
- careless in writing the variance as standard deviation without the square

10	**(a)**	In a certain university, there are 110 Co-Curricular Activities (CCAs) clustered into 4 categories. There are 22 Arts and Culture CCAs, 16 Community Service CCAs, 38 Physical Sports CCAs and 34 Special Interest CCAs.		
		(i)	Albert wishes to find out about approaches to training of the Physical Sports CCAs, so he sends a questionnaire to 22 Physical Sports CCAs. Explain whether these 22 Physical Sports CCAs form a sample or a population.	[1]
		(ii)	Benedict wishes to investigate the level of student engagement in CCAs, but does not want to obtain the detailed information necessary from all 110 CCAs. Explain how he should carry out his investigation, and why he should do the investigation in this way.	[2]
		(iii)	Find the number of different possible samples of 16 CCAs, with 4 CCAs chosen from each category.	[2]
	(b)	The probability of events A, B and C occurring are given by $\mathrm{P}(A)=\frac{8}{25}$, $\mathrm{P}(B)=\frac{61}{100}$ and $\mathrm{P}(C)=\frac{7}{20}$ respectively. It is also known that $\mathrm{P}(A\cap C)=\frac{11}{100}$, $\mathrm{P}(B\mid C)=\frac{13}{35}$ and $\mathrm{P}(A\cap B\cap C)=\frac{7}{100}$.		
		(i)	Find $\mathrm{P}(B\cap C)$.	[2]
		(ii)	If events A and B are independent, find $\mathrm{P}(A\cap B\cap C')$.	[2]
		(iii)	If events A and B are **not** known to be independent, find the greatest and least possible values of $\mathrm{P}(A\cap B\cap C')$.	[4]

Solution:

(ai) These 22 CCAs form a sample since this number is less than the population size of Physical Sports CCAs which is 38.

(aii) Benedict should select a <u>random sample</u> from the 110 CCAs: every CCA has an equal probability to be selected to be in the sample and the CCAS are selected independently. (If define without stating "random", need 2 full points.)
A random sample helps to <u>avoid bias</u>.

(aiii) Number of samples $= {}^{22}C_4 \times {}^{16}C_4 \times {}^{38}C_4 \times {}^{34}C_4$

$= 4.56\times10^{16}$ (3 sf)

(bi) $\mathrm{P}(B\mid C)=\frac{13}{35}$

$$\frac{\mathrm{P}(B\cap C)}{\mathrm{P}(C)}=\frac{13}{35}$$

$$\frac{\mathrm{P}(B\cap C)}{\frac{7}{20}}=\frac{13}{35}$$

$$P(B \cap C) = \frac{13}{100}$$

(bii) $P(A \cap B) = P(A) \times P(B)$

$$= \frac{8}{25} \times \frac{61}{100}$$

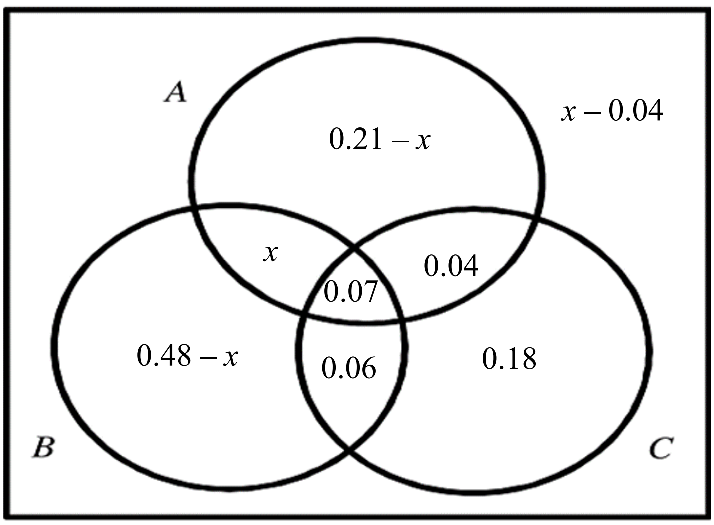
$$= \frac{122}{625}$$

$P(A \cap B \cap C')$

$$= \frac{122}{625} - \frac{7}{100}$$

$$= \frac{313}{2500}$$

(biii)



Let $\mathrm{P}(A \cap B \cap C') = x$ where $0 \le x \le 1 \ldots (1)$

Then

$$\mathrm{P}(A \cap B' \cap C')$$
$$= \frac{8}{25} - \frac{11}{100} - x$$
$$= \frac{21}{100} - x$$

$$\mathrm{P}(A' \cap B \cap C')$$
$$= \frac{61}{100} - \frac{13}{100} - x$$
$$= \frac{12}{25} - x$$

$$\mathrm{P}(A \cup B \cup C)$$
$$= \mathrm{P}(C) + \mathrm{P}(A \cap B' \cap C') + \mathrm{P}(A' \cap B \cap C') + \mathrm{P}(A \cap B \cap C)$$
$$= \frac{7}{20} + \left(\frac{21}{100} - x\right) + \left(\frac{12}{25} - x\right) + x$$
$$= \frac{26}{25} - x$$

Consider the probabilities $\mathrm{P}(A \cap B' \cap C')$, $\mathrm{P}(A' \cap B \cap C')$ and $\mathrm{P}(A \cup B \cup C)$,

$0 \le \frac{21}{100} - x \le 1$ and $0 \le \frac{12}{25} - x \le 1$ and $0 \le \frac{26}{25} - x \le 1$

$-\frac{79}{100} \le x \le \frac{21}{100} \ldots (2)$ $\quad -\frac{13}{25} \le x \le \frac{12}{25} \ldots (3)$ $\quad \frac{1}{25} \le x \le \frac{26}{25} \ldots (4)$

Taking intersection of inequalities (1) to (4), $\frac{1}{25} \le x \le \frac{21}{100}$

min $\mathrm{P}(A \cap B \cap C') = \frac{1}{25}$ and max $\mathrm{P}(A \cap B \cap C') = \frac{21}{100}$

Examiner's Report:

(a)(i) Most students were able to see that the 22 CCAs form a sample. However, there were quite a handful who were not able to state clearly the reason for that. Some compared to the total number of 110 CCAs. Some were vague and just said CCAs. Some did not even give an explanation.

(a)(ii) This part was quite poorly done. Many students tried to explain in detail how random sampling could be carried out but did not state clearly that the sample should be selected randomly. Those who did not state "random" but used "equal probability" and "independent" usually missed out on 1 of the definitions.
A lot of students did not mention "avoid bias" or tried to explain about biasness but was incomplete.

(a)(iii) Most students were able to identify the need to use ${}^{22}C_4, {}^{16}C_4, {}^{38}C_4, {}^{34}C_4$. However, many changed from multiplication to addition probably because they found the answer to be too big and thought it might not be possible. There were some students who used !, not understanding the fact that there is no need for arrangement in this question.