ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS | **HIGHER 2** | **9758/01**

Wednesday | **28 August 2019** | **3 hrs**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:____(_____) **C.G.:** __________

TUTOR'S NAME: ____

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: _______________________

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

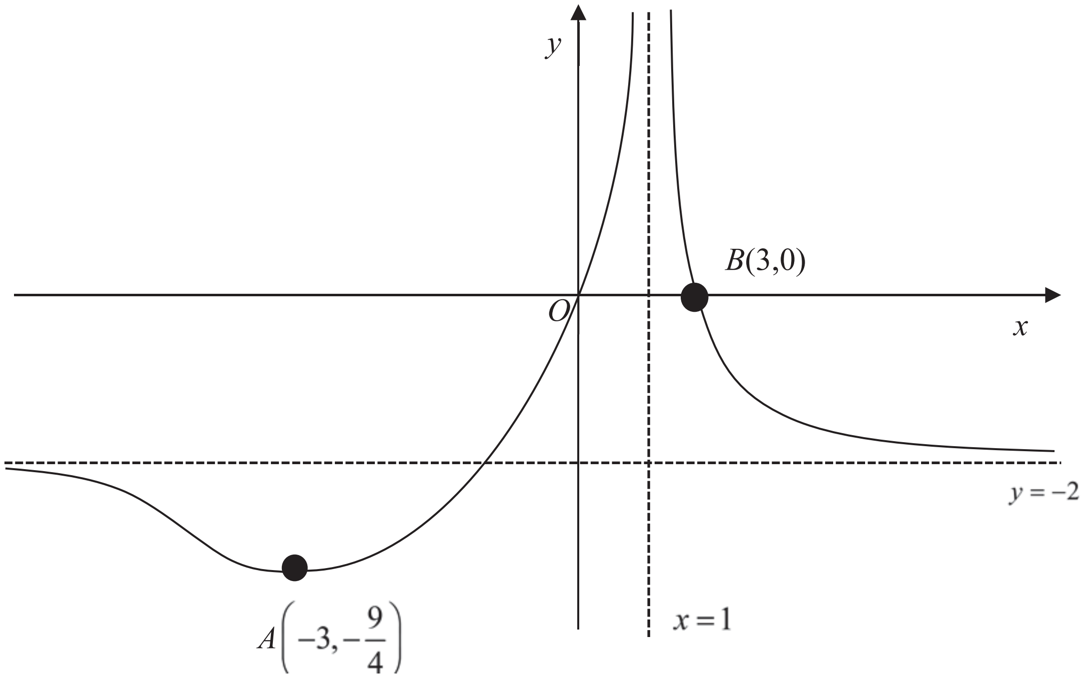
The number of marks is given in brackets [ ] at the end of each question or part question.

Question	1	2	3	4	5	6	7	8	9	10	TOTAL
Marks											
	6	7	10	9	10	10	12	12	12	12	100

This document consists of **27** printed pages and **1** blank page including this page.

[Turn Over]

1 The diagram below shows the graph of $y = 2\mathrm{f}(3-x)$. The graph passes through the origin O, and two other points $A\left(-3,-\frac{9}{4}\right)$ and $B(3,0)$. The equations of the vertical and horizontal asymptotes are $x=1$ and $y=-2$ respectively.



(a) State the range of values of k such that the equation $\mathrm{f}(3-x)=k$ has exactly two negative roots. **[1]**

(b) By stating a sequence of two transformations which transforms the graph of $y=2\mathrm{f}(3-x)$ onto $y=\mathrm{f}(3+x)$, find the coordinates of the minimum point on the graph of $y=\mathrm{f}(3+x)$. Also, write down the equations of the vertical asymptote(s) and horizontal asymptote(s) of $y=\mathrm{f}(3+x)$. **[5]**

2 **(i)** On the same axes, sketch the curves with equations $y=\left|2x^2+6x+4\right|$ and $y=3-4x$, indicating any intercepts with the axes and points of intersection. Hence solve the inequality $3-4x<\left|2x^2+6x+4\right|$. **[4]**

(ii) Find the exact area bounded by the graphs of $y=3-4x$, $y=\left|2x^2+6x+4\right|$, $x=-3$ and $x=-1$. **[3]**

[Turn Over

3 The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \frac{x-4}{x-1}\ ,\ x \in \mathbb{R}, x \neq 1$$

$$\mathrm{g}: x \mapsto x^2 + 2x + 2\ ,\ x \in \mathbb{R}, x > -1$$

(i) Show that f has an inverse. **[1]**

(ii) Show that $\mathrm{f} = \mathrm{f}^{-1}$ and hence evaluate $\mathrm{f}^{101}(101)$. **[5]**

(iii) Prove that the composite function fg exists and find its range. **[4]**

4 It is given that $y = \sqrt{e^{\cos x}}$.

(i) Show that $2\frac{dy}{dx} + y\sin x = 0$. Hence find the Maclaurin's expansion of y up to and including the term in x^2. **[4]**

Deduce the series expansion for $e^{\sin^2\left(\frac{x}{2}\right)}$ up to and including the term in x^2. **[3]**

(ii) Using the series expansion from **(ii)**, estimate the value of $\int_0^{\sqrt{2}} e^{\sin^2\left(\frac{x}{2}\right)} dx$ correct to 3 decimal places. **[2]**

5 A curve C is determined by the parametric equations

$$x = at^2, \quad y = 2at, \quad \text{where } a > 0.$$

(i) Sketch C. **[1]**

(ii) Find the equation of the normal at a point P, with non-zero parameter p. **[2]**

Show that the normal at the point P meets C again at another point Q, with parameter q, where $q = -p - \frac{2}{p}$. Hence show that $|PQ|^2 = \frac{16a^2}{p^4}(p^2+1)^3$. **[4]**

(iii) Another point R on C with parameter r, is the point of intersection of C and the circle with diameter PQ. By considering the gradients of PR and QR, show that $p^2 - r^2 + 2\left(\frac{r}{p}\right) = 2$. **[3]**

[Turn Over

6 **(a)** **(i)** Express $p=-1-\sqrt{3}\mathrm{i}$ in exponential form. **[1]**

(ii) Without the use of a calculator, find the two smallest positive whole number values of n for which $\dfrac{(p^*)^n}{\mathrm{i}p}$ is a purely imaginary number. **[4]**

(b) Without the use of a calculator, solve the simultaneous equations

$$z-w+6+7\mathrm{i}=0 \text{ and } 2w-\mathrm{i}z^*-19-3\mathrm{i}=0,$$

giving z and w in the form $x+y\mathrm{i}$ where x and y are real. **[5]**

7 The position vectors, relative to an origin O, at time t in seconds, of the particles P and Q are $(\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + 0\mathbf{k}$ and $\left(\frac{3}{2}\cos\left(t+\frac{\pi}{4}\right)\right)\mathbf{i}+\left(3\sin\left(t+\frac{\pi}{4}\right)\right)\mathbf{j}+$ $\left(\frac{3\sqrt{3}}{2}\cos\left(t+\frac{\pi}{4}\right)\right)\mathbf{k}$ respectively, where $0\le t\le 2\pi$.

(i) Find $|\overrightarrow{OP}|$ and $|\overrightarrow{OQ}|$. **[2]**

(ii) Find the cartesian equation of the path traced by the point P relative to the origin O and hence give a geometrical description of the motion of P. **[2]**

(iii) Let θ be the angle POQ at time t. By using scalar product, show that

$$\cos\theta=\frac{3\sqrt{2}}{8}-\frac{1}{4}\cos\left(2t+\frac{\pi}{4}\right).$$ **[3]**

(iv) Given that the length of projection of $\overrightarrow{OQ}$ onto $\overrightarrow{OP}$ is $\sqrt{5}$ units, find the acute angle θ and the corresponding values of time t. **[5]**

[Turn Over

8 **(a)** Meredith owns a set of screwdrivers numbered 1 to 17 in decreasing lengths. The lengths of the screwdrivers form a geometric progression. It is given that the total length of the longest 3 screwdrivers is equal to three times the total length of the 5 shortest screwdrivers. It is also given that the total length of all the odd-numbered screwdrivers is 120 cm. Find the total length of all the screwdrivers, giving your answer correct to 2 decimal places. **[4]**

(b) Meredith is building a DIY workbench, and she needs to secure several screws by twisting them with a screwdriver drill. Each time Meredith presses the button on the drill, the screw is rotated clockwise by u_n radians, where n is the number of times the button is pressed. Each press rotates the screw more than the previous twist, and on the first press, the screw is rotated by $\frac{2\pi}{3}$ radians. It is given that $\cos u_{n+1} = \frac{1}{2}\cos u_n - \frac{\sqrt{3}}{2}\sin u_n$ and $\sin u_{n+1} = \frac{1}{2}\sin u_n + \frac{\sqrt{3}}{2}\cos u_n$ for all $n \geq 1$.

(i) By considering $\cos\left(u_{n+1} - u_n\right)$ or otherwise, and assuming that the increase in rotation in successive twists is less than π radians, prove that $\{u_n\}$ is an arithmetic progression with common difference $\frac{\pi}{3}$ radians. **[3]**

(ii) Each screw requires at least 25 complete revolutions to ensure that it does not fall out. Find the minimum number of times Meredith has to press the drill button to ensure the screw is fixed in place. **[3]**

(iii) The distance the screw is driven into the workbench on the nth press of the drill, d_n, is proportional to the angle of rotation u_n. If the total distance the screw is driven into the workbench after 21 presses is 144mm, find the distance the screw is driven into the workbench on the first press. **[2]**

[Turn Over

9 **(i)** By using the substitution $x = 15\sin\theta + 15$, find the $\int_0^{15} \sqrt{15^2 - (x-15)^2}\,dx$ leaving your answer in terms of π . **[5]**

(ii) A sculptor decides to make a stool by carving from a cylindrical block of base radius

30 cm and height 35 cm using a 3D carving machine. The design of the stool based on the piecewise function $g(x)$ where

$$g(x) = \begin{cases} 30 - \frac{2}{3}\sqrt{15^2 - (x-15)^2} & \text{for } 0 \le x \le 15 \\ 30 & \text{for } 15 < x < 35. \end{cases}$$

The figure below shows the 3D image of the stool after the design ran through a 3D machine simulator.

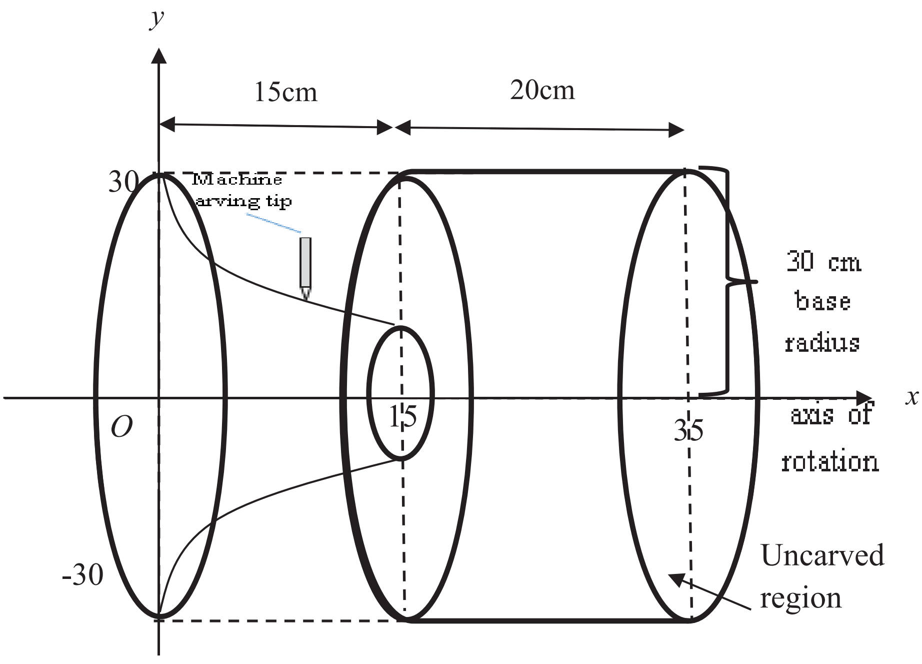


Figure 1: 3D Image of the stool

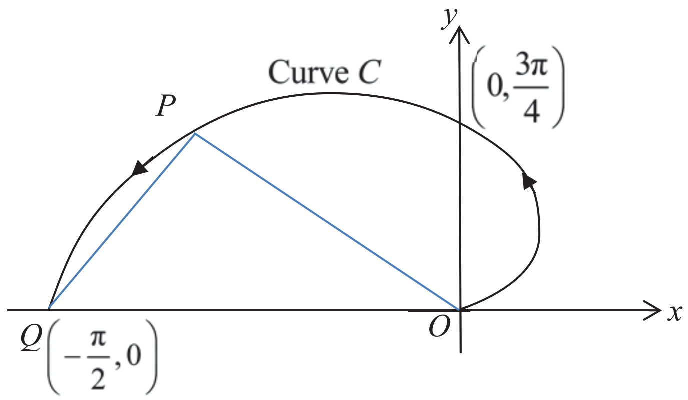
(a) Find the exact area bounded by the curve $y = g(x)$, $x = 15$ and the x-axis and y-axis. **[3]**

(b) The curve defined by the function $y = g(x)$ when rotated 2π radians about the x–axis gives the shape of the stool that the sculptor desires, as shown in Figure 1.
Find the exact volume of the stool. **[4]**

[Turn Over

10 The diagram below shows a curve C with parametric equations given by

$$x = \theta\cos 2\theta, \quad y = 3\theta\sin 2\theta, \qquad \text{for } 0 \le \theta \le \frac{\pi}{2}.$$



The area bounded by curve C and the x-axis is a plot of land which is owned by a farmer Mr Green where he used to grow vegetables. Over the past weeks, vegetables were mysteriously missing and Mr Green decided to install an automated moving surveillance camera which moves along the boundary of the farmland in an anticlockwise direction along the curve C starting from point O and ending at point Q before moving in a clockwise direction along the curve C back to O.

At a particular instant t seconds, the camera is located at a point P with parameter θ on the curve C. You may assume that the camera is at O initially. The camera should be orientated so that the field of view should span from O to Q exactly as shown.

(i) Assuming that the camera is moving at a speed given by $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.01$ radians/sec, find the rate of change of the area of the triangle OPQ, A when $\theta = \frac{\pi}{6}$. **[4]**

(ii) Using differentiation, find the value of θ that would maximize A and explain why A is a maximum for that value of θ. Hence find this value of A and the coordinates of the point P corresponding to the location of the camera at that instant. **[5]**

(iii) For the image to be 'balanced', triangle OPQ is isosceles. Find the coordinates of the location where the camera should be. **[3]**

End of Paper

ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAM

MATHEMATICS **HIGHER 2** **9758/02**

Monday **16 September 2019** **3 hr**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

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TUTOR'S NAME: ____

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M												
	9	11	12	8	6	6	8	8	8	12	12	100

This document consists of **26** printed pages including this page.

[Turn Over]

Section A: Pure Mathematics [40 marks]

1 The curve C has equation $y=\dfrac{x^2+ax+b}{x+c}$ where a, b and c are constants. The line $x=-1$ is an asymptote to C and the range of values that y can take is given by $y\le 0$ or $y\ge 4$.

(i) State the value of c and show that $a=4$ and $b=4$. **[4]**

(ii) Sketch C indicating clearly the equations of the asymptotes and coordinates of the turning points and axial intercepts. **[3]**

(iii) State the coordinates of the point of intersection of the asymptotes. Hence state the range of values of k such that the line $y=k(x+1)+2$ cuts C at two distinct points. **[2]**

2 A water tank contains 1 cubic meter of water initially. The volume of water in the tank at time t seconds is V cubic metres. Water flows out of the tank at a rate proportional to the volume of water in the tank and at the same time, water is added to the tank at a constant rate of k cubic metres per second.

(i) Show that $\dfrac{dV}{dt}=k\left(1-\dfrac{a}{k}V\right)$, where a is a positive constant. **[2]**

Hence find V in terms of t. **[5]**

(ii) Sketch the solution curve for V against t, such that

(a) $a<k$.

(b) $a>k$.

For cases **(a)** & **(b)**, describe and explain what would happen to the volume of water, V in the tank eventually. **[4]**

3 The plane π_1 has equation $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=10$, and the coordinates of A and B are (2, a, 2), (1, 0, 3) respectively, where a is a constant.

(i) Verify that B lies on π_1. **[1]**

(ii) Given that A does not lie on π_1, state the possible range of values for a. **[1]**

(iii) Given that $a=9$, find the coordinates of the foot of the perpendicular from A to π_1. Hence, or otherwise, find the vector equation of the line of reflection of the line AB in π_1. **[5]**

The plane π_2 has equation $\mathbf{r}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=4$.

(iv) Find the acute angle between π_1 and π_2. **[2]**

(v) Find the cartesian equations of the planes such that the perpendicular distance from each plane to π_2 is $\frac{5\sqrt{2}}{2}$. **[3]**

[Turn Over]

4 **(a)** Given that $f(r) = \frac{r}{2^r}$, by considering $f(r+1) - f(r)$, find $\sum_{r=1}^{n} \frac{1-r}{2^{r+1}}$. **[3]**

(b) **(i)** Cauchy's root test states that a series of the form $\sum_{r=0}^{\infty} a_r$ (where $a_r > 0$ for all r) converges when $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$, and diverges when $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$. When $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, the test is inconclusive. Using the test and given that $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$ for all positive p, explain why the series $\sum_{r=0}^{\infty} \frac{2^r r^x}{3^r}$ converges for all positive values of x. **[3]**

(ii) By considering $(1-y)^{-2} = 1 + 2y + 3y^2 + 4y^3 + \ldots$, evaluate $\sum_{r=0}^{\infty} \frac{2^r r^x}{3^r}$ for the case when $x = 1$. **[2]**

Section B: Probability and Statistics [60 marks]

5 Seng Ann Joo Cooperative sells granulated sugar in packets. These packets come in two sizes: standard and large. The masses, in grams, of these packets are normally distributed with mean and standard deviation as shown in the table below.

	Mean	Standard Deviation
Standard	520	8
Large	1030	11

(i) Find the probability that two standard packets weigh more than a large packet. **[3]**

(ii) Find the probability that the mean mass of two standard packets and one large packet of sugar is between 680g and 700g. **[3]**

6 A university drama club contains 3 Biology students, 4 History students, and 6 Literature students. 5 students are to be selected as the cast of an upcoming production.

(i) In how many ways can the 5 cast members be selected so that there are at most 2 Biology students? **[2]**

(ii) Find the probability that, amongst the cast members, the number of History students exceeds the number of Literature students, given that there are at most 2 Biology students. **[4]**

[Turn Over]

7 **(i)** The discrete random variable X takes values 1, 2, 3,… , n, where n is a positive integer greater than 1, with equal probabilities.

Find, in terms of n, the mean μ, and the variance, σ^2, of X. **[4]**

[You may use the result $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.]

Let $n = 6$. An observation of X is defined as an *outlier* if $|X - \mu| > \sigma$.

(ii) 20 observations of X are made. Find the probability that there are at least 8 observations that are outliers. **[4]**

8 In a game of chance, a player has to draw a counter from a bag containing n red counters and $(40-n)$ blue counters before throwing a fair die. If a red counter is drawn, she throws a six-sided die, with faces labelled 1 to 6. If a blue counter is drawn, she throws a ten-sided die, with faces labelled 1 to 10. She wins the game if the uppermost face of the die thrown shows a number that is a perfect square.

(i) Given that $n = 15$, find the exact probability that a player wins the game. Hence, find the probability that, when 3 people play this game, exactly 2 won. **[3]**

(ii) For a general value of n, the probability that a winning player drew a blue counter is denoted by $f(n)$. Show that $f(n) = a + \frac{b}{360+n}$, where a and b are constants to be determined. Without further working, explain why f is a decreasing function for $0 \le n \le 40$, and interpret what this statement means in the context of the question. **[5]**

9 Many different interest groups, such as the lumber industry, ecologists, and foresters, benefit from being able to predict the volume of a tree from its diameter. The following table of 10 shortleaf pines is part of the data set concerning the diameter of a tree, x, in inches and volume of a tree y, in cubic feet.

Diameter (x inches)	5.0	5.6	7.5	9.1	9.9	10.3	11.5	12.5	16.0	18.3
Volume (y cubic feet)	3.0	7.2	10.3	17.0	23.1	27.4	26.0	41.3	65.9	97.9

(Bruce and Schumacher, 1935)

(i) Draw a scatter diagram to illustrate the data, labelling the axes clearly. **[2]**

It is thought that the volume of trees with different diameters can be modelled by one of the formulae

[Turn Over]

$$y = a + bx \quad \text{or} \quad \ln y = c + d \ln x$$

where a, b, c and d are constants.

(ii) Find the value of the product moment correlation coefficient between
(a) y and x,
(b) $\ln y$ and $\ln x$.
Leave your answers correct to 5 decimal places **[2]**

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of the models is the better model. **[1]**

(iv) It is required to estimate the value of y for which $x = 20$. Find the equation of a suitable regression line and use it to find the required estimate, correct to 1 decimal place. Explain whether your estimate is reliable. **[3]**

10 A factory manufactures a large number of erasers in a variety of colours. Each box of erasers contains 36 randomly chosen erasers. On average, 20% of erasers in the box are blue.

(i) State, in context, two assumptions needed for the number of blue erasers in a box to be well modelled by a binomial distribution. **[2]**

(ii) Find the probability that a randomly chosen box of erasers contain at most six blue erasers. **[1]**

200 randomly chosen boxes are packed into a carton. A carton is considered acceptable if at least 40% of the boxes contain at most six blue erasers each.
(iii) Find the probability that a randomly chosen carton is acceptable. **[3]**

The cartons are exported by sea. Over a one-year period, there are 30 shipments of 150 cartons each.
(iv) Using a suitable approximation, find the probability that the mean number of acceptable cartons per shipment for the year is less than 80. **[3]**

The owner decided to change the proportion of blue erasers to p. A box of erasers is chosen.

(v) Write down in terms of p, the probability that the box contains exactly one blue eraser. **[1]**

(vi) The probability that a box contains exactly one blue eraser is twice the probability that the box contains exactly two blue erasers. Write an equation in terms of p, and hence find the value of p. **[2]**

[Turn Over]

11 The time T seconds required for a computer to boot up, from the moment it is switched on, is a normally distributed random variable. The specifications for the computer state that the population mean time should not be more than 30 seconds. A Quality Control inspector checks the boot up time using a sample of 25 randomly chosen computers.

A particular sample yielded $\sum t = 802.5$ and $\sum t^2 = 26360.25$.

(i) Calculate the unbiased estimates of the population mean and variance. **[2]**

(ii) What do you understand by the term "unbiased estimate"? **[1]**

(iii) Test, at the 5% level of significance level, whether the specification is being met. Explain in the context of the question, the meaning of "5% level of significance". **[5]**

(iv) Find the range of values of $\overline{t}$ such that the specification will be met in the test carried out in part **(iii)**. **[1]**

(v) A new Quality Control policy is that when the specification is not met, all the computers will be sent back to the manufacturer for upgrading. The inspector tested a second random sample of 25 computers, and the boot up time, y seconds, of each computer is measured, with $\overline{y} = 32.4$. Using a hypothesis test at the 5% level of significance, find the range of values of the population standard deviation such that the computers will not be sent back for upgrading. **[3]**

End of Paper

ANNEX

H2 MA 2019 JC2 Prelim (Paper 1 and Paper 2)

Paper 1

Select topic from the dropdown list. If the question consists of multiple topics, choose 1 topic.

QN	TOPIC (H2) Paper 1	ANSWERS (Exclude graphs and text answers)
1	Graphs & Transformations	(a) $-\frac{9}{8}<k<-1$ (b) $\left(3,-\frac{9}{8}\right)$, $x=-1$, $y=-1$
2	Integration & Applications	(ii) 20
3	Functions	(ii) $\mathrm{f}^{-1}(x)=1-\frac{3}{x-1}=\frac{x-4}{x-1}$, $\frac{97}{100}$ (iii) $(-\infty,1)$
4	Maclaurin & Binomial Series	(i) $y=\sqrt{\mathrm{e}}-\frac{\sqrt{\mathrm{e}}}{4}x^2+\ldots$ (ii) $1+\frac{1}{4}x^2+\ldots$ (iii) 1.650
5	Integration & Applications	(ii) $y=2ap-p(x-ap^2)$ (iii) $\frac{16a^2}{p^4}(p^2+1)^3$
6	Complex Numbers	(a)(i) $2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ (ii) $n=2$ and $n=5$ (b) $w=7+2\mathrm{i}$ and $z=1-5\mathrm{i}$
7	Vectors	(i) 3 (iv) 0.730, $t=0.910, 1.45, 4.05, 4.59$
8	APGP	(a) r = 0.882854, 222.38 (b)(i) $\frac{\pi}{3}$ (ii) 16 (iii) $\frac{8}{7}$ mm
9	Integration & Applications	(a)(i) $\frac{225}{4}\pi$ (a)(ii) $\left(450-\frac{75}{2}\pi\right)$ cm^2 (b) $\left(32500\pi-2250\pi^2\right)$ cm^3

10	Differentiation & Applications	(i) 0.0327 (ii) 2.14 units^2, $(-0.449, 2.73)$ (iii) $P(-0.785, 2.55)$
11	H2 Prelim P1 Q11 Topic	Nil
12	H2 Prelim P1 Q12 Topic	Nil
13	H2 Prelim P1 Q13 Topic	Nil
14	H2 Prelim P1 Q14 Topic	Nil

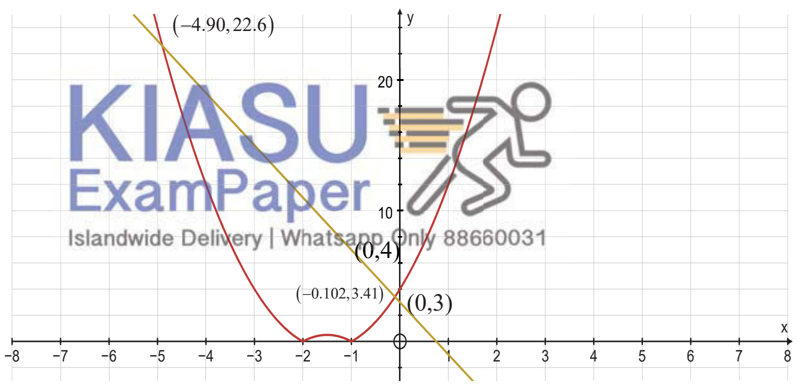
Paper 2

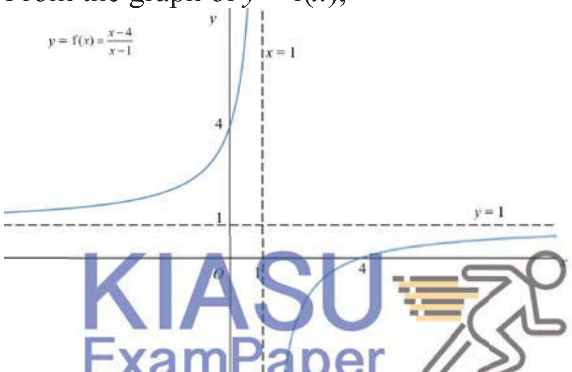
Select topic from the dropdown list. If the question consists of multiple topics, choose 1 topic.

QN	TOPIC (H2) Paper 2	ANSWERS (Exclude graphs and text answers)
1	Graphs & Transformations	(i) $a=4, \quad b=4$
2	Differential Equations	(i) $V=\frac{1}{a}(k-(k-a)e^{-at})$
3	Vectors	(ii) $a\in\mathbb{R}, a\neq -2.$ (iii) $l_{BA'}: \mathbf{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+\mu\begin{pmatrix}3\\7\\5\end{pmatrix},\ \mu\in\mathbb{R}.$ (iv) $31.5°$ (v) $x+z=9$ or $x+z=-1$
4	Sigma Notation & MOD	(a) $\frac{n+1}{2^{n+1}}-\frac{1}{2}$ (b)(ii) 6
5	Normal Distribution	0.737, 0.943
6	PnC & Probability	(i) 1242 (ii) 0.210
7	DRV	(i) $\frac{n+1}{2}, \frac{n^2-1}{12}$ (ii) 0.339
8	PnC & Probability	(i) $\frac{5}{16}, \frac{825}{4096}$ (ii) $-9+\frac{3600}{360+n}$
9	Correlation & Regression	(ii) (a) 0.96346 (ii) (b) 0.98710 (iii) second model (iv) 121
10	Binomial Distribution	(ii) 0.401 (iii) 0.535 (iv) 0.423 (v) $36p(1-p)^{35}$

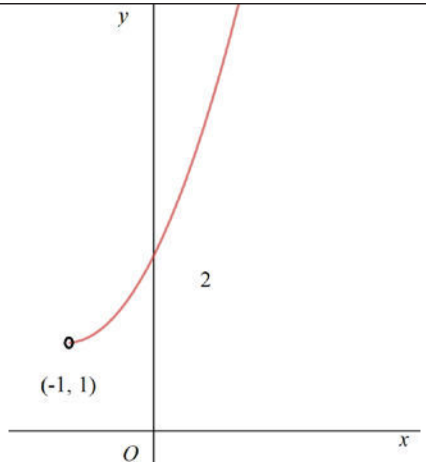
		(vi) $p=\frac{1}{36}$
11	Hypothesis Testing	(i) 32.1, 25 (iii) p-value = 0.0179 , reject H_0. (iv) $0<\overline{t}<31.6$. (v) $\sigma>7.30$
12	H2 Prelim P2 Q12 Topic	Nil
13	H2 Prelim P2 Q13 Topic	Nil
14	H2 Prelim P2 Q14 Topic	Nil

2019 H2 Math Prelim Paper 1 Solutions

Qn	Solutions
1(a)	$\mathrm{f}(3-x)=k$ Range of values of k is $-\frac{9}{8}<k<-1$.
1(b)	The graph of $y=2\mathrm{f}(3-x)$ undergoes 1) A reflection in the y-axis followed by 2) A scaling of ½ unit parallel to the y-axis $A\left(-3,-\frac{9}{4}\right)\xrightarrow{\text{(1) reflection in the } y\text{-axis}}A_1\left(3,-\frac{9}{4}\right)$ $\xrightarrow{\text{(2) scaling of a factor } \frac{1}{2} \text{ parallel to } y\text{-axis}}A_2\left(3,-\frac{9}{8}\right)$ Coordinates of the minimum point on the graph of $y=\mathrm{f}(3+x)$ are $\left(3,-\frac{9}{8}\right)$. Equation of vertical asymptote: $x=-1$ Equation of horizontal asymptote : $y=-1$
2(i)	

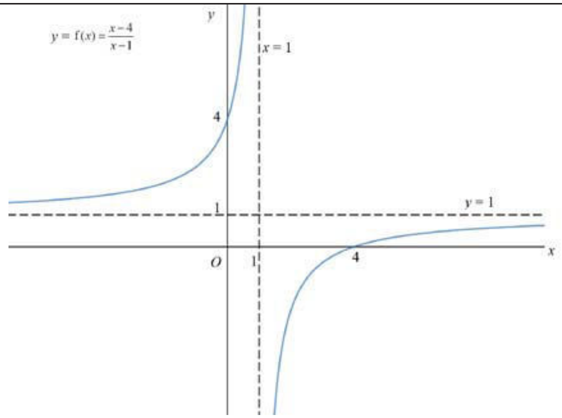
(ii)	$\int_{-3}^{-1}\left(3-4x-\lvert 2x^2+6x+4\rvert\right)\mathrm{d}x$ $=\int_{-3}^{-2}\left(3-4x-2x^2-6x-4\right)\mathrm{d}x+\int_{-2}^{-1}\left(3-4x+2x^2+6x+4\right)\mathrm{d}x$ $=\int_{-3}^{-2}\left(-2x^2-10x-1\right)\mathrm{d}x+\int_{-2}^{-1}\left(2x^2+2x+7\right)\mathrm{d}x$ $=\left[-\frac{2}{3}x^3-5x^2-x\right]_{-3}^{-2}+\left[\frac{2}{3}x^3+x^2+7x\right]_{-2}^{-1}$ $=(\frac{16}{3}-20+2)-(18-45+3)+(-\frac{2}{3}+1-7)-(-\frac{16}{3}+4-14)$ $=20.$
3(i)	$\mathrm{f}(x)=\frac{x-4}{x-1}=1-\frac{3}{x-1}$ From the graph of $y=\mathrm{f}(x)$,  Since every horizontal line $y=h, h\in\mathbb{R}, h\neq 1$ cuts the graph of f at exactly one point, f is an one-one function. Therefore f^{-1} exists.
(ii)	Let $y=1-\frac{3}{x-1}$

	$\frac{3}{x-1}=1-y$ $x-1=\frac{3}{1-y}$ $x=1+\frac{3}{1-y}$ $x=1-\frac{3}{y-1}$ Since $y=\mathrm{f}(x),\ x=\mathrm{f}^{-1}(y)$. $\therefore\ \mathrm{f}^{-1}(y)=1-\frac{3}{y-1}$ Hence $\mathrm{f}^{-1}(x)=1-\frac{3}{x-1}=\frac{x-4}{x-1},\ x\in\mathbb{R},x\neq 1$ Since the $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=(-\infty,1)\cup(1,\infty)=D_{\mathrm{f}}$ Since $\mathrm{f}^{-1}=\mathrm{f}$, $\mathrm{f}^2(x)=\mathrm{ff}^{-1}(x)=x,\ldots,\mathrm{f}^{100}(x)=x$ $\mathrm{f}^{101}(101)=\mathrm{f}\left(\mathrm{f}^{100}(101)\right)$ $=\mathrm{f}(101)$ $=\frac{101-4}{101-1}=\frac{97}{100}$
(iii)	$\mathrm{g}:x\mapsto x^2+2x+2,\ x\in\mathbb{R},x>-1$ $\mathrm{g}(x)=(x+1)^2+1$ From the graph of g,

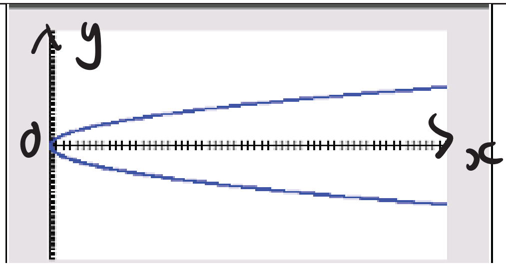


Range of g = $(1, \infty)$
Domain of f = $\mathbb{R} \setminus \{1\}$
Since $(1, \infty) \subseteq \mathbb{R} \setminus \{1\}$
i.e. Range of g $\subseteq$ Domain of f
Therefore the composite function fg exists.

To find range of composite function fg:

	 $(-1,\infty)\xrightarrow{\text{g}}(1,\infty)\xrightarrow{\text{f}}(-\infty,1)$ Range of fg is $(-\infty,1)$.
4	$y=\sqrt{\mathrm{e}^{\cos x}}$ --- (1) $y^2=\mathrm{e}^{\cos x}$ Differentiate with respect to x, $2y\frac{\mathrm{d}y}{\mathrm{d}x}=(-\sin x)\mathrm{e}^{\cos x}$ $y\left(2\frac{\mathrm{d}y}{\mathrm{d}x}+y\sin x\right)=0$ Since $y=\sqrt{\mathrm{e}^{\cos x}}>0$, $2\frac{\mathrm{d}y}{\mathrm{d}x}+y\sin x=0$ (Shown) --- (1) Differentiate with respect to x, $2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(\sin x)\frac{\mathrm{d}y}{\mathrm{d}x}+y\cos x=0$

	When $x = 0$, $y = \sqrt{e}$ from (1) $\frac{dy}{dx} = 0$ from (2) $\frac{d^2y}{dx^2} = -\frac{\sqrt{e}}{2}$ $\therefore y = \sqrt{e} - \frac{\sqrt{e}}{4}x^2 + \ldots$
(ii)	$e^{\sin^2\left(\frac{x}{2}\right)} = e^{\frac{1-\cos x}{2}}$ $= \frac{e^{\frac{1}{2}}}{e^{\frac{1}{2}\cos x}}$ $\approx \frac{e^{\frac{1}{2}}}{e^{\frac{1}{2}} - \frac{e^{\frac{1}{2}}}{4}x^2}$ $= \frac{1}{1-\frac{1}{4}x^2}$ $= \left(1-\frac{1}{4}x^2\right)^{-1}$ $= 1+\frac{1}{4}x^2+\ldots$
(iii)	$\int_0^{\sqrt{2}} e^{\sin^2\left(\frac{x}{2}\right)} dx \approx \int_0^{\sqrt{2}} \left(1+\frac{1}{4}x^2\right) dx$ $=1.649915$ $=1.650$

5(i)	
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{1}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)} = \frac{2a}{2at} = \frac{1}{t}$ Gradient of normal = -*t*. Equation of normal at a point *P* is given by $\frac{y-2ap}{x-ap^2} = -p$ $\Rightarrow y = 2ap - p(x - ap^2)$
(iii)	If the normal at point *P* meets *C* again at point *Q*,

$2aq = 2ap - p(aq^2 - ap^2)$

$pq^2 + 2q - (2p + p^3) = 0$

$$q = \frac{-2 \pm \sqrt{4 + 4p(2p + p^3)}}{2p}$$

$$= \frac{-2 \pm \sqrt{4 + 8p^2 + 4p^4}}{2p}$$

$$= \frac{-2 \pm \sqrt{(2p^2 + 2)^2}}{2p}$$

$$= \frac{-2 + (2p^2 + 2)}{2p} \text{ or } \frac{-2 - (2p^2 + 2)}{2p}$$

$= p$ (rejected as it is the point P) or $-p - \dfrac{2}{p}$

Therefore Q will meet C again with $q = -p - \dfrac{2}{p}$.

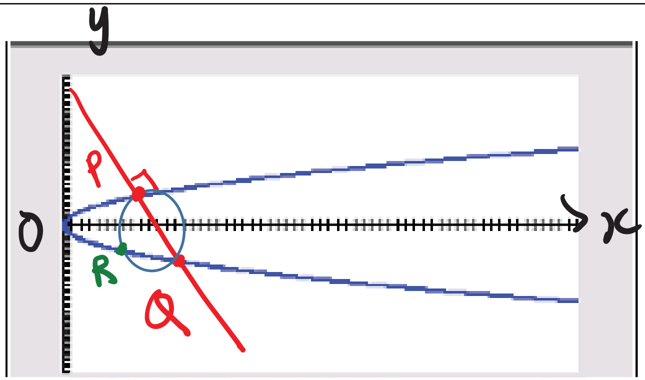
Coordinates of $P = (ap^2, 2ap)$

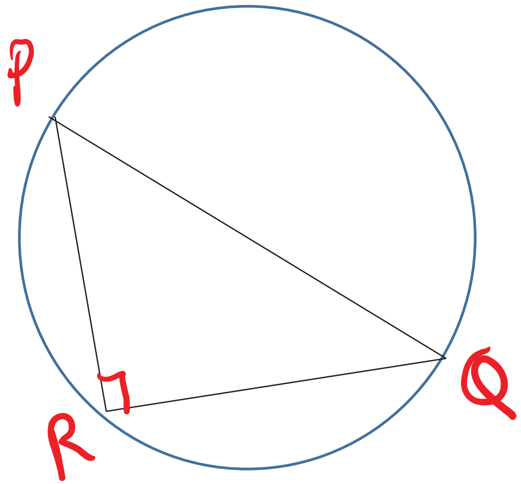
Coordinates of $Q = \left(a(p + \dfrac{2}{p})^2, -2a\left(\dfrac{p^2 + 2}{p}\right)\right)$

$|PQ|^2$

$$= (ap^2 - a((p + \frac{2}{p})^2))^2 + (2ap + 2a(\frac{p^2 + 2}{p}))^2$$

$$= (ap^2 - a(p^2 + 4 + \frac{4}{p^2}))^2 + (4ap + \frac{4a}{p})^2$$

	$=16a^2(1+\frac{1}{p^2})^2+16a^2(\frac{p^2+1}{p})^2$ $=16a^2(\frac{(p^2+1)^2}{p^4}+(\frac{p^2+1}{p})^2)$ $=\frac{16a^2}{p^4}((p^2+1)^2+p^2(p^2+1)^2)$ $=\frac{16a^2}{p^4}(p^2+1)^3$
	



Note that PR and QR are perpendicular to each other (angle PRQ is 90^0 – angle in a semi-circle).

Gradient of PR = $\dfrac{2ap-2ar}{ap^2-ar^2}=\dfrac{2}{p+r}$

Gradient of QR = $\dfrac{-2a\left(\dfrac{p^2+2}{p}\right)-2ar}{a\left(\dfrac{p^2+2}{p}\right)^2-ar^2}=-\dfrac{2}{\left(\dfrac{p^2+2}{p}\right)-r}$

$-\dfrac{2}{\left(\dfrac{p^2+2}{p}\right)-r}\bullet\dfrac{2}{p+r}=-1$

	$(p+r)(p+\frac{2}{p}-r)=4$ $p^2-r^2+\frac{2r}{p}=2$. (Shown).
6(a) (i)	$p=-1-\sqrt{3}\text{i}$ $=2\text{e}^{\text{i}\left(-\frac{2\pi}{3}\right)}$
(ii)	$\frac{(p^*)^n}{\text{i}p}=\frac{2^n\text{e}^{\text{i}\left(\frac{2n\pi}{3}\right)}}{2\text{e}^{\text{i}\left(-\frac{2\pi}{3}+\frac{\pi}{2}\right)}}=2^{n-1}\text{e}^{\text{i}\left(\frac{2n\pi}{3}+\frac{\pi}{6}\right)}$ For $\frac{(p^*)^n}{ip}$ to be purely imaginary, $\frac{2n\pi}{3}+\frac{\pi}{6}=(2k+1)\frac{\pi}{2}$ where $k\in\mathbb{Z}$ $\frac{2n\pi}{3}+\frac{\pi}{6}=k\pi+\frac{\pi}{2}$ $\frac{2n\pi}{3}=k\pi+\frac{\pi}{3}$ $2n=3k+1$ $n=\frac{3k+1}{2}$ For the two smallest $n\in\mathbb{Z}^+$, $k=1$ and $k=3$. $n=2$ and $n=5$

6 (b)	$z = w - 6 - 7\text{i}$ --- (1) $2w - \text{i}z^* - 19 - 3\text{i} = 0$--- (2) Substitute (1) into (2): $2w - \text{i}(w - 6 - 7\text{i})^* - 19 - 3\text{i} = 0$ $2w - \text{i}(w^* - 6 + 7\text{i}) - 19 - 3\text{i} = 0$ Let $w = x + \text{i}y$, $2(x + \text{i}y) - \text{i}((x - \text{i}y) - 6 + 7\text{i}) - 19 - 3\text{i} = 0$ $2(x + \text{i}y) - \text{i}((x - \text{i}y) - 6 + 7\text{i}) - 19 - 3\text{i} = 0$ $2x + 2y\text{i} - \text{i}x - y + 6\text{i} + 7 - 19 - 3\text{i} = 0$ $2x - y - 12 + \text{i}(2y - x + 3) = 0$ Compare real and imaginary components, $2x - y = 12$ ---------- (1) $2y - x = -3$ ---------- (2) From (2), $x = 2y + 3$ -------- (3) Substituting (3) into (1) $2(2y + 3) - y = 12$ $3y = 6$ $y = 2$ When $y = 2$, $x = 2(2) + 3 = 7$ $w = 7 + 2\text{i}$ and $z = 7 + 2\text{i} - 6 - 7\text{i} = 1 - 5\text{i}$
7(i)	$\overrightarrow{OP} = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$ $\lvert\overrightarrow{OP}\rvert = \sqrt{\cos^2 t + \sin^2 t} = 1$

	$\|\overrightarrow{OQ}\|^2 = \frac{9}{4}\cos^2(t+\frac{\pi}{4}) + 9\sin^2(t+\frac{\pi}{4}) + \frac{27}{4}\cos^2(t+\frac{\pi}{4})$ $= 9\cos^2(t+\frac{\pi}{4}) + 9\sin^2(t+\frac{\pi}{4})$ $= 9$ $\Rightarrow \|\overrightarrow{OQ}\| = 3.$
(ii)	$x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi$ The cartesian equation is given by $x^2 + y^2 = 1$. Since $z = 0$, P lies on a circle centre at O and radius 1 unit in the x-y plane.
(iii)	Using scalar product, $\overrightarrow{OP} \bullet \overrightarrow{OQ} = \|\overrightarrow{OP}\|\|\overrightarrow{OQ}\|\cos\theta$ $\Rightarrow 3\cos\theta = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \bullet \begin{pmatrix} \frac{3}{2}\cos\left(t+\frac{\pi}{4}\right) \\ 3\sin\left(t+\frac{\pi}{4}\right) \\ \frac{3\sqrt{3}}{2}\cos\left(t+\frac{\pi}{4}\right) \end{pmatrix}$ $= \frac{3}{2}\cos t\cos\left(t+\frac{\pi}{4}\right) + 3\sin t\sin\left(t+\frac{\pi}{4}\right)$

	$\Rightarrow 3\cos\theta = \frac{3}{4}\left[\cos\left(2t+\frac{\pi}{4}\right)+\cos\frac{\pi}{4}\right]-\frac{3}{2}\left[\cos\left(2t+\frac{\pi}{4}\right)-\cos\frac{\pi}{4}\right]$ $\Rightarrow 3\cos\theta = -\frac{3}{4}\cos\left(2t+\frac{\pi}{4}\right)+\frac{3\sqrt{2}}{8}+\frac{3\sqrt{2}}{4}$ $\Rightarrow \cos\theta = \frac{3\sqrt{2}}{8}-\frac{1}{4}\cos\left(2t+\frac{\pi}{4}\right)$ (Shown).
(iv)	The length of projection of $\overrightarrow{OQ}$ onto $\overrightarrow{OP}$ is $\lvert\overrightarrow{OQ}\bullet\overrightarrow{OP}\rvert$ since $\overrightarrow{OP}$ is a unit vector. Given that the length of projection of $\overrightarrow{OQ}$ onto $\overrightarrow{OP}$ is $\sqrt{5}$ units, $\lvert\overrightarrow{OQ}\bullet\overrightarrow{OP}\rvert=\sqrt{5}$ $\Rightarrow \lvert\overrightarrow{OQ}\rvert\cos\theta=\sqrt{5}$ $\Rightarrow \cos\theta=\frac{\sqrt{5}}{3}$ $\Rightarrow \theta = 0.730$ (since θ is acute). Solving $\frac{3\sqrt{2}}{8}-\frac{1}{4}\cos\left(2t+\frac{\pi}{4}\right)=\frac{\sqrt{5}}{3}$ $\cos(2t+\frac{\pi}{4})=-0.86010$ $2t+\frac{\pi}{4}=0.53532$ (basic angle) $2t+\frac{\pi}{4}=\pi-0.53532, \pi+0.53532, \pi-0.53532+2\pi, \pi+0.53532+2\pi$ $t=0.910, 1.45, 4.05, 4.59$

8(a)	Let the length of the first screwdriver be b, and the common ratio be r. $b+br+br^2=3\left(br^{12}+br^{13}+br^{14}+br^{15}+br^{16}\right)$ $1+r+r^2-3r^{12}-3r^{13}-3r^{14}-3r^{15}-3r^{16}=0$ Using the GC, r = 0.882854 (6 s.f.) since $0<r<1$. There are 9 odd numbered screwdrivers, with common ratio of lengths being r^2. Total length = $\dfrac{b\left(1-\left(r^2\right)^9\right)}{1-r^2}=120$ $\dfrac{b\left(1-r^{18}\right)}{1-r^2}=120$ Using the GC, b = 29.6122 (6 s.f.) The total length is $\dfrac{b\left(1-r^{17}\right)}{1-r}=222.38$ cm $\quad$(to 2 d.p.)
8 (b)(i)	$\cos\left(u_{n+1}-u_n\right)=\cos u_{n+1}\cos u_n+\sin u_{n+1}\sin u_n$ $=\dfrac{1}{2}\cos^2 u_n-\dfrac{\sqrt{3}}{2}\sin u_n\cos u_n$ $+\dfrac{1}{2}\sin^2 u_n+\dfrac{\sqrt{3}}{2}\sin u_n\cos u_n$ $=\dfrac{1}{2}$ $\therefore u_{n+1}-u_n=\cos^{-1}\dfrac{1}{2}=\dfrac{\pi}{3}$ (since the difference is less than π) is a constant independent of n. Hence $\{u_n\}$ is an arithmetic progression.

	The common difference is $\frac{\pi}{3}$. **<u>Alternatively</u>** $\cos u_{n+1} = \frac{1}{2}\cos u_n - \frac{\sqrt{3}}{2}\sin u_n$ $= \cos u_n \cos\frac{\pi}{3} - \sin u_n \sin\frac{\pi}{3}$ $= \cos\left(u_n + \frac{\pi}{3}\right)$ $\therefore u_{n+1} = u_n + \frac{\pi}{3}$ since the difference is less than π. $u_{n+1} - u_n$ is a constant, therefore $\{u_n\}$ is an arithmetic progression. The common difference is $\frac{\pi}{3}$.
(ii)	The total angle rotated over n twists is $\frac{n}{2}\left(2\left(\frac{2\pi}{3}\right)+(n-1)\frac{\pi}{3}\right)$. $\frac{n}{2}\left(\frac{4\pi}{3}+(n-1)\frac{\pi}{3}\right) \geq 25\times(2\pi)$ $\frac{2n\pi}{3}+\frac{\pi}{6}n(n-1)-50\pi \geq 0$ $4n+n(n-1)-300 \geq 0$ $n^2+3n-300 \geq 0$ Using the GC,

n	$n^2+3n-300$
15	-30

16	4
17	40

The minimum number of buttons presses required is 16.

(iii) Total angle rotated after 21 button presses is

$$\frac{21}{2}\left(2\left(\frac{2\pi}{3}\right)+(20)\frac{\pi}{3}\right)$$

$$=14\pi+70\pi$$

$$=84\pi$$

$84\pi k=144$, where k is a positive real constant.

The first button press rotates the screw by $\frac{2\pi}{3}$.

$$d_1=ku_1=\frac{144}{84\pi}\left(\frac{2\pi}{3}\right)=\frac{8}{7}$$

The first button press drills the screw in by $\frac{8}{7}$ mm (1.14mm).

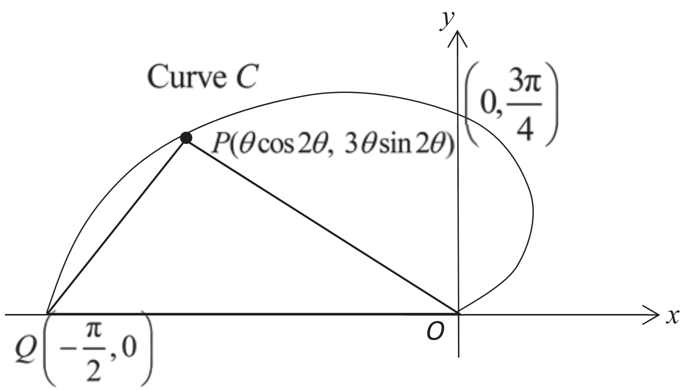
9(i) $x=15\sin\theta+15$

$$\frac{dx}{d\theta}=15\cos\theta$$

$$\int_0^{15}\sqrt{15^2-(x-15)^2}\,dx$$

	$=\int_{-\frac{\pi}{2}}^{0}\left(\sqrt{15^2-(15\sin\theta)^2}\right)(15\cos\theta)\mathrm{d}\theta$ $=225\left[\int_{-\frac{\pi}{2}}^{0}(\cos\theta)(\cos\theta)\mathrm{d}\theta\right]$ $=\frac{225}{2}\left[\int_{-\frac{\pi}{2}}^{0}\cos^2\theta\ \mathrm{d}\theta\right]$ $=\frac{225}{2}\left[\int_{-\frac{\pi}{2}}^{0}(\cos 2\theta+1)\ \mathrm{d}\theta\right]$ $=\frac{225}{2}\left[\frac{\sin 2\theta}{2}+\theta\right]_{-\frac{\pi}{2}}^{0}$ $=\frac{225}{2}\left(\frac{\pi}{2}\right)$ $=\frac{225}{4}\pi$

9(ii) (a)	Required area $=\int_0^{15} \mathrm{g}(x)\mathrm{d}x$ $=\int_0^{15}\left(30-\frac{2}{3}\sqrt{15^2-(x-15)^2}\right)\mathrm{d}x$ $=\left[30x\right]_0^{15}-\frac{2}{3}\int_0^{15}\sqrt{15^2-(x-15)^2}\,\mathrm{d}x$ $=450-\frac{2}{3}\left(\frac{225}{4}\pi\right)$ from (i) $=\left(450-\frac{75}{2}\pi\right)$ cm^2
(b)	Required Volume $=\pi(30)^2(20)+\pi\int_0^{15} y^2\mathrm{d}x$ $=\pi(30)^2(20)+\pi\int_0^{15}\left(30-\frac{2}{3}\sqrt{15^2-(x-15)^2}\right)^2 \mathrm{d}x$ $=\pi(30)^2(20)+$ $\pi\int_0^{15}\left(900-30(2)\left(\frac{2}{3}\right)\sqrt{15^2-(x-15)^2}+\left(\frac{4}{9}\right)\left(225-(x-15)^2\right)\right)\mathrm{d}x$ $=18000\pi+\pi\int_0^{15}\left(1000-40\sqrt{15^2-(x-15)^2}-\left(\frac{4}{9}\right)(x-15)^2\right)\mathrm{d}x$ $=18000\pi+\pi\left[1000x-\left(\frac{4}{9}\right)\frac{(x-15)^3}{3}\right]_0^{15}-40\pi\left(\frac{225}{4}\pi\right)$ from (i) $=\left(32500\pi-2250\pi^2\right)$ cm^3.

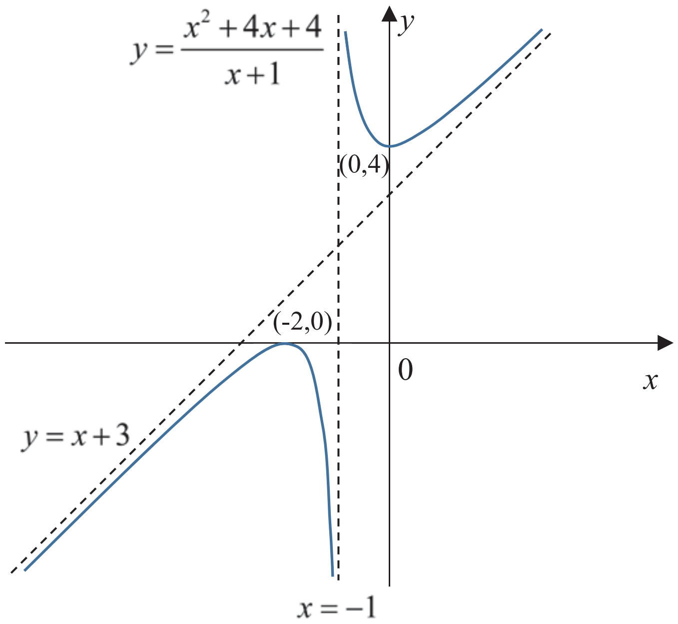
10(i)	 Point P has coordinates $(\theta\cos 2\theta,\ 3\theta\sin 2\theta)$. Let area of triangle OPQ be A. $A = \frac{1}{2}\left(\frac{\pi}{2}\right)(3\theta\sin 2\theta) = \frac{3\pi}{4}(\theta\sin 2\theta)$ $\frac{dA}{d\theta} = \frac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta)$ $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$ $= \frac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta)(0.01)$ When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt}$ =0.0327 units^2/s (3 s.f.)

(ii)	When $\frac{\mathrm{d}A}{\mathrm{d}\theta}=0$, $\frac{3\pi}{4}\left(2\theta\cos 2\theta+\sin 2\theta\right)=0$ Since $\frac{3\pi}{4}\neq 0$, $2\theta\cos 2\theta+\sin 2\theta=0$ Using GC, $\theta=1.0144$ (5 s.f.) $=1.01$ (3 s.f.) $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=\frac{3\pi}{4}\left(-4\theta\sin 2\theta+2\cos 2\theta+2\cos 2\theta\right)$ When $\theta=1.0144$, $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=-12.7<0$ $\therefore\ \theta=1.0144$ will result in maximum A. When $\theta=1.0144$, $A=\frac{3\pi}{4}(1.0144)\sin(2\times 1.0144)$ $=2.14$ units^2 (3 s.f.)

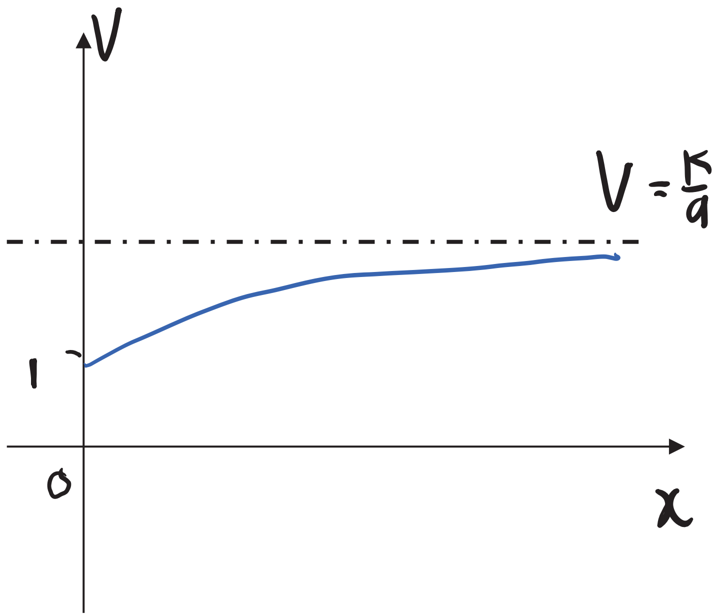
	When $\theta = 1.0144$, $x = 1.0144\cos\left[2(1.0144)\right] = -0.449$ $y = 3(1.0144)\sin\left[2(1.0144)\right] = 2.73$ $\therefore$ Location of the camera is at a point with coordinates $(-0.449,\ 2.73)$
(iii)	For triangle OPQ to be an isosceles triangle, $x = -\dfrac{\pi}{2} \div 2 = -\dfrac{\pi}{4}$ $-\dfrac{\pi}{4} = \theta\cos 2\theta$ Using GC, $\theta = 1.1581$ $y = 3(1.1581)\sin(2\times 1.1581) = 2.55$ $\therefore$ coordinates of $P(-0.785,\ 2.55)$

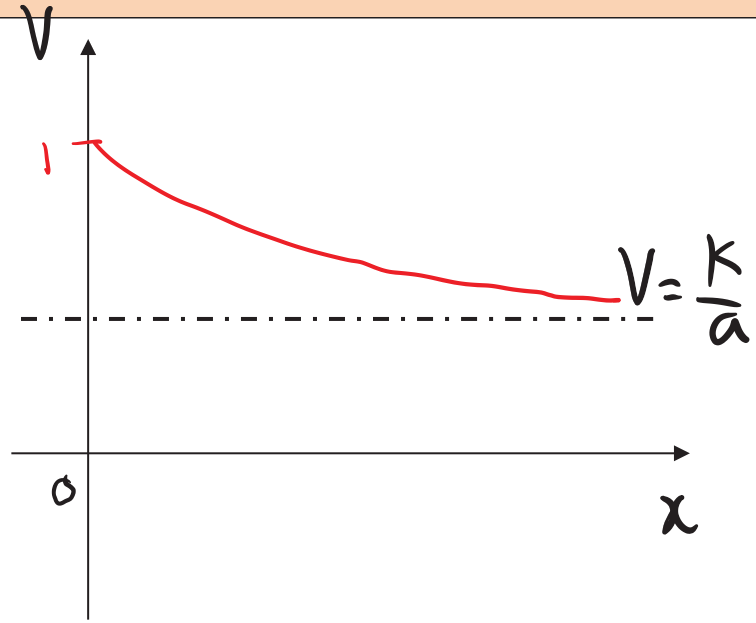
2019 SAJC H2 Math Paper 2 Solutions

Qn	Solution
1(i)	Since $x=-1$ is a vertical asymptote, $c=1$ $y=\dfrac{x^2+ax+b}{x+1}$ $y(x+1)=x^2+ax+b$ $x^2+(a-y)x+(b-y)=0$ The values that y can take satisfy the inequality: $(a-y)^2-4(b-y)\geq 0$ $y^2+(4-2a)y+(a^2-4b)\geq 0$ Since $y\leq 0$ or $y\geq 4$: $y=0$ and $y=4$ are roots to the equation $y^2+(4-2a)y+(a^2-4b)=0$ ....(1) Substituting $y=0$ and $y=4$ into (1) and solving: $a^2-4b=0$ $16+(4-2a)(4)=0$ $a=4,\quad b=4$
(ii)	$y=\dfrac{x^2+4x+4}{x+1}=x+3+\dfrac{1}{x+1}$

Qn	Solution
	
(iii)	Point of intersection: $(-1,2)$ For all $k\in\mathbb{R}$, the line $y=k(x+1)+2$ passes through the point (-1,2). Hence the line will cut C for $k>1$.
2(i)	Let the volume of water in the tank be V cubic metres at t seconds. $\frac{dV}{dt}=\frac{dV_{IN}}{dt}-\frac{dV_{OUT}}{dt}$ $=k-aV$ $=k\left(1-\frac{a}{k}V\right)$. (Shown) where $k>0$, and $a>0$.

Qn	Solution
	$\int \frac{1}{1-\frac{a}{k}V}\,\mathrm{d}V = \int k\,\mathrm{d}t$ $-\frac{k}{a}\ln\left|1-\frac{a}{k}V\right| = kt + C$ $\ln\left|1-\frac{a}{k}V\right| = -at - \frac{aC}{k}$ $1-\frac{a}{k}V = \pm e^{-\frac{ac}{k}}e^{-at} = Ae^{-at}$ $V = \frac{k}{a}\left(1 - Ae^{-at}\right)$ C is an arbitrary constant and $A = \pm e^{-\frac{aC}{k}}$ At $t = 0$, $V = 1$, $A = 1 - \frac{a}{k}$ $V = \frac{k}{a}\left(1-(1-\frac{a}{k})e^{-at}\right) = \frac{1}{a}(k-(k-a)e^{-at})$ **Case (i) if $a < k$**

Qn	Solution
	 The volume of water in the water tank, V, increases from one cubic meter and approach $\frac{k}{a}$ cubic meters eventually. **Case (ii) if $a > k$**

Qn	Solution
	 The volume of water in the water tank, V, decreases from one cubic meter and approach $\frac{k}{a}$ cubic meters eventually.

3(i)	$\overrightarrow{OB}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=1+0+9=10$ B lies on π_1.
(ii)	$\begin{pmatrix}2\\a\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}\neq 10$ $2-a+6\neq 10$ $a\neq -2$ The range of values is $a\in\mathbb{R}, a\neq -2$.
(iii)	Let the foot of perpendicular be F. The line through AF has vector equation $l_{AF}:\mathbf{r}=\begin{pmatrix}2\\9\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix},\ \lambda\in\mathbb{R}.$ Since F lies on l_{AF}, $\overrightarrow{OF}=\begin{pmatrix}2\\9\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ for some fixed $\lambda\in\mathbb{R}$ Since F lies on π_1, $\overrightarrow{OF}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=10$ $\therefore\left[\begin{pmatrix}2\\9\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=10$ $2-9+6+11\lambda=10$ $\lambda=1$

$\overrightarrow{OF} = \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$

The coordinates of F are $(3,8,5)$.

Let the point of reflection of A about π_1 be A'.

$\frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2} = \overrightarrow{OF}$

$\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA}$

$= \begin{pmatrix} 6 \\ 16 \\ 10 \end{pmatrix} - \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$

$\overrightarrow{BA'} = \overrightarrow{OA'} - \overrightarrow{OB}$

$= \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$

Line of reflection, $l_{BA'}$, has vector equation

$l_{BA'} : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}, \ \mu \in \mathbb{R}.$

(iv)	Acute angle between π_1 and π_2 is $\cos^{-1}\left\|\frac{\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\-1\\3\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\0\\1\end{pmatrix}\right\|}\right\| = \cos^{-1}\left\|\frac{4}{\sqrt{11}\sqrt{2}}\right\| = 31.5° \ (1 \text{ d.p.})$
(v)	The desired planes have equation $\mathbf{r}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} = D$, where D is the constant to be determined. Distance between the planes is given by $\left\|\frac{D}{\sqrt{2}} - \frac{4}{\sqrt{2}}\right\| = \frac{5}{\sqrt{2}}$ $\frac{D}{\sqrt{2}} - \frac{4}{\sqrt{2}} = \pm\frac{5}{\sqrt{2}}$ $D = 4 \pm 5$ $D = 9$ or $D = -1$ The possible equations are $x + z = 9$ or $x + z = -1$. **<u>Alternative Solution</u>** Let a point D on the desired plane have coordinates (x, y, z). Then

	$\dfrac{\left\|\overrightarrow{AD}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right\|}{\sqrt{2}}=\dfrac{5}{\sqrt{2}}$ $\left\|\begin{pmatrix}x-1\\y\\z-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right\|=5$ $\|x-1+z-3\|=5$ $x+z-4=\pm 5$ $x+z=4\pm 5$ The possible equations are $x+z=9$ or $x+z=-1$.
4(a)	$\mathrm{f}(r+1)-\mathrm{f}(r)=\dfrac{r+1}{2^{r+1}}-\dfrac{r}{2^{r}}$ $=\dfrac{r+1-2r}{2^{r+1}}$ $=\dfrac{1-r}{2^{r+1}}$

	$\sum_{r=1}^{n}\frac{1-r}{2^{r+1}}=\sum_{r=1}^{n}\left[f(r+1)-f(r)\right]$ $=\begin{bmatrix} f(2)-f(1) \\ +f(3)-f(2) \\ +f(4)-f(3) \\ \vdots \\ +f(n-1)-f(n-2) \\ +f(n)-f(n-1) \\ +f(n+1)-f(n) \end{bmatrix}$ $=f(n+1)-f(1)$ $=\frac{n+1}{2^{n+1}}-\frac{1}{2}$
(b)(i)	$a_n=\frac{2^n n^x}{3^n}$ $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{2^n n^x}{3^n}}$ $=\lim_{n\to\infty}\left(\frac{2\sqrt[n]{n^x}}{3}\right)$ $=\frac{2}{3}\lim_{n\to\infty}\left(\sqrt[n]{n^x}\right)$ $=\frac{2}{3}(1)$ since $\lim_{n\to\infty}\left(\sqrt[n]{n^x}\right)=1$ $=\frac{2}{3}<1$

	Since $\lim_{n\to\infty}\sqrt[n]{a_n}=\frac{2}{3}<1$, by the Cauchy Test, the series converges for all real values of x
(b)(ii)	$\sum_{r=0}^{\infty}\frac{2^r r}{3^r}=0+1\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)^2+3\left(\frac{2}{3}\right)^3+4\left(\frac{2}{3}\right)^4+\ldots$ $=\frac{2}{3}\left[1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)^3+\ldots\right]$ $=\frac{2}{3}\left(1-\frac{2}{3}\right)^{-2}$ since $1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)^3+\ldots=\left(1-\frac{2}{3}\right)^{-2}$ with $y=\frac{2}{3}$ $=6$
5	Let X be the mass of a standard packet of sugar in grams. Let Y be the mass of a large packet of sugar in grams. $X\sim\mathrm{N}\left(520,8^2\right)$ $Y\sim\mathrm{N}\left(1030,11^2\right)$ $X_1+X_2-Y\sim\mathrm{N}(10,249)$ $\mathrm{P}(X_1+X_2>Y)=\mathrm{P}(X_1+X_2-Y>0)$ $=0.73687$ $=0.737$ (3 s.f.)
	$\frac{X_1+X_2+Y}{3}\sim\mathrm{N}\left(690,\frac{83}{3}\right)$ $\mathrm{P}\left(680<\frac{X_1+X_2+Y}{3}<700\right)=0.94272$ $=0.943$ (3 s.f.)
6(i)	Total number of ways to select 5 members $={}^{13}C_5$ Number of ways to select 5 members with 3 Biology students $={}^{10}C_2$ Number of ways to select 5 members with at most 2 Biology students $={}^{13}C_5-{}^{10}C_2=1242$

(ii)	Let the number of Biology, History, and Literature students be B, H, L respectively. $P(H>L \mid B\le 2)=\dfrac{P((H>L)\cap(B\le 2))}{P(B\le 2)}$ $=\dfrac{n((H>L)\cap(H+L\ge 3))}{n(B\le 2)}$ Number of ways to select cast members with H > L when there are 3 humanities students = ${}^{3}C_2\times\left({}^{4}C_2\times{}^{6}C_1+{}^{4}C_3\times{}^{6}C_0\right)=120$ Number of ways to select cast members with H > L when there are 4 humanities students = ${}^{3}C_1\times\left({}^{4}C_3\times{}^{6}C_1+{}^{4}C_4\times{}^{6}C_0\right)=75$ Number of ways to select cast members with H > L when there are 5 humanities students = ${}^{3}C_0\times\left({}^{4}C_3\times{}^{6}C_2+{}^{4}C_4\times{}^{6}C_1\right)=66$ Required Probability $=\dfrac{n((H>L)\cap(H+L\ge 3))}{n(B\le 2)}$ $=\dfrac{120+75+66}{1242}$ $=\dfrac{261}{1242}=0.210\ \ (3\text{ s.f.})$
7(i)	$P(X=1)=P(X=2)=....=P(X=n)=\dfrac{1}{n}$ $E(X)=\sum_{x=1}^{n}xP(X=x)=\sum_{x=1}^{n}x\cdot\left(\dfrac{1}{n}\right)$ $=\dfrac{1}{n}\left(\dfrac{n(n+1)}{2}\right)$ $=\dfrac{n+1}{2}$ $Var(X)=E(X^2)-(E(X))^2$

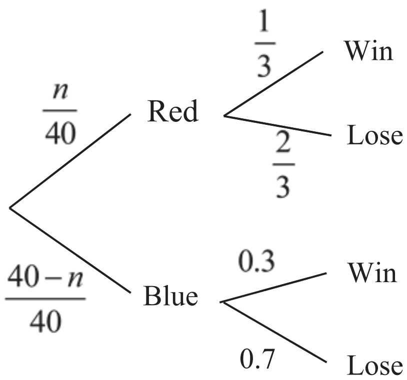
	$\mathrm{E}(X^2)=\sum_{x=1}^{n}x^2\cdot\left(\frac{1}{n}\right)$ $=\frac{1}{n}\left(\frac{n}{6}(n+1)(2n+1)\right)$ $=\frac{(n+1)(2n+1)}{6}$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2$ $=\frac{(n+1)(2n+1)}{6}-\left(\frac{n+1}{2}\right)^2$ $=\frac{(n+1)}{12}\left(2(2n+1)-3(n+1)\right)$ $=\frac{(n+1)(n-1)}{12}$ $=\frac{n^2-1}{12}$
(ii)	$\mathrm{E}(X)=\frac{7}{2}$ and $\mathrm{Var}(X)=\frac{6^2-1}{12}=\frac{35}{12}$ $\mathrm{P}\left(\lvert X-\mu\rvert>\sigma\right)=\mathrm{P}\left(\left\lvert X-\frac{7}{2}\right\rvert>\sqrt{\frac{35}{12}}\right)$ $=\mathrm{P}\left(X>\frac{7}{2}+\sqrt{\frac{35}{12}}\right)+\mathrm{P}\left(X<\frac{7}{2}-\sqrt{\frac{35}{12}}\right)$ $=\mathrm{P}(X>5.21)+\mathrm{P}(X<1.79)$ $=\mathrm{P}(X=6)+\mathrm{P}(X=1)$ $=\frac{2}{6}=\frac{1}{3}$

Let S be the random variable "no. of observations, out of 20, such that the total score is an outlier".

$S \sim \text{B}(20, \frac{1}{3})$

$\text{P}(S \geq 8) = 1 - \text{P}(S \leq 7) = 0.339$

8(i)



P(a player wins the game)

$= \frac{15}{40} \times \frac{1}{3} + \frac{25}{40} \times \frac{3}{10}$

$= \frac{5}{16}$

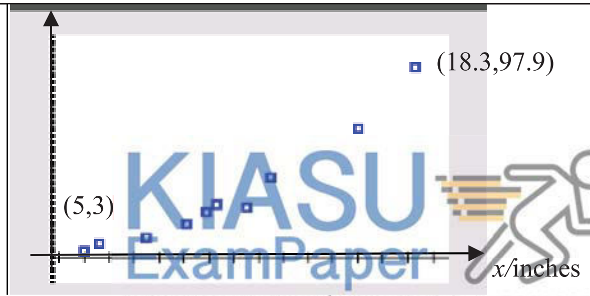
P(exactly 2 of 3 players win)

$= {}^{3}C_{2} \left(\frac{5}{16}\right)^{2} \left(\frac{11}{16}\right)$

$= \frac{825}{4096}$

Alternatively,

	Let X be the random variable "the number of people who wins the game out of 3" $X \sim B(3, \frac{5}{16})$ $P(X = 2) = 0.201$ (to 3 s.f.)
(ii)	P(a player wins the game) $= \frac{n}{40} \times \frac{1}{3} + \frac{40-n}{40} \times \frac{3}{10}$ $= \frac{10n}{1200} + \frac{360-9n}{1200}$ $= \frac{360+n}{1200}$ $f(n)$ = P(player draws blue | player wins) $= \frac{P(\text{player draws blue and wins})}{P(\text{player wins})}$ $= \frac{\frac{40-n}{40} \times \frac{3}{10}}{\frac{360+n}{1200}}$

	$= \dfrac{120-3n}{400} \times \dfrac{1200}{360+n}$ $= \dfrac{3(120-3n)}{360+n}$ $= \dfrac{360-9n}{360+n}$ $= \dfrac{-9(360+n)+3600}{360+n}$ $= -9 + \dfrac{3600}{360+n}$ As n increases, $\dfrac{3600}{360+n}$ decreases, hence f(n) decreases. Hence f is decreasing for all n, $0 \le n \le 40$. This means that as the number of red counters increase, the probability that a winning player drew a blue counter decreases.
9(i)	
(ii)	(a) product moment correlation coefficient, $r = 0.96346$ (b) product moment correlation coefficient, $r = 0.98710$

(iii)	The second model $\ln y = c + d\ln x$ is the better model because its product moment correlation coefficient is closer to one as compared to the product moment correlation coefficient of the first model. From the scatter plot, it can be seen that the data seems to indicate a non-linear (curvilinear) relationship between y and x. Hence the model $y = a + bx$ is not appropriate.
(iv)	We have to use the regression line $\ln y$ against $\ln x$. From GC, the equation is $\ln y = -2.5866 + 2.4665\ln x$ When $x = 20$, $\ln y = -2.5866 + 2.4665\ln 20$ $y = e^{4.8025} = 121.82 = 121$ $x = 20$ is outside the data range and hence the relationship $\ln y = c + d\ln x$ may not hold. Hence the estimate may not be reliable.
10(i)	Assumptions 1. Every eraser is equally likely to be blue. 2. The colour of a randomly selected eraser is independent of the colour of other erasers.
(ii)	Let Y be the number of blue erasers, out of 36. $Y \sim B(36, 0.20)$ $P(Y \le 6) = 0.40069 \approx 0.401$
(iii)	Let W be the number of boxes that contain at most six blue erasers, out of 200. $W \sim B(200, 0.40069)$ $P(W \ge 40\% \text{ of } 200) = P(W \ge 80) = 1 - P(W \le 79)$ $= 0.53477 \approx 0.535$
(iv)	Let T denote the number of cartons where each carton contains at least 40% of the boxes that contains at most six blue erasers per box. $T \sim B(150, 0.53477)$ $E(T) = 150 \times 0.53477 = 80.216$ $\text{Var}(T) = 150 \times 0.53477 \times (1 - 0.53477) = 37.319$

	Since n is large $(n=30)$, by the Central Limit Theorem, $\overline{T}=\frac{T_1+T_2+...+T_{30}}{30} \sim N(80.216, \frac{37.319}{30})$ approximately. $P(\overline{T}<80)=0.423219 \approx 0.423$ (3 sig. fig.)
(v)	Let R be the number of blue erasers, out of 36. $R \sim B(36, p)$ $P(R=1)=\binom{36}{1}p^1(1-p)^{35}=36p(1-p)^{35}$
(vi)	$P(R=2)=\binom{36}{2}p^2(1-p)^{34}=630p^2(1-p)^{34}$ $P(R=1)=2P(R=2)$ $36p(1-p)^{35}=2\times 630p^2(1-p)^{34}$ $36(1-p)=1260p$ $1-p=35p$ $1=36p$ $p=\frac{1}{36}$

11(i)	Let T be the random variable " time taken in seconds for a computer to boot up", with population mean μ. Unbiased estimate of the population mean, $\bar{t}=\frac{802.5}{25}=32.1$ Unbiased estimate of the population variance, $s^2=\frac{1}{24}\left[26360.25-\frac{802.5^2}{25}\right]=25$
(ii)	A statistic is said to be an unbiased estimate of a given parameter when the mean of the sampling distribution of the statistic can be shown to be equal to the parameter being estimated. For example, $\mathrm{E}(\bar{X})=\mu$.
(iii)	Test H_0: $\mu=30$ against $H_1:\mu>30$ at the 5% level of significance. Under H_0, $\bar{T}\sim N(30,\frac{25}{25})$. Using GC, $\bar{t}$ = 32.1 gives rise to z_{calc} = 2.1 and p-value = 0.0179 Since p-value = 0.0179 ≤ 0.05, we reject H_0 and conclude that there is sufficient evidence at the 5% significance level that the specification is not being met (<u>or</u> the computer requires more than 30 seconds to boot up). "5% significance level" is the probability of wrongly concluding that the mean boot up time for the computer is more than 30 seconds when in fact it is not more than 30 seconds.
(iv)	The critical value for the test is 31.645. For the specification to be met, H_0 is not rejected. $\bar{t}<31.6$ (3 s.f.) Since $\bar{t}>0$, Answer is $0<\bar{t}<31.6$.
(v)	Under H_0, $\bar{Y}\sim N\left(30,\frac{\sigma^2}{25}\right)$

$$Z = \frac{\overline{Y} - (30)}{\frac{\sigma}{\sqrt{25}}} \sim \mathrm{N}(0,1)$$

Using a 1 – tailed z test,

$$z_{\text{calc}} = \frac{32.4 - 30}{\frac{\sigma}{5}} = \frac{12}{\sigma}, \; z_{\text{crit}} = 1.64485$$

In order not to reject H_0,

$$z_{\text{calc}} < 1.64485$$

$$\frac{12}{\sigma} < 1.64485$$

$$\sigma > 7.2955$$

$$\sigma > 7.30$$