National Junior College
2016 – 2017 H2 Further Mathematics
Topic F3: Further Differential Equations (Lecture Notes)

Key Questions to Answer:

1. How do we solve differential equations of the following forms analytically?

   (i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}\left(x\right)\mathrm{g}\left(y\right)$,

   (ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{P}(x)y = \mathrm{Q}(x).$,

   (iii) $a\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$,

   (iv) $a\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = \mathrm{f}\left(x\right)$, where $\mathrm{f}\left(x\right)$ is a polynomial or $p\mathrm{e}^{kx}$ or $p\cos\left(kx\right) + q\sin\left(kx\right)$, including equations that can be reduced to the above by means of a given substitution.

2. How do we sketch a family of solution curves of a differential equation?

3. How do we determine the equilibrium points and draw the phase lines of autonomous differential equations?

4. How do we model and solve problems related to the spread of diseases or population growth, with competition and harvesting?

5. What is the relationship between the solution of a nonhomogeneous equation and the associated homogeneous equation?

6. How do we model and solve problems related to the motion of particles that involves resistance, free or driven oscillation and damping?

§1 Analytic Solutions of First Order Differential Equations

1.1 Separation of Variables

A first-order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y)$$

is said to be **separable** or to have **separable variables.**

Example 1.1.1

Solve $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=y$, expressing y in terms of x.

Solution:

$$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}=y$$
$$\Rightarrow \int \frac{\mathrm{d}y}{y}=\int \frac{\mathrm{d}x}{1+x}$$
$$\Rightarrow \ln|y|=\ln|1+x|+C$$
$$\Rightarrow \ln\left|\frac{y}{1+x}\right|=C$$
$$\Rightarrow \left|\frac{y}{1+x}\right|=\mathrm{e}^{C}$$
$$\Rightarrow \frac{y}{1+x}=\pm\mathrm{e}^{C}$$
$$\Rightarrow \frac{y}{1+x}=A, A=\pm\mathrm{e}^{C}$$
$$\Rightarrow y=A(1+x)$$

Example 1.1.2

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x}=y^2-4$, expressing y in terms of x.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y^2-4$$
$$\Rightarrow \int \frac{\mathrm{d}y}{y^2-4}=\int 1\ \mathrm{d}x$$
$$\Rightarrow \int \frac{\mathrm{d}y}{y^2-2^2}=x+C$$
$$\Rightarrow \frac{1}{2(2)}\ln\left|\frac{y-2}{y+2}\right|=x+C$$
$$\Rightarrow \ln\left|\frac{y-2}{y+2}\right|=4x+C'$$
$$\Rightarrow \frac{y-2}{y+2}=\pm\mathrm{e}^{4x+C'}=A\mathrm{e}^{4x}, A=\pm\mathrm{e}^{C'}$$
$$\Rightarrow y-2=A\mathrm{e}^{4x}(y+2)$$
$$\Rightarrow y-Ay\mathrm{e}^{4x}=2A\mathrm{e}^{4x}+2$$
$$\Rightarrow y(1-A\mathrm{e}^{4x})=2(1+A\mathrm{e}^{4x})$$
$$\Rightarrow y=\frac{2\left(1+A\mathrm{e}^{4x}\right)}{1-A\mathrm{e}^{4x}}$$

Example 1.1.3

The gradient of a curve at any point (x, y) is given by the expression $\frac{\mathrm{d}y}{\mathrm{d}x} = y - \frac{y}{x}, x \neq 0$. Find the equation of the curve if it passes through the point $\left(2, \mathrm{e}^2\right)$.

Solution:

Since the curve passes through $\left(2, \mathrm{e}^2\right)$, $\mathrm{e}^2 = \frac{A\mathrm{e}^2}{2} \Rightarrow A = 2 \Rightarrow y = \frac{2\mathrm{e}^x}{x}$

Example 1.1.4

Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{y}$.

Sketch 3 members of the family of solution curves.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{y}$$

$$\int y \ \mathrm{d}y = \int -2x \ \mathrm{d}x$$

$$\frac{y^2}{2} = -x^2 + C$$

$$\frac{y^2}{2C} + \frac{x^2}{C} = 1$$

Example 1.1.5

Show that the differential equation $2xy\frac{\mathrm{d}y}{\mathrm{d}x}=x^2+2y^2$ can be reduced by means of the substitution $y=vx$ to $2vx\frac{\mathrm{d}v}{\mathrm{d}x}=1$. Hence, find the general solution, giving y^2 explicitly in terms of *x*.

Solution:

1.2 Linear Equations and Integrating Factor

A linear differential equation takes the general form

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_0(x)y=\mathrm{g}(x)\text{, where } y^{(n)}=\frac{\mathrm{d}^n y}{\mathrm{d}x^n}.$$

Every first-order <u>linear</u> differential equation can be expressed in the standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{P}(x)y=\mathrm{Q}(x).$$

For a first-order differential equation of the above form, the function

$$\mathrm{I}(x)=\mathrm{e}^{\int \mathrm{P}(x)\,\mathrm{d}x}$$

is defined to be the **integrating factor** of the differential equation.

The following proof illustrates how the integrating factor can be used to solve a linear first-order differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{P}(x)y=\mathrm{Q}(x).$

Proof:

Multiplying both sides of the differential equation by $\mathrm{I}(x)=\mathrm{e}^{\int \mathrm{P}(x)\,\mathrm{d}x}$, we obtain

$$y=\frac{\int e^{\int P(x)\,dx}Q(x)\,dx}{e^{\int P(x)\,dx}}=\frac{\int I(x)Q(x)\,dx}{I(x)},\ \text{where } I(x)=e^{\int P(x)\,dx}.$$

This procedure in solving a differential equation is called the **method of integrating factor**.

Example 1.2.1

Solve the differential equation $x\frac{dy}{dx}+3y=5x^2$ for $x>0$, expressing y in terms of x.

Solution:

Step 1: Rewrite the differential equation in standard form.

$$x\frac{dy}{dx}+3y=5x^2 \Rightarrow \frac{dy}{dx}+\left(\frac{3}{x}\right)y=5x$$

Step 2: Compute the integrating factor.

$$I\left(x\right)=e^{\int \frac{3}{x}\,dx}=e^{3\ln x}=x^3$$

Step 3: Multiply both sides of the differential equation (<u>in standard form)</u> by the integrating factor.

$$x^3\frac{dy}{dx}+3x^2y=5x^4$$

Step 4: Rewrite the equation into an exact differential.

$$\frac{d}{dx}\left(x^3y\right)=5x^4$$

Step 5: Solve the differential equation by direct integration.

$$x^3y=\int 5x^4dx$$

$$x^3y=x^5+C$$

$$y=x^2+\frac{C}{x^3}$$

Will the solution be different if $x<0$?

Example 1.2.2

Express x in terms of t given that $t\dfrac{\mathrm{d}x}{\mathrm{d}t}-x=t^3$ and $x = 1$ when $t = 1$.

Solution:

Example 1.2.3

Find the solution of the differential equation $\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x}-y\cos x=\sin^3 x$, given that $y = 1$ when $x = \dfrac{\pi}{2}$.

Solution:

Example 1.2.4

By means of the substitution $\frac{1}{y^2} = -2z$, show that the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = y^3$ may be reduced to $\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{2z}{x} = 1$. Hence find the solution of $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = y^3$, given that $y = 2$ when $x = 1$.

Solution:

$$\frac{1}{y^2} = -2z \Rightarrow -\frac{2}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x} = -2\frac{\mathrm{d}z}{\mathrm{d}x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = y^3\frac{\mathrm{d}z}{\mathrm{d}x}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = y^3 \Rightarrow y^3\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{y}{x} = y^3$$

$$\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} + \frac{1}{xy^2} = 1 \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} - \frac{2z}{x} = 1 \qquad \text{(shown)}$$

§2 Equilibrium Points and Phase Lines of Autonomous Differential Equations

2.1 Equilibrium Point

Definition (Equilibrium Point) A real number c is called an **equilibrium point** of the autonomous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(y)$ if it is a **zero** of f, i.e. **f(*c*) = 0**.

An equilibrium point is also called a **critical point** or a **stationary point.** The solution ***y*(*x*) = *c*** is a **constant** solution and is called the **equilibrium solution** of the autonomous differential equation.

Note: An **autonomous** differential equation does not explicitly depend on the independent variable.

For example, the equilibrium points of $\frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y)$ are 0 and 1.

2.2 Phase Line

A **phase line** is a simple one-dimensional picture that captures all of the information provided by the differential equation in a single vertical line.

The phase line of an autonomous equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(y)$ is constructed as follows:

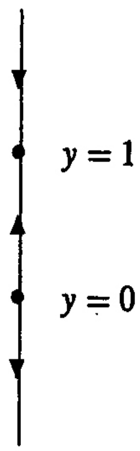
(1) Find the equilibrium points by solving $\mathrm{f}(y) = 0$.

(2) Draw a vertical line and divide the line into various regions by marking all the equilibrium points on it.

(3) For each region where $\mathrm{f}(y) > 0$, draw an arrow on the line pointing **upwards**. The solution curve $y(x)$ should be **increasing** in this case.

(4) For each region where $\mathrm{f}(y) < 0$, draw an arrow on the line pointing **downwards.** The solution curve $y(x)$ should be **decreasing** in this case.

The diagram below shows the phase line for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y)$.



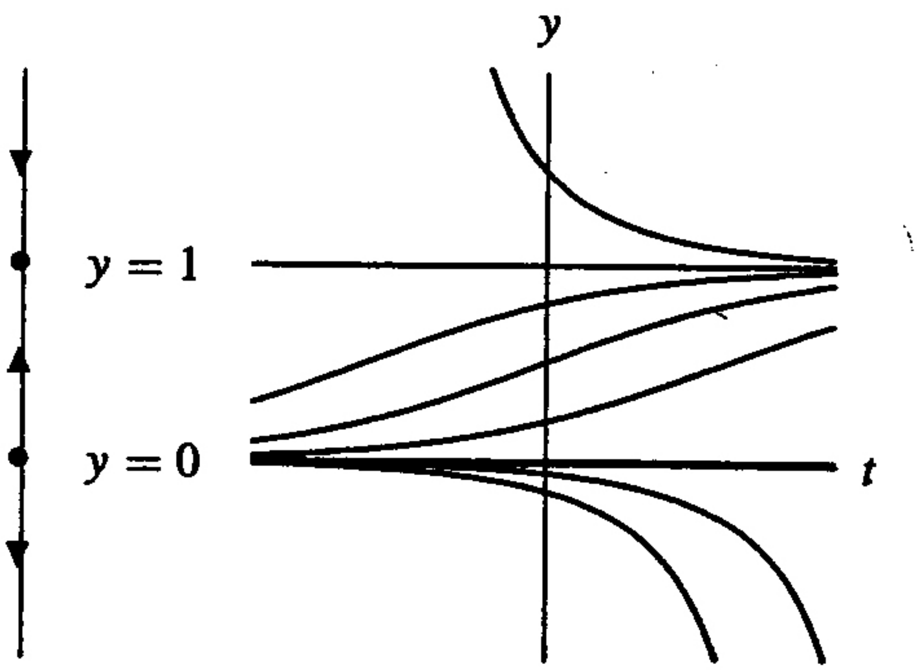
As shown in the diagram,

- $y = 0$ and $y = 1$ are the equilibrium points;
- when $y < 0$, $\mathrm{f}(y) < 0$, thus the arrow is pointing downwards in the phase line;
- when $0 < y < 1$, $\mathrm{f}(y) > 0$, thus the arrow is pointing upwards in the phase line;
- when $y > 1$, $\mathrm{f}(y) < 0$, thus the arrow is pointing downwards in the phase line.

2.3 Sketching Solution Curves Using Phase Lines

Phase lines are useful in showing how a solution behaves as the variable on the horizontal axis (say x) increases. With the help of the phase lines, we can sketch the solution curves as we know whether $y(x)$ is increasing, decreasing, or remains constant in the various regions. However, phase lines do not show how quickly or slowly the solution curves increase or decrease with respect to x.

The diagram below illustrates how the phase line of the above differential equation $\frac{dy}{dt} = y(1-y)$ can be used to sketch the solution curves.



As shown in the diagram,

- y = 0 and y = 1 are the equilibrium points and $y(t) = 0$ and $y(t) = 1$ are the equilibrium (constant) solutions;
- when $y < 0$, the arrow is pointing downwards in the phase line and the particular solution curves shown are decreasing and bounded above such that $y \to 0$ as $t \to -\infty$;
- when $0 < y < 1$, the arrow is pointing upwards in the phase line and the particular solution curves shown are increasing and bounded above and below by the two equilibrium points such that $y \to 1$ as $t \to +\infty$ and $y \to 0$ as $t \to -\infty$;
- when $y > 1$, the arrow is pointing downwards in the phase line and the particular solution curve shown is decreasing and bounded below such that $y \to 1$ as $t \to +\infty$.

Example 2.3.1

Find the equilibrium points and draw the phase line for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=y^2-1$. Hence, on a single diagram, sketch the solution curves, which pass through the point

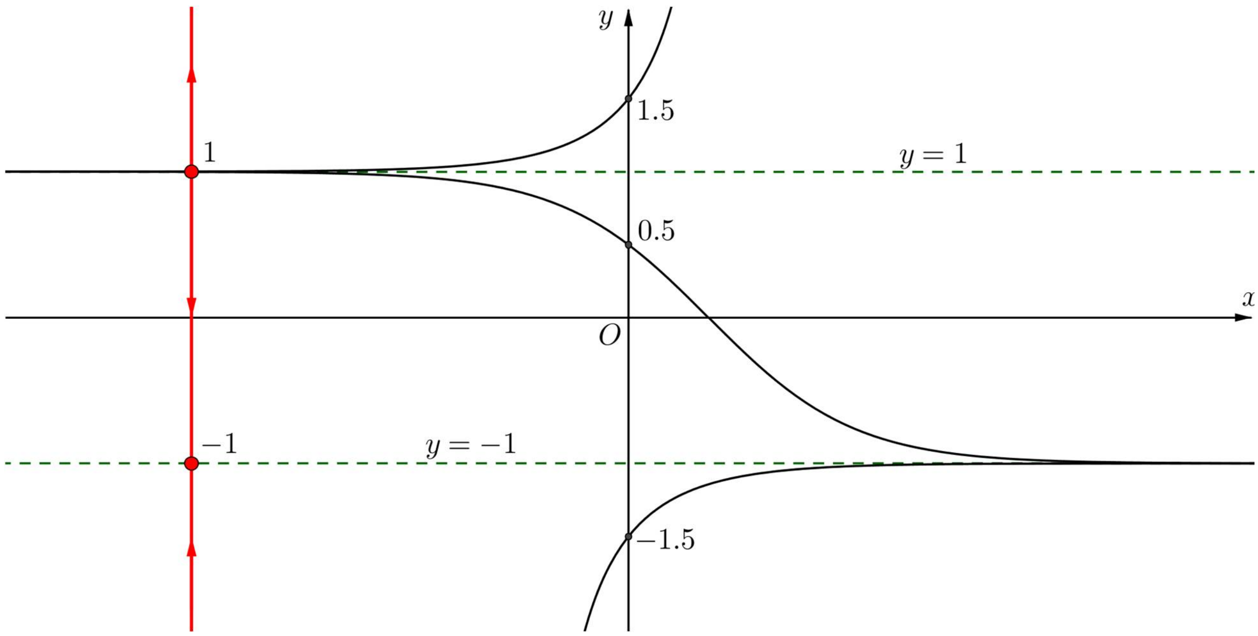
(i) (0, −1.5), **(ii)** (0, 0.5), **(iii)** (0, 1.5).

State the behaviour of y in **(ii)** as $x\to-\infty$ and $x\to+\infty$.

Solution:

Let $\mathrm{f}(y)=y^2-1$. then $\mathrm{f}(y)=0 \Rightarrow y^2-1=0 \Rightarrow y=\pm 1$

$\therefore$ The equilibrium points are 1 and -1.



For the solution curve in **(ii)**, $y\to 1$ and $y\to -1$ as $x\to-\infty$ and $x\to+\infty$ respectively.

§3 Population Dynamics

In population dynamics, we study how population changes with time under different conditions or in different environments. In reality, population dynamics can be very complex. Several assumptions have to be made in order to develop a mathematical model. We will introduce **three** basic models of population growth, namely, *exponential growth*, *logistic growth* and *logistic growth with harvesting*.

3.1 Exponential Growth (Malthus) Model

The Malthus (exponential growth) model introduced by Thomas Robert Malthus (an English economist) in 1798, assumes that *the growth rate of a population at a particular time is proportional to its total population at that time.*

This means that if there are more people at a particular time, then the growth rate of the population at that time will be higher than if there were fewer people. This model assumes that the size of the population is not limited by space and resources (e.g. food), and also ignores random fluctuations such as epidemics, natural disasters and migration effects.

We shall now look at how the exponential model is established.

<u>Real World Problem</u>

Consider the rats infestation problem at Bukit Batok during December 2014. Suppose the National Environment Agency wishes to formulate a mathematical model which describes the growth of the population of the rats in the area, if nothing is done about the problem.

<u>Make Assumptions</u>

- Assume there is no movement of rats in or out of Bukit Batok Area.
- The per-capita birth rate and the per-capita death rate of the population are constants. (In more complex models, these rates may vary with time and space.)
- Population is large enough to ignore random fluctuations. E.g. no sudden disease strikes the rats.

<u>Represent the Problem in a Mathematical Form and Solving the Mathematical Problem</u>

Independent variable : time, t
Dependent variable : size of population, P

Per-capita birth rate, β = number of births per unit population per unit time (constant).
Per-capita death rate, α = number of deaths per unit population per unit time (constant).

Rate of change of population = Rate of births – Rate of deaths

i.e. $$\frac{\mathrm{d}P}{\mathrm{d}t} = \beta P - \alpha P = (\beta - \alpha)P$$

Let $k = \beta - \alpha$ and we have the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

Hence, the rate of growth of a population at a particular time is proportional to its total population at the time.

Analytic Solution

This equation is separable and we can solve it analytically. Separating variables and integrating, we obtain:

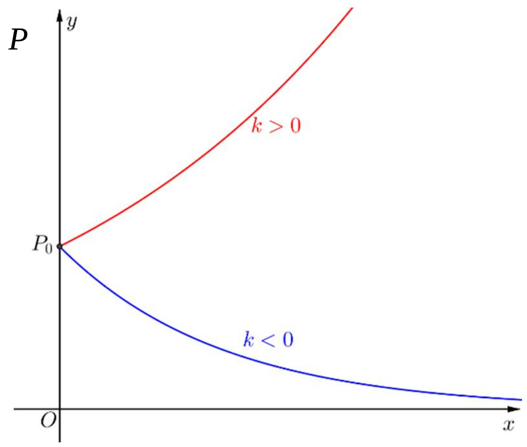
$$\int \frac{1}{P} \cdot \frac{\mathrm{d}P}{\mathrm{d}t}\,\mathrm{d}t = \int k\,\mathrm{d}t$$
$$\ln P = kt + C \qquad (\because P > 0)$$
$$P = \mathrm{e}^{kt+C} = A\mathrm{e}^{kt} \qquad \left(A = \mathrm{e}^{C}\right)$$

If at $t = 0$, $P = P_0$ (i.e. initial population), then we get $P = P_0\mathrm{e}^{kt}$

k is sometimes called the **net growth rate**.

If $k > 0$, there is growth in the population. If $k < 0$, there is decay in the population.

Graph of Solution Curves



When $k < 0$, the population **declines and tend to 0** as t increases. i.e. the rats in the area will die out over time.
When $k > 0$, the model suggests that the population will increase exponentially. However, in reality, the population will eventually be limited by some factor, e.g. limited food resources.

Question: What happens when $k = 0$?

Example 3.1.1

In 1800, the world's population was approximately 1 billion. In 1900, it was 1.7 billion. If the population P (in billions) at time t (in years, measured from 1800) obeys the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$, estimate the world's population in the year 2000. Find also the time taken, correct to the nearest 0.1 years, for the population to reach 2 billion.

Solution:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \Rightarrow P = P_0\mathrm{e}^{kt}$$

The time taken for the population to reach 2 billion is approximately **130.6 years** from 1800.

3.2 Logistic Growth (Verhulst) Model

Pierre François Verhulst (1804 – 1849) was a Belgian demographer who generalised the Malthus model by allowing for the fact that populations encounter internal competition as they grow within a closed environment, and this competition has a tendency to retard the rate of growth. His idea is that while the population will continue to grow as time goes on, the rate at which it does so will also get slower and eventually reach saturation.

In the Verhulst (logistic growth) model, the idea of limited resources and competition is taken into consideration. In other words, *crowding effects* become important. One way to incorporate these effects is to consider how the assumption on the per-capita death rate may be modified.

Instead of a constant per-capita death rate (as in the exponential growth model), we now assume a *linear* dependence of the per-capita death rate with the population size:

$$\text{Per-capita death rate} = \alpha + rP(t),$$

where α and γ are positive constants.

Using this assumption, the exponential growth model now becomes

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= \beta P - (\alpha + \gamma P)P \\ &= (\beta - \alpha)P - \gamma P^2 \\ &= kP - \gamma P^2\end{aligned}$$

which may be rewritten as

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)}$$

for some constants k and $N\left(= \frac{k}{\gamma}\right)$. This equation is also known as the **logistic equation** and the constant N is known as the **carrying capacity** of the environment.

Observe that when P is small compared to N (i.e. $\frac{P}{N} \to 0$), the equation reduces to the exponential growth model, which is reasonable since P is expected to increase exponentially as long as resources are abundant.

Analytic Solution of the Logistic Equation

We begin by first writing the equation in the form $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$.

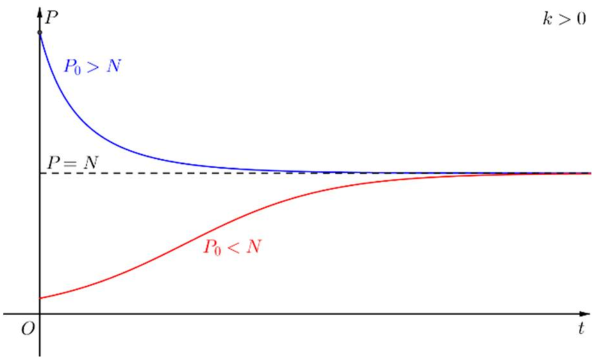
This equation is separable and we can solve it analytically. Separating variables and integrating, we obtain:

Making P the subject, we obtain the solution of the logistic equation as

$$P(t) = \frac{NP_0}{P_0 + (N - P_0)\mathrm{e}^{-kt}}.$$

Graph of Solution Curves of the Logistic Equation

Some typical solutions for the cases where $k > 0$ and $k < 0$ are shown below:



Note that for $P > 0$,

- if $k > 0$, $P \to N$ as $t \to \infty$;
- if $k < 0$ (now $N < 0$) $P \to 0$ as $t \to \infty$ provided $P_0 > 0$.

This is a more realistic approach than the exponential growth model, which predicts that populations will grow exponentially, and without a bound – a prospective that defies physical limitations.

Example 3.2.1

A junior college has 4,000 students. On the first day of the semester, a group of 4 students thought they heard their mathematics lecturer say that everyone would receive an A-grade for the course. The next day, a carefully conducted survey of the entire student population revealed that by now 80 students had heard this rumour. If the rumour spreads according to the logistic equation, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay(b - y),$$

where y is the number of students who have heard the rumour, t is the number of days after the day which the rumour started, and a and b are constants. At what time will 90% of the students have heard the rumour?

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay(b - y) = aby\left(1 - \frac{y}{b}\right)$$

Comparing this differential equation to the logistic equation, and since a and b are constants, it follows that $b = 4000$ (carrying capacity, which in this case is the total number of students).

$$t = -\frac{1}{4000a}\ln\frac{4}{9(3996)} = \text{ 3.02 days (3 sig. fig.)}$$

3.3 Logistic Equation with Harvesting

Recall the rats infestation problem at Bukit Batok during December 2014. On 22 Dec 2014, it was reported that "An estimated 180 rats have been caught, five days into an operation to rid an area beside Bukit Batok MRT station of the rodents." This process of catching the rats over a period of time is termed as **harvesting**[1].

The simplest way to include harvesting into the logistic equation is to **assume** that the population is **continually** being harvested at a **constant rate**, H. The equation thus becomes:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{N}\right)-H \ ,$$

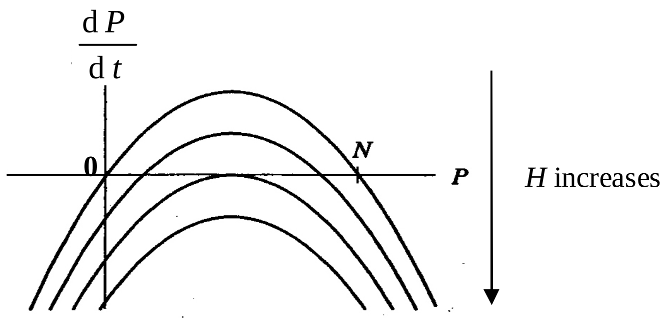
where k is the net growth rate and N is the carrying capacity of the environment.

We could, if we wish, solve the above equation analytically. However, for models like this, the more critical question is whether the population will survive or face extinction. That is, we are more interested in studying the long-term behaviour of the population under different values of H, assuming that k and N are *positive* constants.

[1] *Harvesting* refers to the removal of a certain number of individuals from the population of a species over a period of time. This is traditionally a result of hunting or gathering natural resources for use. In particular, we may wish to know the amount of resources that should be harvested so that the resource will not face extinction. (The fact that there are over 750 plants and animals on the endangered species list indicates that humans are not always cognizant of how their actions will affect plants and animals.

To do so, we look for equilibrium solutions and assuming that one can do nothing about the net growth rate (k) and the carrying capacity (N), we can then treat H as a real parameter in the model, and study its effects on the solutions of the model.

The graph of $\frac{\mathrm{d}P}{\mathrm{d}t}$ against P for different values of H is shown below:



By setting $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$, we can determine the equilibrium points as:

$$kP\left(1 - \frac{P}{N}\right) - H = 0 \Rightarrow P = \frac{N}{2} \pm \sqrt{\frac{N^2}{4} - \frac{HN}{k}}$$

The following can be observed.

1. When the value of H increases, the graph of $\frac{\mathrm{d}P}{\mathrm{d}t}$ against P moves down.

2. When the value of H increases, the equilibrium points

$$P = \frac{N}{2} + \sqrt{\frac{N^2}{4} - \frac{HN}{k}} \text{ and } P = \frac{N}{2} - \sqrt{\frac{N^2}{4} - \frac{HN}{k}}$$

move towards each other.

3. (i) When $H = 0$, the equilibrium points are $P = 0$ and $P = N$.

(ii) When $\frac{N^2}{4} - \frac{HN}{k} > 0$, i.e. $H < \frac{kN}{4}$, there are 2 equilibrium points.

(iii) When $\frac{N^2}{4} - \frac{HN}{k} < 0$, i.e. $H > \frac{kN}{4}$, there are no equilibrium points.

For case (iii), $\frac{\mathrm{d}P}{\mathrm{d}t}$ is negative for all values of P. In this scenario, the harvesting rate is large enough to ensure that the population will decrease regardless of the size of the initial population. The species will eventually become extinct when the population reaches zero.

Example 3.3.1

Let $P(t)$ be the population of a certain animal species. It was observed that if the population was large, the rate of growth decreases or even becomes negative; if the population was too small, fertile adults run the risk of not being able to find suitable mates so that the rate of growth is again negative. A suggested model for this population is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right), \ k > 0 .$$

The carrying capacity N indicates when the population is too big and the sparsity parameter M indicates when the population is too small.

(a) Find the equilibrium points and sketch a phase line for the above model.

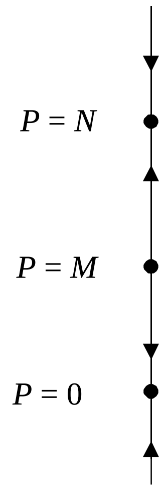
(b) Assume $N = 100$, $M = 1$ and $k = 1$. Hence, sketch a graph of the solution which satisfies the initial condition $P(0) = 20$, to show the behaviour of the population of the animal species in the long run.

(c) Assume that the animals are emigrating with a fixed rate E. Write down the new differential equation and find the set of values of E at which there are two, one or no equilibrium points for $P > 0$. Leave your answers to 2 decimal places.

Solution:

(a) For equilibrium points, $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right) = 0$

$\Rightarrow \quad P = 0$, $P = N$ or $P = M$



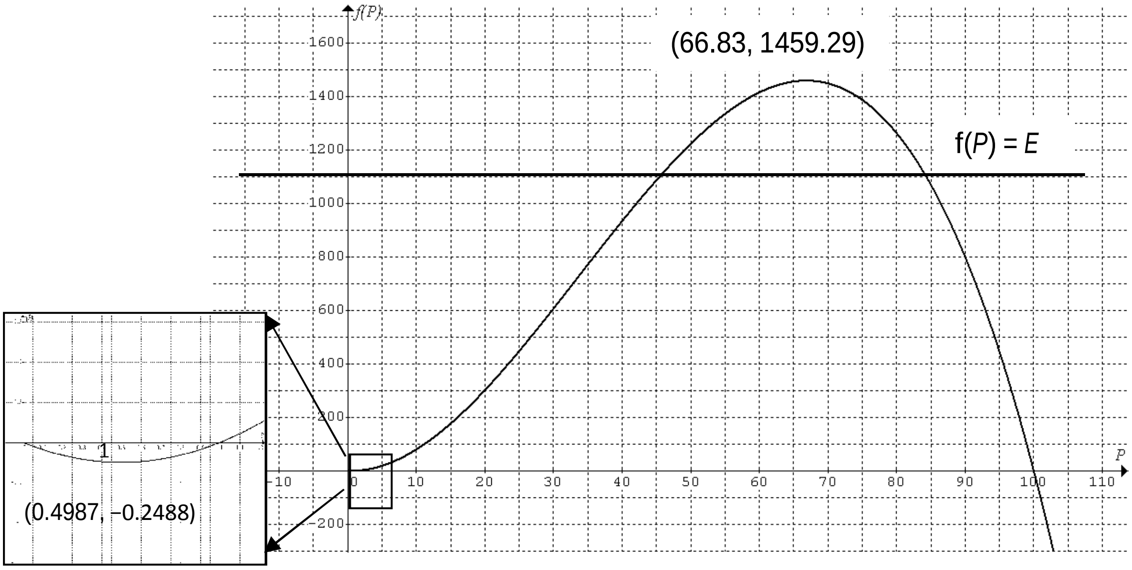
(b) Given: $N = 100$, $M = 1$ and $k = 1$.

(c) If the animals are emigrating with a fixed rate E, then the differential equation becomes

$$\frac{dP}{dt} = kP\left(1-\frac{P}{N}\right)\left(\frac{P}{M}-1\right)-E$$

Consider $\frac{dP}{dt} = kP\left(1-\frac{P}{N}\right)\left(\frac{P}{M}-1\right)-E=0 \Rightarrow kP\left(1-\frac{P}{N}\right)\left(\frac{P}{M}-1\right)=E$

Let $f(P) = kP\left(1-\frac{P}{N}\right)\left(\frac{P}{M}-1\right)=E$. When N = 100, M = 1 and k = 1, we have $f(P) = P\left(1-\frac{P}{100}\right)(P-1)=E$.



The graph of f(P) has a maximum turning point at (66.83, 1459.29) and a minimum turning point at (0.4987, −0.2488).

Hence, for

(i) $E > 1459.29$, there are no equilibrium points for $P > 0$.

(ii) $E = 1459.29$, there is only one equilibrium point for $P > 0$.

(iii) $0 < E < 1459.29$, there are two equilibrium points for $P > 0$.

Note: When $E > 1459.29$, $\frac{dP}{dt} < 0$ for all positive values of P. Thus, the population P decreases and becomes zero in a finite amount of time. As such, the species will become extinct.

Example 3.3.2

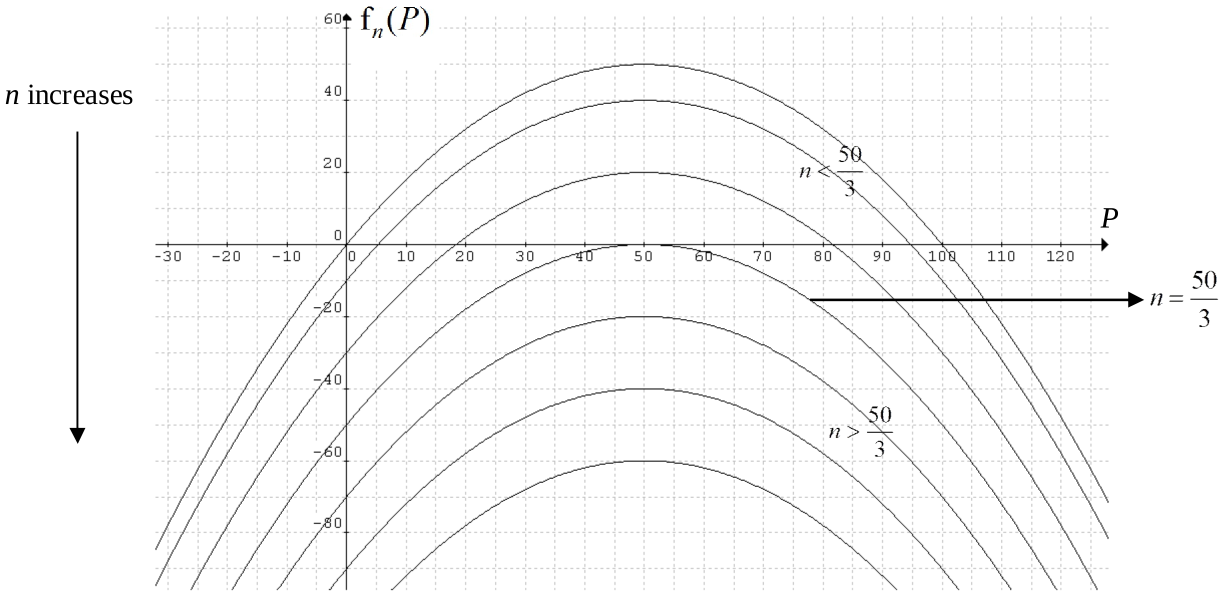
Consider the population model $\frac{\mathrm{d}P}{\mathrm{d}t}=2P-\frac{P^2}{50}$ for a species of fish in a lake. Suppose it is decided that fishing will be allowed, but it is unclear how many fishing licenses should be issued. Suppose the average catch of a fisherman with a license is 3 fish per year.

(a) What is the largest number of licenses that can be issued if the fish are to have a chance to survive in the lake?

(b) Suppose the number of fishing licenses in part (a) is issued. Discuss the long-term behaviour of the fish population for different initial population.

Solution:

(a) Let n denote the number of licenses issued.



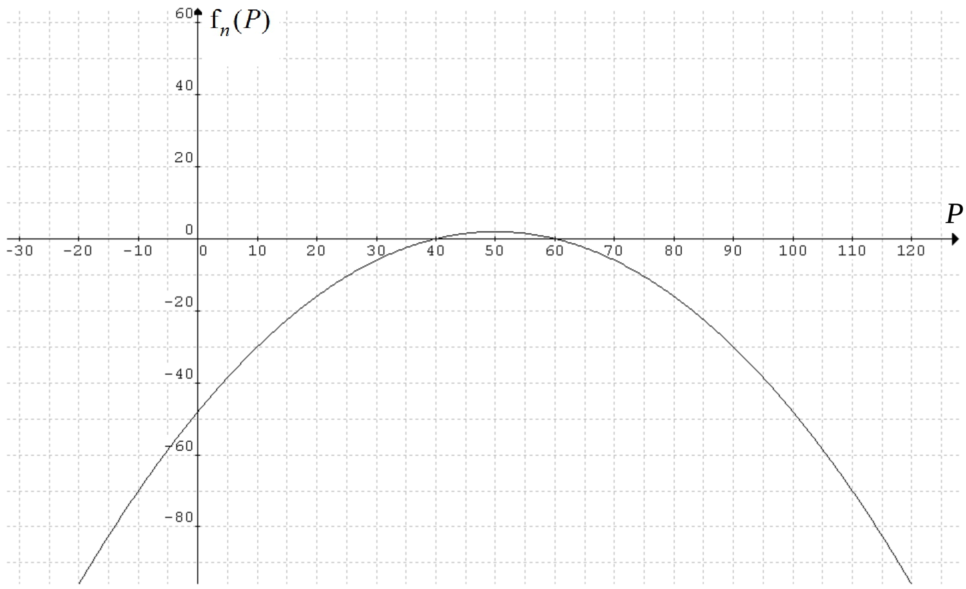
Note:

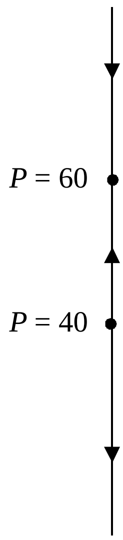
(i) When $2500-150n<0$, i.e. $n>\frac{50}{3}$, there are no equilibrium points and $\frac{\mathrm{d}P}{\mathrm{d}t}<0$. In other words, if the number of licenses issued is more than $\frac{50}{3}$, the species of fish will be wiped out in a finite period of time.

(ii) When $2500-150n>0$, i.e. $n<\frac{50}{3}$, there are two equilibrium points. In other words, if the number of licenses issued is less than $\frac{50}{3}$, the species of fish will eventually stabilise at a number greater than 50. However, this number will be smaller with more licenses issued.

Thus, the largest number of licenses that can be issued for the fishes to have a chance to survive is **16**.

(b) When n = 16, the equilibrium points are





(i)

(ii)

(iii)

§4 Analytic Solutions of Second Order Differential Equations

4.1 Introduction

A second order differential equation is one that contains the term $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ as the highest derivative. The following are examples of second order differential equations:

(a) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{f}(x)$

(b) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{g}(y)$

(c) $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$

(d) $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = \mathrm{f}(x)$

Which of the above types of differential equations do you already know how to solve?

4.2 Homogenous Linear Second Order Differential Equations with Constant Coefficients

A second order differential equation of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0 \qquad \text{---} \qquad (1)$$

where a, b and c are real constants is said to be a **homogenous linear** second order ordinary differential equation with constant coefficients.

Note that we sometimes use the notations y' and y'' to represent the first and second derivative of y with respect to x respectively. Equation (1) may then be written as

$$ay'' + by' + cy = 0 \qquad \text{---} \qquad (2)$$

We shall now look at how to solve such a differential equation analytically.

Definition
Two functions defined on an open interval I are said to be **linearly independent** on I if neither function is a constant multiple of the other.

Example 4.2.1

Which of the following pairs of functions are linearly independent on the entire real line?

(a) $\sin x$ and $\cos x$

(b) $\sin 2x$ and $\sin x \cos x$

(c) e^{x} and e^{-2x}

Solution:

$\dfrac{\sin x}{\cos x} = \tan x$, $\qquad \dfrac{e^{x}}{e^{-2x}} = e^{3x}$

For (a) and (c), the ratio of each pair does not give a constant-valued function on the entire real line and hence, they are linearly independent on the real line.

$\dfrac{\sin 2x}{\sin x \cos x} = \dfrac{2\sin x \cos x}{\sin x \cos x} = 2$ (constant on the entire real line)

$\Rightarrow$ $\sin 2x$ and $\sin x \cos x$ are linearly dependent on the real line.

Theorem 4.2.1

If y_1 and y_2 are **linearly independent** solutions of the differential equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$, then the **general solution** is $y = Ay_1 + By_2$, where A and B are arbitrary constants. Note that y_1 and y_2 are functions of x.

Proof:

Hence, $y = Ay_1 + By_2$ is a solution to $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$.

Strictly speaking, we have only proven that $y = Ay_1 + By_2$ is a solution to the differential equation, and not that it must be the general solution. We shall accept that $y = Ay_1 + By_2$ is the form of the general solution without proof.

One natural question to ask now will be what will be the forms of y_1 and y_2. It turns out that a function of the form $y = \mathrm{e}^{mx}$, where m is a constant seems to be a possible candidate.

If $y = \mathrm{e}^{mx}$ is to be a solution, then $\frac{\mathrm{d}y}{\mathrm{d}x} = m\mathrm{e}^{mx}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = m^2\mathrm{e}^{mx}$.
Substituting these expressions into equation (1), we see that $y = \mathrm{e}^{mx}$ if

$$am^2\mathrm{e}^{mx} + bm\mathrm{e}^{mx} + c\mathrm{e}^{mx} = 0$$

or $$\left(am^2 + bm + c\right)\mathrm{e}^{mx} = 0 \qquad \text{---} \qquad (5)$$

Since e^{mx} is never zero, thus $y = \mathrm{e}^{mx}$ is a solution to equation (1) if m is a root of the equation

$$am^2 + bm + c = 0 \qquad \text{---} \qquad (6)$$

Equation (6) is known as the **characteristic equation** or **auxiliary equation** of $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$

Note that it can be obtained from the differential equation by replacing $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ with m^2, $\frac{\mathrm{d}y}{\mathrm{d}x}$ with m and y with 1.

The characteristic equation is a quadratic equation whose roots can be easily obtained. Let the roots be m_1 and m_2. It is well know that the roots of a quadratic equation with real coefficients may be classified into three cases:

Case 1: The roots are real and distinct.
Case 2: The roots are complex conjugate pairs.
Case 3: The roots are real and equal.

Depending on the nature of the roots of the characteristic equation, we will obtain different forms of the general solutions. We will discuss each case in turn.

Case 1: The roots are real and distinct.

In this case, the roots m_1 and m_2 of the characteristic equation are real and distinct and so $y_1 = \mathrm{e}^{m_1x}$ and $y_2 = \mathrm{e}^{m_2x}$ are two linearly independent solutions of equation (1). By Theorem 4.1, we have the following result.

Result 4.2.2

If the roots m_1 and m_2 of the characteristic equation $am^2 + bm + c = 0$ are real and distinct, then the general solution of $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$ is $y = A\mathrm{e}^{m_1x} + B\mathrm{e}^{m_2x}$, where A and B are arbitrary constants.

Example 4.2.2

Solve the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+3y=0$.

Solution:

Characteristic equation:

$$m^2+4m+3=0$$
$$(m+3)(m+1)=0$$
$$m=-1 \text{ or } -3$$

General solution: $y=A\mathrm{e}^{-x}+B\mathrm{e}^{-3x}$

Case 2: The roots are complex conjugate pairs.

In this case, the roots m_1 and m_2 of the characteristic equation are complex numbers and we can write

$$m_1=\alpha+\mathrm{i}\beta \qquad \text{and} \qquad m_2=\alpha-\mathrm{i}\beta$$

where α and β are real numbers. (Taking $\sqrt{-1}=\mathrm{i}$, e.g. $\sqrt{-16}=4\mathrm{i}$)

From $y=A\mathrm{e}^{m_1x}+B\mathrm{e}^{m_2x}$, and using Euler's formula (i.e. $\mathrm{e}^{\mathrm{i}\theta}=\cos\theta+\mathrm{i}\sin\theta$), we obtain

$$\begin{aligned}
y &= A\mathrm{e}^{m_1x}+B\mathrm{e}^{m_2x}\\
&= A\mathrm{e}^{(\alpha+\mathrm{i}\beta)x}+B\mathrm{e}^{(\alpha-\mathrm{i}\beta)x}\\
&= \mathrm{e}^{\alpha x}\left[A\mathrm{e}^{\mathrm{i}\beta x}+B\mathrm{e}^{-\mathrm{i}\beta x}\right]\\
&= \mathrm{e}^{\alpha x}\left[A(\cos\beta x+\mathrm{i}\sin\beta x)+B(\cos\beta x-\mathrm{i}\sin\beta x)\right]\\
&= \mathrm{e}^{\alpha x}\left[(A+B)\cos\beta x+\mathrm{i}(A-B)\sin\beta x\right]\\
&= \mathrm{e}^{\alpha x}\left[C\cos\beta x+D\sin\beta x\right], \quad \text{where } C=A+B, \quad D=\mathrm{i}(A-B)
\end{aligned}$$

Result 4.2.3

If the roots m_1 and m_2 of the characteristic equation $am^2+bm+c=0$ are the complex numbers $m_1=\alpha+\mathrm{i}\beta$ and $m_2=\alpha-\mathrm{i}\beta$, then the general solution of $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0$ is $y=\mathrm{e}^{\alpha x}(A\cos\beta x+B\sin\beta x)$, where A and B are arbitrary constants.

Example 4.2.3

Solve the differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+5y=0$.

Solution:

Characteristic equation:

$$m^2+4m+5=0$$

Case 3: ***The roots are real and equal.***

If $m_1 = m_2 = m$, then, we immediately only have one solution of the form $y=\mathrm{e}^{mx}$. We need to find another solution. It turns out that $y=x\mathrm{e}^{mx}$ is also a solution in this case which we shall verify.

In solving the characteristic equation, we get

$$am^2+bm+c=0 \Rightarrow m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Since the roots are real and equal, $b^2-4ac=0$ and we obtain

$$m=\frac{-b}{2a}\Rightarrow 2ma+b=0$$

From $y=x\mathrm{e}^{mx}$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{mx}+mx\mathrm{e}^{mx} \text{ and } \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=m\mathrm{e}^{mx}+m\mathrm{e}^{mx}+m^2x\mathrm{e}^{mx}=2m\mathrm{e}^{mx}+m^2x\mathrm{e}^{mx}$$

Substitute into LHS of equation (1), we obtain

$$\begin{aligned}\text{LHS}&=a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy\\&=a\left(2m\mathrm{e}^{mx}+m^2x\mathrm{e}^{mx}\right)+b\left(\mathrm{e}^{mx}+mx\mathrm{e}^{mx}\right)+cx\mathrm{e}^{mx}\\&=\left(2ma+b\right)\mathrm{e}^{mx}+\left(am^2+bm+c\right)x\mathrm{e}^{mx}\\&=\left(0\right)\mathrm{e}^{mx}+\left(0\right)x\mathrm{e}^{mx}\\&=0\\&=\text{RHS}\end{aligned}$$

Using **Theorem 4.2.1**, we have the following result.

Result 4.2.4

If the roots of the characteristic equation $am^2 + bm + c = 0$ are real and equal such that $m_1 = m_2 = m$, then the general solution of $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$ is $y = (A + Bx)\mathrm{e}^{mx}$, where A and B are arbitrary constants.

Example 4.2.4

Solve the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$.

Solution:

Characteristic equation:

General solution:

Example 4.2.5

Show by means of the substitution $z = y^3$ that the differential equation

$$3y^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 6y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y^3 = 0$$

can be reduced to the form

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} + a\frac{\mathrm{d}z}{\mathrm{d}x} + bz = 0$$

where a and b are real numbers to be determined.

Hence, or otherwise, find the general solution of the given differential equation, expressing y in terms of x.

Solution:

Characteristic equation:

General solution:

4.3 Non-Homogenous Linear Second Order Differential Equations with Constant Coefficients

The general form for a non-homogeneous second order differential equation with constant coefficients is

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=\mathrm{f}\left(x\right) \qquad \text{---} \qquad (7)$$

where *a, b* and *c* are real constants and $\mathrm{f}\left(x\right)\neq 0$.

Now, we call its homogeneous equivalent (i.e. equation (1) under Section 4.2) the associated homogeneous equation. The general solution to the associated homogeneous equation is known as the **complementary function**, which we will denote by y_c.

Any function, y_p, that satisfies equation (7) is known as its **particular solution** or **particular integral**.

Theorem 4.3.1

Let y_p be any particular solution of the non-homogeneous linear second order differential equation $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=\mathrm{f}\left(x\right)$, and let y_1 and y_2 be linearly independent solutions of the associated homogeneous differential equation $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0$. Then the general solution is $y=c_1y_1+c_2y_2+y_p$, where c_1 and c_2 are arbitrary constants.

Thus, the general solution of the non-homogeneous linear second order differential equation can be written as $y=y_c+y_p$, where $y_c=c_1y_1+c_2y_2$.

As discussed earlier, y_c may be obtained by solving the characteristic equation of the associated homogeneous differential equation. Now, the question is, how do we find y_p? One way is to use the **method of undetermined coefficients**.

Method of Undetermined Coefficients

The main idea behind this method is to guess a possible form for y_p by looking at the type of function that makes up the RHS of equation (7), i.e. $\mathrm{f}(x)$. We may choose a possible y_p that is kind of similar to $\mathrm{f}(x)$ in some ways. The chosen y_p would probably contain some unknown (hence, 'undetermined') coefficients. We then substitute this chosen y_p into equation (7) and by equating like terms, we determine the values of these coefficients.

This method would have a good chance of success if $\mathrm{f}(x)$ is a constant, a polynomial function, an exponential function, a sine or cosine function, or some sums of products of these functions.

The following table shows some common $\mathrm{f}(x)$ and their corresponding suitable particular integrals:

$\mathrm{f}(x)$	Trial function for particular integral
Polynomial: $\mathrm{f}(x)=a_0+a_1x+a_2x^2+...+a_nx^n$	$y_p=\alpha_0+\alpha_1x+...+\alpha_nx^n$, where $\alpha_i, 0\le i\le n$ are constants to be determined.
Exponential function: $\mathrm{f}(x)=k_0e^{ax}$	$y_p=ke^{ax}$, where k is a constant to be determined.
Trigonometric functions: $\mathrm{f}(x)=a\cos kx+b\sin kx$	$y_p=\alpha\cos kx+\beta\sin kx$, where α and β are constants to be determined.

For example,

$\mathrm{f}(x)$	Trial function for particular integral
$\mathrm{f}(x)=3x-4$	
$\mathrm{f}(x)=2x^2+12$	
$\mathrm{f}(x)=5\mathrm{e}^{-2x}$	
$\mathrm{f}(x)=2\sin x-3\cos x$	
$\mathrm{f}(x)=3x-4\mathrm{e}^{x}$	
$\mathrm{f}(x)=-2\sin 2x$	

Note:

If the particular integral contains a term found in the complementary function, then the choice for the particular integral must be multiplied by x, repeatedly, until the chosen particular integral is no longer a solution of the corresponding homogeneous equation.
For example, if $\mathrm{f}(x)=10\mathrm{e}^{x}$ and $y_c=A\mathrm{e}^{x}+B\mathrm{e}^{-3x}$, then we should choose $y_p=kx\mathrm{e}^{x}$.

Example 4.3.1

Find the general solution of

(a) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\dfrac{\mathrm{d}y}{\mathrm{d}x}=6x$,

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\dfrac{\mathrm{d}y}{\mathrm{d}x}+4y=16\mathrm{e}^{-2x}$,

(c) $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\dfrac{\mathrm{d}x}{\mathrm{d}t}+5x=\sin 2t.$

Solution:

(a) Characteristics equation: $m^2+3m=0 \Rightarrow m=0$ or -3
Complementary function: $y_c=A+B\mathrm{e}^{-3x}$

Let the particular integral $y_p=ax^2+bx$. Then $y_p'=2ax+b$ and $y_p''=2a.$
Substituting into the differential equation:

$y_p'+3y_p''=6x$

$2a+3\left(2ax+b\right)=6x$

$6ax+2a+3b=6x$

$\therefore a=1, b=-\dfrac{2}{3}$

Thus $y_p=x^2-\dfrac{2}{3}x.$

General solution is $y=y_c+y_p=A+B\mathrm{e}^{-3x}+x^2-\dfrac{2}{3}x.$

(b) Characteristics equation: $m^2+4m+4=0 \Rightarrow m=-2$
Complementary function: $y_c=\left(Ax+B\right)\mathrm{e}^{-2x}$

General solution is $y=\left(Ax+b\right)\mathrm{e}^{-2x}+8x^2\mathrm{e}^{-2x}$

(c)

§5 Modelling with Second Order Differential Equations

In this section, we will examine how the motion of a vibrating spring (under various circumstances) can be modelled by second order differential equations and henceforth describe this motion completely by solving these differential equations.

5.1 Free Undamped Motion

A **spring** is an elastic object which is typically used to store energy due to resilience and subsequently release it to absorb shock, or to maintain a force between contacting surfaces.

For example, suspension systems in vehicles (that comprise multiple springs) contribute to the vehicles' road holding/handling and braking for good active safety and driving pleasure, and keeping vehicle occupants comfortable and reasonably well isolated from road noise, bumps, and vibrations.

Consider an elastic string or a spring in both its natural state, as well as when it is suspended with a mass hanging from it as shown in Figure 1.

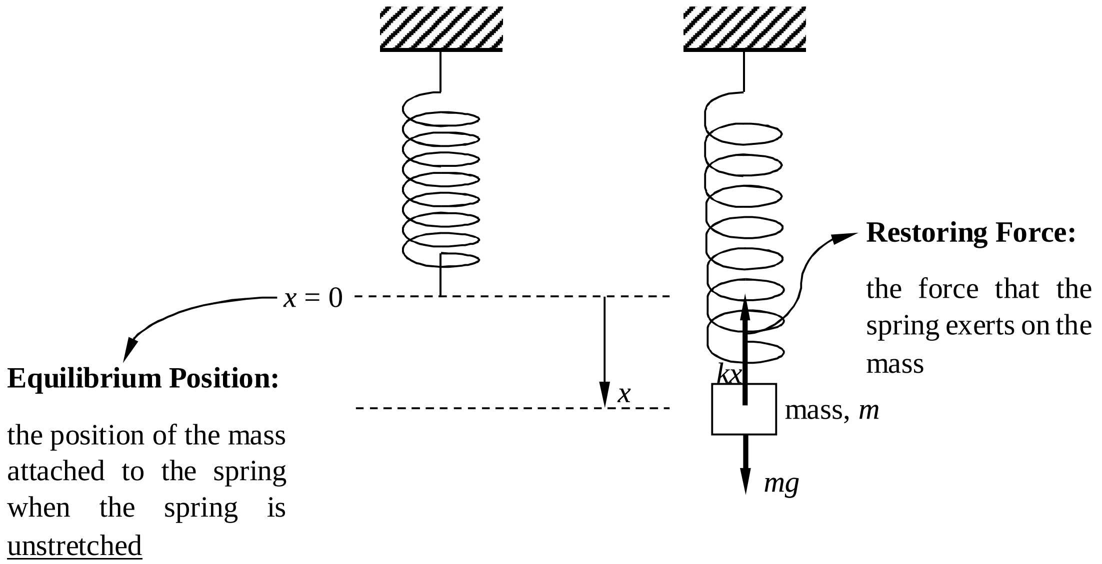


Figure 1: Elastic Spring

Now, suppose the initial length of the spring is at $x = 0$, the equilibrium position. Adding on a mass will stretch the spring by a certain distance as shown in Figure 1. Let x be the displacement of the mass, m, from the equilibrium position, where the positive direction is in the same direction as the gravitational acceleration, g. The spring will tend to restore to its original length by reacting with a *restoring force*.

In general, we can describe the restoring force using a physical principle known as *Hooke's Law.*

Hooke's Law

The restoring force F_R exerted by an elastic spring on a mass is **directly proportional** to the displacement x of the mass m from its equilibrium position. In other words,

$$F_R = -kx$$

for some positive constant k (known as the *spring constant*).

Note that the negative sign on the right hand side of Hooke's Law indicates that the restoring force F_R acts in the opposite direction of the displacement x.

Before we discuss how to describe the motion of the mass in Figure 1 mathematically, let us first consider another spring-mass system as shown in Figure 2.

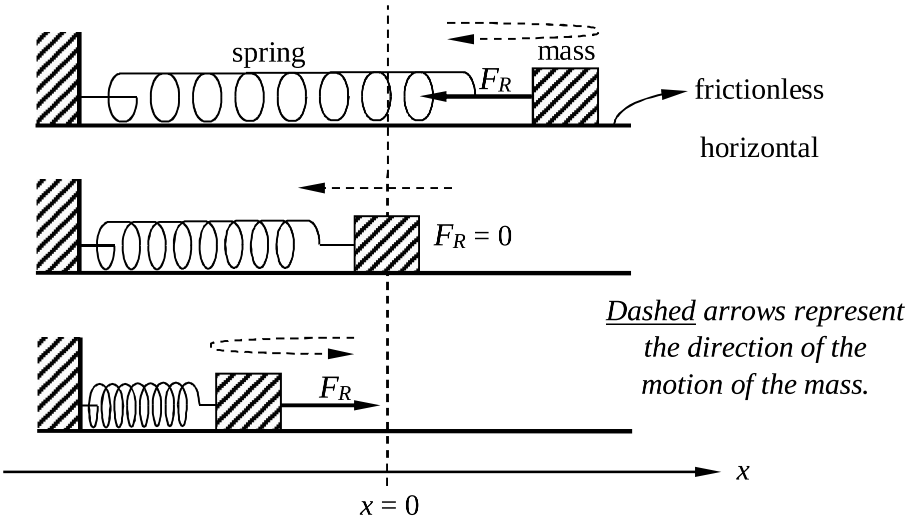


Figure 2: A spring-mass system

Here, a mass is attached to one end of a spring whose other end is fixed to a vertical wall. The mass is allowed to move on a smooth, frictionless horizontal surface. Suppose this spring-mass system is set in motion by displacing the mass from its resting position and letting it oscillate.

Using Newton's second law (which states that the net force on a body is equal to its mass multiplied by its acceleration), the equation of motion of the mass may be written as

$$mx'' = F = -kx,$$

where x'' is the second derivative of x with respect to time, which represents the *acceleration* of the mass due to its motion.

Rewriting the equation, we obtain

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{k}{m}x = 0,$$

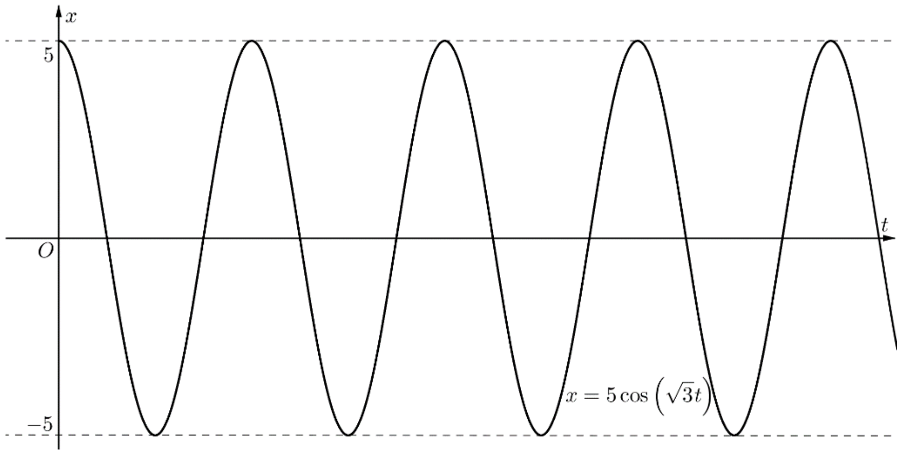
which is a linear homogeneous second order differential equation with constant coefficients.

Hence, by finding its characteristic equation, we can obtain the general solution as

$$x = A\cos(\omega t - \phi),$$

where A is the *amplitude* (i.e. maximum distance from the equilibrium position), $\omega = \sqrt{k/m}$ is the *angular frequency*, and ϕ is the *phase angle*. This equation models the motion of the mass, which in this case, will oscillate about the equilibrium position freely without resistance. This motion is known as **simple harmonic motion**.

Depending on the initial conditions, the values of A and ϕ can be found. If the constants k and m are known, we can calculate the angular frequency ω. The graph below shows the solution curve for the case when $k = 6$ and $m = 2$, and $A = 5$ and $\phi = 0$.



As one would expect, the mass will oscillate indefinitely and the motion repeats itself i.e. it is *periodic*. The period (the time taken for the mass to complete one full oscillation) is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}\ .$$

The *frequency* (the number of complete cycles per second) of the motion, measured in Hertz (Hz), is given by

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\ .$$

Note that both the period and frequency of the motion is independent of the amplitude A and the phase angle ϕ .

5.2 Free Damped Motion

Another important model that uses the homogeneous linear second order differential equation is the spring-mass-dashpot system.

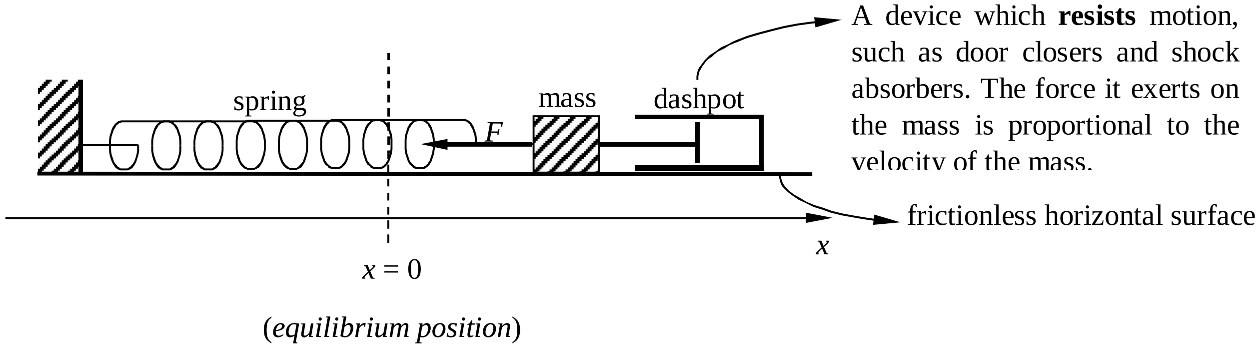


Figure 3: A Spring-mass-dashpot system

The equation of motion for the mass m shown above is

$$\underbrace{mx''}_{\text{net force on mass}} = \underbrace{-kx}_{\substack{\text{restoring force}\\\text{exerted by spring}}} + \underbrace{\left(-cx'\right)}_{\substack{\text{force exerted}\\\text{by dashpot}}},$$

where c is a positive constant called the *damping constant* of the dashpot. The higher the value of c, the higher the damping or higher resistance to motion is. Rewriting, the above equation gives

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{c}{m}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{k}{m}x=0,$$

which is a homogeneous linear second order differential equation with constant coefficients. Note that the right hand side of the equation is zero, which indicates that no external force is exerted on the system to force the motion. Consequently, the motion is referred as a *free damped* motion or free damped oscillation.

Using the characteristic equation, the equation can be solved. As discussed in the previous chapter, the solution depends on the roots of the characteristic equation (whether they are (a) real and distinct, (b) real and equal, or (c) complex conjugate pairs).

Example 5.2.1

Consider a spring-mass-dashpot system modelled by the equation

$$x'' + bx' + 10x = 0,$$

where the coefficient b determines the amount or degree of damping. It is also given that the initial displacement of the mass from its equilibrium position and its initial velocity are 1 m and 1 ms^{-1} respectively. For each of the following cases, find x in terms of t and sketch the corresponding solution curve.

(a) $b = 7$,
(b) $b = 2$,
(c) $b = 2\sqrt{10}$.

Solution:

(a) $x'' + 7x' + 10x = 0$

Characteristic equation: $m^2 + 7m + 10 = 0$

$\Rightarrow (m+2)(m+5) = 0$

$\Rightarrow m = -2$ or $m = -5$

Hence the general solution is given by $x = A\mathrm{e}^{-2t} + B\mathrm{e}^{-5t}$.

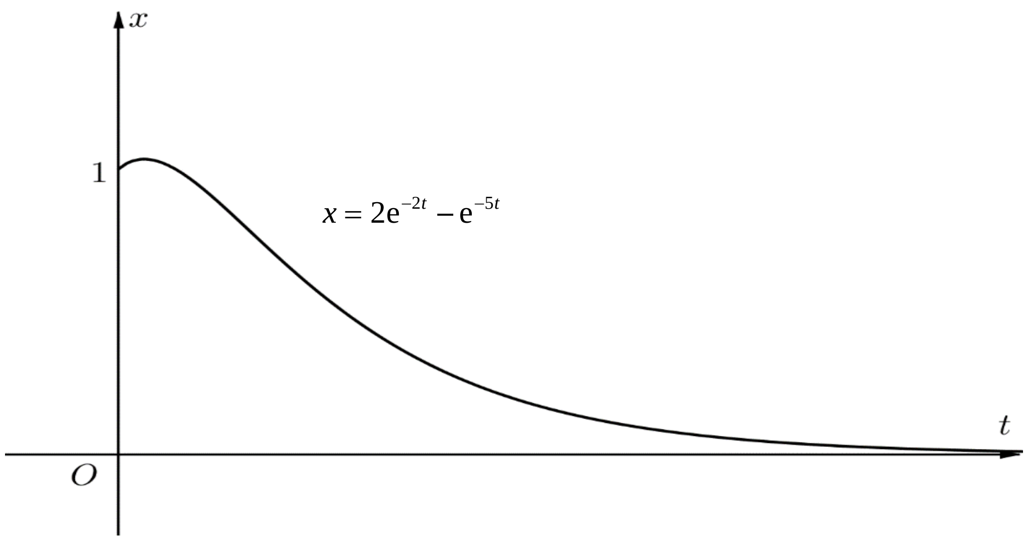
When t = 0, x = 1 $\Rightarrow 1 = A + B \ \ ---(1)$

$x = A\mathrm{e}^{-2t} + B\mathrm{e}^{-5t} \Rightarrow x' = -2A\mathrm{e}^{-2t} - 5B\mathrm{e}^{-5t}$

When t = 0, x' = 1 $\Rightarrow 1 = -2A - 5B \ \ ---(2)$

Solving (1) and (2) simultaneously, A = 2, B = –1. Hence $x = 2\mathrm{e}^{-2t} - \mathrm{e}^{-5t}$.

The graph is as shown below. Note that the motion is **not** oscillatory.



As can been observed from the graph, the displacement x approaches zero in a gradual manner. This signifies that the damping provided by the dashpot is quite large compared to the restoring force exerted by the spring. In this case, we say that the motion is **over-damped**.

(b) $x'' + 2x' + 10x = 0$

Characteristic equation: $m^2 + 2m + 10 = 0$

$$\Rightarrow m = \frac{-2 \pm \sqrt{2^2 - 4(1)(10)}}{2(1)}$$

$$\Rightarrow m = -1 \pm 3\text{i}$$

Hence the general solution is given by $x = \text{e}^{-t}(A\cos 3t + B\sin 3t)$.

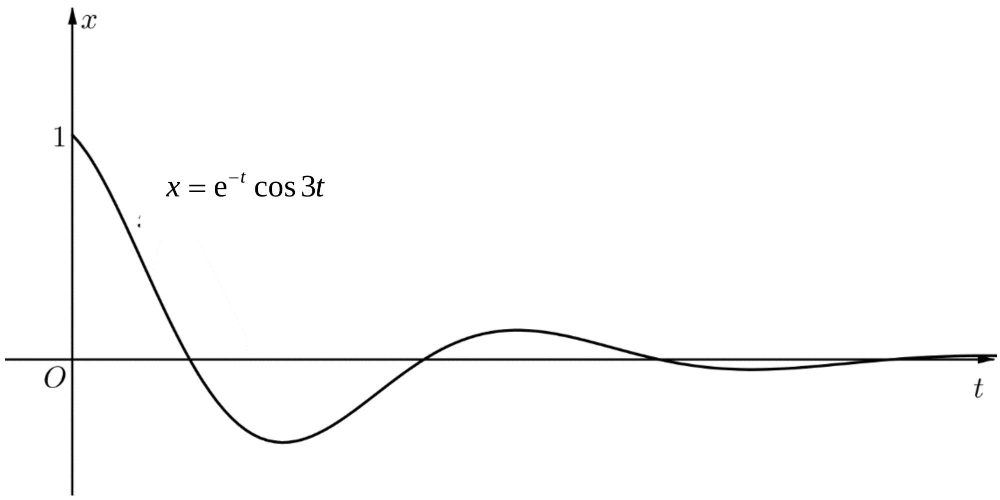
When $t = 0$, $x = 1$ $\Rightarrow 1 = A$

$$\begin{aligned} x' &= -\text{e}^{-t}(A\cos 3t + B\sin 3t) + \text{e}^{-t}(-3A\sin 3t + 3B\cos 3t) \\ &= \text{e}^{-t}\left((B - 3A)\sin 3t + (A + 3B)\cos 3t\right) \end{aligned}$$

When $t = 0$, $x' = 1$ $\Rightarrow 1 = 1 + 3B$ $\Rightarrow B = 0$

Hence $x = \text{e}^{-t}\cos 3t$.

The graph is as shown below. Note that the motion is **not** periodic.



As can been observed from the graph, the mass moves past the equilibrium position, then oscillates about it but with a smaller amplitude after each oscillation. As time goes on, the oscillations eventually taper off. In this case, because there is insufficient damping (since *b* has been reduced) to resist the motion, the system is said to be **under-damped**.

(c) $x'' + 2\sqrt{10}x' + 10x = 0$

Characteristic equation: $m^2 + 2\sqrt{10}m + 10 = 0$

$\Rightarrow \left(m + \sqrt{10}\right)^2 = 0$

$\Rightarrow m = -\sqrt{10}$ (repeated roots)

Hence the general solution is given by $x = Ate^{-\sqrt{10}t} + Be^{-\sqrt{10}t}$.

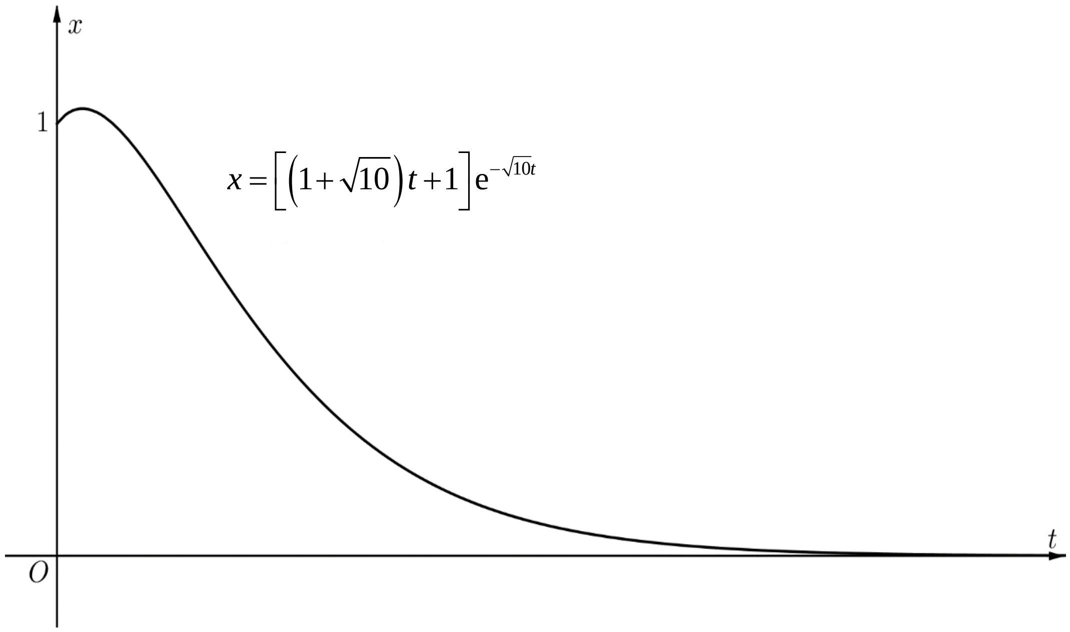
When $t = 0$, $x = 1$ $\Rightarrow 1 = B$ $---(1)$

$x = Ate^{-\sqrt{10}t} + Be^{-\sqrt{10}t} \Rightarrow x' = A\left(e^{-\sqrt{10}t} - \sqrt{10}te^{-\sqrt{10}t}\right) - B\sqrt{10}e^{-\sqrt{10}t}$

When $t = 0$, $x' = 1$ $\Rightarrow 1 = A - \sqrt{10} \Rightarrow A = 1 + \sqrt{10}$

Hence $x = \left[\left(1+\sqrt{10}\right)t + 1\right]e^{-\sqrt{10}t}$.

The graph is as shown below. Note that the motion is **not** oscillatory.



As can been observed from the graph, the system moves very quickly towards zero and remains close to it without any oscillation. This is the case where the motion is said to be **critically damped**.

It is useful to note that critically damped systems are similar to over-damped systems and are not oscillatory. In many mechanical vibration systems where damping is employed to reduce or control oscillations, one hopes to achieve critical damping.

5.2 Driven Motion

A non-homogeneous linear second order differential equation with constant coefficients used to represent a mechanical system may be written as

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+b\frac{\mathrm{d}x}{\mathrm{d}t}+cx=F(t) \qquad \text{---} \qquad (8)$$

where *a*, *b* and *c* are real constants, $x(t)$ is the *response function* and $F(t)$ is the *input function*. So far, we have considered free undamped and damped motion. These motions are called 'free' because there is no 'input function'. In other words, $F(t)=0$ in the equation of the motion, and this means that there is no force driving the motion.

Damped Driven Motion

Now, suppose the RHS of equation (8) is not zero, i.e. $F(t)\neq 0$, so that there is now an external force acting on the mass that is vibrating or allowed to vibrate. For instance, in a spring-mass-dashpot system, instead of having the spring fixed to a stationary wall or support, a driving force is moving the spring, as illustrated below.

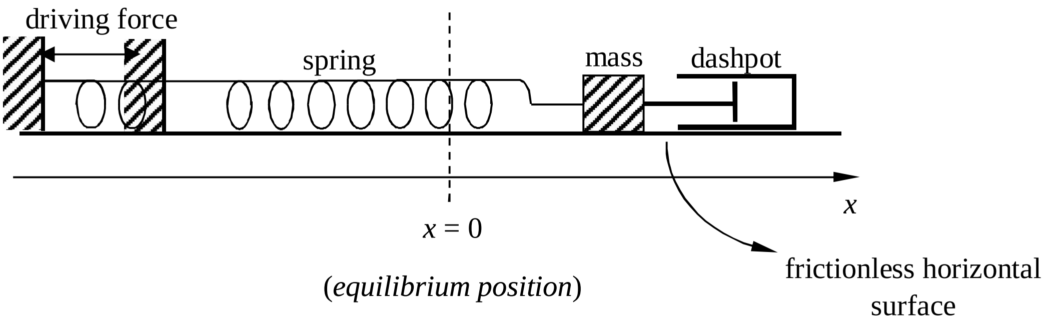


Figure 4: A Spring-mass-dashpot system with driven motion

This gives rise to a system with a **damped driven** or **forced motion**.

Undamped Driven Motion

Consider now the case of a spring-mass system without damping, and driven by some external force. The figure below depicts a mass hanging from a spring without any resisting force. The support to which the spring is attached is moved under some kind of force, and in response, the mass will move and oscillate.

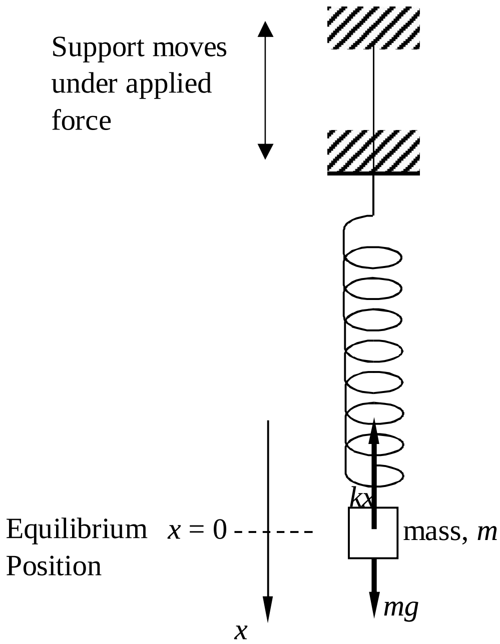


Figure 5: Elastic spring with driven motion

This system where the motion is undamped but driven by some force may be modelled using the same equation (8), with $b = 0$.

Example 5.3.1

A 10 kg mass is attached to a spring having a spring constant of 140 Nm^{-1}. The mass is started in motion from the equilibrium position with an initial velocity of 1 ms^{-1} in the upward direction and with an applied external force $F(t)=5\sin t$. Find an expression for the position of the mass, *x*, at any time *t*, if the force due to air resistance is $-90\dot{x}$ N.

Solution:

The equation of motion for the mass:

$$10\ddot{x}+90\dot{x}+140x=5\sin t$$

$$\Rightarrow \ddot{x}+9\dot{x}+14x=\frac{1}{2}\sin t$$

Characteristics equation: $m^2+9m+14=0 \Rightarrow m=-2 \text{ or } -7$

Complementary function: $y_c = A\mathrm{e}^{-2x}+B\mathrm{e}^{-7x}$

Let $y_p = a\cos t + b\sin t$.

Then $y_p' = -a\sin t + b\cos t$ and $y_p'' = -a\cos t - b\sin t$.

$-a\cos t - b\sin t + 9(-a\sin t + b\cos t) + 14(a\cos t + b\sin t) = \frac{1}{2}\sin t$

$(13b-9a)\sin t + (9b+13a)\cos t = \frac{1}{2}\sin t$

$\therefore 9b+13a=0 \Rightarrow b=-\frac{13}{9}a$

and $13\left(-\frac{13}{9}a\right)-9a=\frac{1}{2} \Rightarrow a=-\frac{9}{500}$

$\therefore b=-\frac{13}{9}\left(-\frac{9}{500}\right)=\frac{13}{500}$

Thus $y_p=\frac{13}{500}\sin t-\frac{9}{500}\cos t.$

General solution is $x=A\mathrm{e}^{-2t}+B\mathrm{e}^{-7t}+\frac{13}{500}\sin t-\frac{9}{500}\cos t.$

$\dot{x}=-2A\mathrm{e}^{-2t}-7B\mathrm{e}^{-7t}+\frac{13}{500}\cos t+\frac{9}{500}\sin t.$

When $t = 0$, $x = 0$ and $\dot{x} = -1$.

$$0 = A + B - \frac{9}{500} \Rightarrow A = \frac{9}{500} - B$$

Also,

$$-1 = -2A - 7B + \frac{13}{500} \Rightarrow 2A + 7B = \frac{513}{500}$$

$$\therefore 2\left(\frac{9}{500} - B\right) + 7B = \frac{513}{500} \Rightarrow 5B = \frac{99}{100} \Rightarrow B = \frac{99}{500}$$

$$\therefore A = \frac{9}{500} - \frac{99}{500} = -\frac{9}{50}$$

Thus, $x = -\frac{9}{50}\mathrm{e}^{-2t} + \frac{99}{500}\mathrm{e}^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t$.