1 An ellipse E has equation

$$9x^2 + 16y^2 = 144.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in term of x and y. [1]

(ii) Show that the point P with coordinates $(4\cos\theta,\ 3\sin\theta)$ lies on E. [1]

(iii) Find the equation of the normal to the ellipse E at the point P, in the form of $y = mx + c$ where m and c are single trigonometric expressions of θ. [3]

Solution:

(i) $9x^2 + 16y^2 = 144 \Rightarrow 18x + 32y\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{9x}{16y}$$

(ii) Since $9x^2 + 16y^2 = 9(4\cos\theta)^2 + 16(3\sin\theta)^2$

$$= 144(\cos^2\theta + \sin^2\theta)$$
$$= 144$$

Hence, point P lies on C.

(iii) Equation of normal at point P:

$$y - 3\sin\theta = \frac{16(3\sin\theta)}{9(4\cos\theta)}(x - 4\cos\theta)$$

$$\Rightarrow 3(\cos\theta)y - 9\cos\theta\sin\theta = 4(\sin\theta)x - 16\cos\theta\sin\theta$$

$$\Rightarrow y = \frac{4}{3}\tan\theta x - \frac{7}{3}\sin\theta$$

2 Consider the equation

$$2z^3 + (1-2\text{i})z^2 - (a+b\text{i})z + 2 + 2\text{i} = 0,$$

where *a* and *b* are real.

Given that -2 is a root of the equation, find the values of *a* and *b*. [3]

Given also that $1+\text{i}$ is another root, find the third root of the equation. [3]

Solution:

Since -2 is a root of the equation,

$\therefore 2(-2)^3 + (1-2\text{i})(-2)^2 - (a+b\text{i})(-2) + 2 + 2\text{i} = 0$

$\Rightarrow \quad -16 + 4 - 8\text{i} + 2a + 2b\text{i} + 2 + 2\text{i} = 0$

$\Rightarrow \quad 10 + 6\text{i} = 2a + 2b\text{i}$

$\therefore$ by comparing coeff., $\quad a = 5, \quad b = 3$

Let $z = \alpha$ be the third root.

$\therefore 2z^3 + (1-2\text{i})z^2 - (5+3\text{i})z + 2 + 2\text{i} = 2(z-\alpha)(z+2)(z-1-\text{i})$

By comparing constant term, we have

$2(-2\alpha)(-1-\text{i}) = 2 + 2\text{i}$

$2(-2\alpha)(-1-\text{i}) = 2 + 2\text{i}$

$\alpha = \dfrac{1}{2}$

3 HIMHEYS' Confectionery recently created three types of chocolate: Organic white, Organic milk and Organic dark, available in 250g bars. Cocoa butter, an essential ingredient in chocolate bars, makes up 25%, 20% and 15% of the mass of an Organic white, Organic milk and Organic dark chocolate bar respectively.

To prepare for the official launch of their chocolate bars at an upcoming Food Expo, HIMHEYS' decides to manufacture a total of 300 bars for the event, with more than 70 bars of each type.

The confectionary intends to use 14kg of cocoa butter in the production of the above batch of chocolate bars. If the number of milk chocolate bars is to be smaller than the number of white chocolate bars, determine how many organic chocolate bars of each type can be produced. [6]

Solution:

Let D, M and W be the number of dark, white and milk chocolate bars (250g organic) respectively.

$D+M+W=300$ -------- (1)

$$\left(\frac{15}{100}\times 250\times D\right)+\left(\frac{20}{100}\times 250\times M\right)+\left(\frac{25}{100}\times 250\times W\right)=14000$$

ie. $3D+4M+5W=1120$-------- (2)

Solving (1) and (2): $D=80+W, \quad M=220-2W$

Given $D>70, \quad W>70, \quad 70<M<W$,

$\Rightarrow \quad 70<220-2W<W$

$\Rightarrow \quad 73.3<W<75$

Since D, M and W are integers, we get $W=74, \quad M=72, \quad D=154$

4 Referred to an origin O, the position vectors of two points A and B are **a** and **b** respectively. The points P on OA and Q on AB are such that $OP = 2PA$ and $5AQ = 4QB$. Show that the equation of the line l passing through P and Q can be written as

$$\mathbf{r} = \frac{2}{3}\mathbf{a} + \lambda(4\mathbf{b} - \mathbf{a}), \text{ where } \lambda \in \mathbb{R}\ .$$ [4]

The point X on l is such that AX is perpendicular to l. If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 1$ and **a** is perpendicular to **b**, show that the position vector of X is $\frac{1}{15}(11\mathbf{a} - 4\mathbf{b})$. [4]

Solution:

Since $OP = 2PA$,

$\overrightarrow{OP} = \frac{2}{3}\mathbf{a}$

Since $5AQ = 4QB$, therefore $AQ{:}QB=4{:}5$

Using Ratio Theorem,

$$\overrightarrow{OQ} = \frac{5\mathbf{a} + 4\mathbf{b}}{9} = \frac{5}{9}\mathbf{a} + \frac{4}{9}\mathbf{b}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -\frac{1}{9}\mathbf{a} + \frac{4}{9}\mathbf{b} = \frac{1}{9}(4\mathbf{b} - \mathbf{a})$$

Since the line passes through P and is // $4\mathbf{b} - \mathbf{a}$

therefore an equation of l is

$$\mathbf{r} = \frac{2}{3}\mathbf{a} + \lambda(4\mathbf{b} - \mathbf{a}) \quad \text{where } \lambda \in \mathbb{R}$$

Since X lies on l, we have $\overrightarrow{OX} = \frac{2}{3}\mathbf{a} + t(4\mathbf{b} - \mathbf{a})$ for a particular value of t

Since AX is perpendicular to l, $\overrightarrow{AX} \bullet (4\mathbf{b} - \mathbf{a}) = 0$

$$\left(\frac{2}{3}\mathbf{a} + t(4\mathbf{b} - \mathbf{a}) - \mathbf{a}\right) \bullet (4\mathbf{b} - \mathbf{a}) = 0$$

$$\left(\left(-\frac{1}{3} - t\right)\mathbf{a} + 4t\mathbf{b}\right) \bullet (4\mathbf{b} - \mathbf{a}) = 0$$

Since a and b are perpendicular, $\therefore \mathbf{a} \bullet \mathbf{b} = 0$

$$\therefore \left(\frac{1}{3} + t\right)|\mathbf{a}|^2 + 16t|\mathbf{b}|^2 = 0$$

$$\therefore \left(\frac{1}{3} + t\right)4 + 16t = 0$$

$$t = -\frac{1}{15}$$

$$\therefore \overrightarrow{OX} = \frac{2}{3}\mathbf{a} - \frac{1}{15}(4\mathbf{b} - \mathbf{a}) = \frac{1}{15}(11\mathbf{a} - 4\mathbf{b})$$

5 The functions f and g are defined by

$$\mathrm{f} : x \mapsto \sqrt{6+x-x^2}, -2 < x < \frac{1}{2},$$

$$\mathrm{g} : x \mapsto \ln\left(9-x^2\right), -3 < x < 3.$$

Determine whether each of the following functions exists and give a definition (including the domain) of the function if it exists.

(a) f^{-1},

(b) gf. [9]

Solution:

(a) Any horizontal line $y = k, k \in \mathbb{R}$ cuts the graph of $y = f(x)$ at most once, so by the horizontal line test, f is 1-1. Therefore f^{-1} exists.

Now Domain of f^{-1}= Range of f = $\left(0, \frac{5}{2}\right)$

Let $y = \sqrt{6+x-x^2} \Rightarrow y^2 = 6+x-x^2$

$\Rightarrow x^2 - x + \left(y^2-6\right) = 0$

$\Rightarrow x = \dfrac{1 \pm \sqrt{1-4\left(y^2-6\right)}}{2} = \dfrac{1 \pm \sqrt{25-4y^2}}{2}$

Since $x \le \frac{1}{2}$, $x = \dfrac{1 - \sqrt{1-4\left(y^2-6\right)}}{2} = \dfrac{1 - \sqrt{25-4y^2}}{2}$

So $\mathrm{f}^{-1} : x \mapsto \dfrac{1-\sqrt{25-4y^2}}{2}, 0 < x < \dfrac{5}{2}$

(b) Range of f = $\left(0, \frac{5}{2}\right)$, Domain of g = (–3,3)

Since Range of f $\subseteq$ Domain of g, $\therefore \mathrm{gf}$ exists.

$$\begin{aligned}\mathrm{gf}(x) &= \mathrm{g}\left(\sqrt{6+x-x^2}\right) \\ &= \ln\left(9-\left(\sqrt{6+x-x^2}\right)^2\right) \\ &= \ln\left(3-x+x^2\right)\end{aligned}$$

So $\mathrm{gf}: x \mapsto \ln\left(3-x+x^2\right), -2 < x < \dfrac{1}{2}$

6 Given that $y=\ln(1+\sin x)$, show that

(i) $e^y \frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$, [2]

(ii) $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+\frac{\mathrm{d}y}{\mathrm{d}x}=0$. [3]

Find the Maclaurin's series for y up to and including the term in x^3. [3]

Hence, or otherwise, show that $\frac{\cos x}{1+\sin x}\approx 1-x+\frac{x^2}{2}$. [2]

Solution:

(i) $y=\ln(1+\sin x) \Rightarrow e^y=1+\sin x$

Diff wrt x, we have

$$e^y \frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$$

(ii) Diff wrt x again, we have

$$e^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+e^y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\sin x$$

Diff wrt x again, we have

$$e^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+2e^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+e^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+e^y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\cos x$$

$$e^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+2e^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+e^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+e^y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-e^y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\therefore \frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

When $x = 0$,

$$y=0,\quad \frac{\mathrm{d}y}{\mathrm{d}x}=1,\quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1,\quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=1$$

$$y=0+1x+\frac{-1}{2!}x^2+\frac{1}{3!}x^3+\cdots$$

$$\therefore y=x-\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots$$

i.e. $\ln(1+\sin x)=x-\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots$

[Hence]

Diff wrt to x, we have

$$\frac{\cos x}{1+\sin x}=1-x+\frac{1}{2}x^2+\cdots$$

[Otherwise] $\frac{\cos x}{1+\sin x}=\left(1-\frac{x^2}{2}\right)(1+x)^{-1}=\left(1-\frac{x^2}{2}\right)\left(1-x+x^2+\cdots\right)=1-x+\frac{1}{2}x^2+\cdots$

7 **(a)** A finite arithmetic progression has n terms and common difference d. The first term is 1 and the sum of the last 5 terms exceeds the sum of the first 4 terms by 193.

(i) Show that $5nd - 21d - 192 = 0$. [3]

(ii) Given also that the 6^{th} term of the progression is 16, find n. [2]

(b) A sequence U is formed in which the n^{th} term is given by e^{t_n} where t_n is the n^{th} term of an arithmetic progression with first term $t_1 = 1$.

(i) Show that U is a geometric progression. [2]

(ii) Given that the sum to infinity of even-numbered terms of U is $\frac{8e}{63}$, find the common ratio of U. [3]

Solution:

Method 1

(a)(i)

$$S_n - S_{n-5} = S_4 + 193$$

$$\frac{n}{2}\left[2+(n-1)d\right] - \frac{n-5}{2}\left[2+(n-6)d\right] = \frac{4}{2}(2+3d)+193$$

$$2n + nd(n-1) - 2(n-5) - (n-5)(n-6)d = 8 + 12d + 386$$

$$n^2d - nd + 10 - n^2d + 11nd - 30d = 12d + 394$$

$$10nd - 42d - 384 = 0$$

$$5nd - 21d - 192 = 0$$

(a)(ii)

$$1 + 5d = 16$$

$$d = 3$$

$$\therefore 5n(3) - 21(3) - 192 = 0$$

$$n = 17$$

Method 2

$$T_n + T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4} = S_4 + 193$$

$$\frac{5}{2}(T_{n-4} + T_n) = \frac{4}{2}(T_1 + T_4) + 193$$

$$\frac{5}{2}(a + (n-5)d + a + (n-1)d) = \frac{4}{2}(2a + 3d) + 193$$

$$\frac{5}{2}(2a + 2nd - 6d) = 4a - 6d + 193$$

$5nd - 21d - 192 = 0$ when a =1

(b)(i)

$u_n = e^{t_n}$ where $t_{n+1} = t_n + d$ (d is the common difference and is a constant)

$u_{n+1} = e^{t_{n+1}} = e^{t_n + d}$

$\frac{u_{n+1}}{u_n} = \frac{e^{t_n + d}}{e^{t_n}} = e^{d}$ which is a constant independent of n

$\therefore U$ is a geometric progression with common ratio $r = e^d$

(b)(ii) $\frac{er}{1-r^2} = \frac{8e}{63}$

$8r^2 + 63r - 8 = 0$

$\therefore r = \frac{1}{8}$ or $r = -8$ (rejected)

8 The point A has position vector 3**j** – 4**k** with respect to an origin O.
The plane π_1 has Cartesian equation $13x-9y+z=15$.

(i) If π_2 is a plane parallel to π_1 and contains the point A, write down the equation of π_2 in scalar product form and find the distance from the origin O to π_2. [3]

(ii) Hence or otherwise, find the distance between π_1 and π_2. State with clear explanations whether O and A are on the same side of π_1. [3]

(iii) Another plane π_3 has Cartesian equation $x+py+3z=q$. If π_2 and π_3 intersect at a line containing A and that π_3 is perpendicular to π_1, find the value of p and q. [4]

Solution:

(i) $\overrightarrow{OA}=\begin{pmatrix}0\\3\\-4\end{pmatrix}$ $\pi_2:\mathbf{r}\bullet\begin{pmatrix}13\\-9\\1\end{pmatrix}=\begin{pmatrix}0\\3\\-4\end{pmatrix}\bullet\begin{pmatrix}13\\-9\\1\end{pmatrix}=-27-4=-31\Rightarrow\pi_2:\mathbf{r}\bullet\begin{pmatrix}13\\-9\\1\end{pmatrix}=-31$

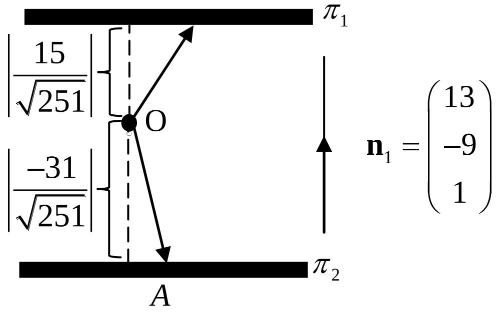
$\pi_2:\mathbf{r}\bullet\begin{pmatrix}13\\-9\\1\end{pmatrix}=-31\Rightarrow\mathbf{r}\bullet\frac{1}{\sqrt{251}}\begin{pmatrix}13\\-9\\1\end{pmatrix}=\frac{-31}{\sqrt{251}}\Rightarrow$ Distance from origin to π_2 is $\frac{31}{\sqrt{251}}$.

(ii)

Since

$\pi_1:\mathbf{r}\bullet\begin{pmatrix}13\\-9\\1\end{pmatrix}=15\Rightarrow\mathbf{r}\bullet\frac{1}{\sqrt{251}}\begin{pmatrix}13\\-9\\1\end{pmatrix}=\frac{15}{\sqrt{251}}>0$ and $\pi_2:\mathbf{r}\bullet\frac{1}{\sqrt{251}}\begin{pmatrix}13\\-9\\1\end{pmatrix}=\frac{-31}{\sqrt{251}}<0$

Hence distance between π_1 & π_2 $=\left|\frac{15}{\sqrt{251}}\right|+\left|\frac{-31}{\sqrt{251}}\right|=\frac{46}{\sqrt{251}}$



And from above diagram, A and O are on the same side of π_1.

(iii) Given π_3 has Cartesian equation $x + py + 3z = q \Rightarrow \pi_3 : \mathrm{r} \bullet \begin{pmatrix} 1 \\ p \\ 3 \end{pmatrix} = q$

Since π_3 is perpendicular to $\pi_1 \Rightarrow \underset{\sim}{n}_1 \perp \underset{\sim}{n}_3$,

$$\Rightarrow \begin{pmatrix} 13 \\ -9 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ p \\ 3 \end{pmatrix} = 0 \quad \Rightarrow \quad 13 - 9p + 3 = 0 \quad \Rightarrow \quad p = \frac{16}{9}$$

and point $A(0, 3, -4)$ lies in π_3,

$$\Rightarrow \quad 3p + 3(-4) = q \quad \Rightarrow \quad q = -\frac{20}{3}$$

9 **(a)** Prove by induction that

$$\sum_{r=1}^{n}\left(\frac{r}{2^r}+1\right)=\left(n+2\right)\left(1-\frac{1}{2^n}\right) .$$ [5]

(b) Show that u^2+u-1 can be written in the form $(u+2)(u+1)-k(u+2)+h$ where h and k are positive constants to be determined. [2]

Hence show that $\displaystyle\sum_{u=1}^{N}\frac{1-u-u^2}{(u+2)!}=\frac{N+1}{(N+2)!}-\frac{1}{2}$. [4]

Solution:

(a)

Let P_n be the statement "$\displaystyle\sum_{r=1}^{n}\left(\frac{r}{2^r}+1\right)=(n+2)\left(1-\frac{1}{2^n}\right),\ n\in\mathbf{Z}^+$"

When n=1

LHS= $\frac{1}{2}+1=\frac{3}{2}$

RHS= (1+2)(1-$\frac{1}{2}$)=$\frac{3}{2}$

$\therefore P_1$ is true

Assume P_k is true for some $k\in\mathbf{Z}^+$, i.e. $\displaystyle\sum_{r=1}^{k}\left(\frac{r}{2^r}+1\right)=(k+2)\left(1-\frac{1}{2^k}\right)$

Want to prove that P_{k+1} is true, i. e, $\displaystyle\sum_{r=1}^{k+1}\left(\frac{r}{2^r}+1\right)=(k+3)\left(1-\frac{1}{2^{k+1}}\right)$

From P_k, we add the (k+1)th term,

$$\begin{aligned}\sum_{r=1}^{k+1}\left(\frac{r}{2^r}+1\right)&=\sum_{r=1}^{k}\left(\frac{r}{2^r}+1\right)+\left(\frac{k+1}{2^{k+1}}+1\right)\\&=(k+2)\left(1-\frac{1}{2^k}\right)+\frac{k+1}{2^{k+1}}+1\\&=(k+3)-[\frac{(k+2)2}{2^{k+1}}-\frac{k+1}{2^{k+1}}]\\&=(k+3)-\frac{k+3}{2^{k+1}}\\&=(k+3)\left(1-\frac{1}{2^{k+1}}\right)\end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true. Therefore by mathematical induction, P_n is true for all n $\in\mathbb{Z}^{+}$

(b)

$u^2+u-1=(u+2)(u+1)-k(u+2)+h$

equating coeff of u: $1=3-k \Rightarrow k=2$

equating constant: $-1=2-2k+h \Rightarrow h=1$

$$\sum_{u=1}^{N}\frac{1-u-u^2}{(u+2)!}=\sum_{u=1}^{N}\frac{-(u^2+u-1)}{(u+2)!}$$

$$=(-1)\sum_{u=1}^{N}\frac{(u+2)(u+1)-2(u+2)+1}{(u+2)!}$$

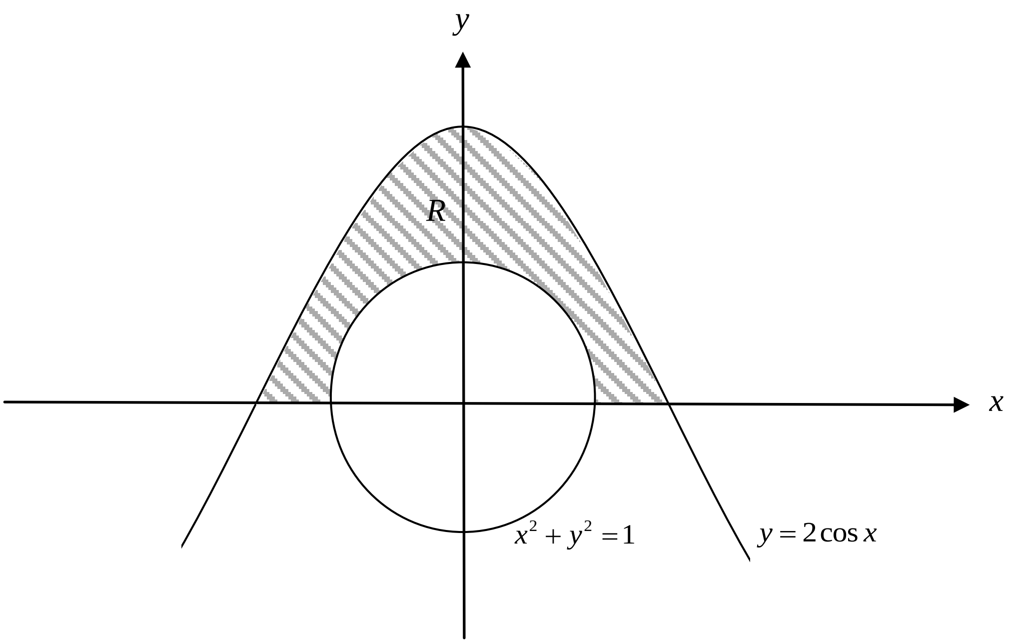
$$=\sum_{u=1}^{N}\frac{-1}{u!}+\frac{2}{(u+1)!}-\frac{1}{(u+2)!}$$

$$\begin{aligned}=\ & -1+\frac{2}{2!}-\frac{1}{3!}\\ & +\left(\frac{-1}{2!}+\frac{2}{3!}-\frac{1}{4!}\right)\\ & +\left(\frac{-1}{3!}+\frac{2}{4!}-\frac{1}{5!}\right)\\ & +\ \dots\\ & +\left(\frac{-1}{(N-2)!}+\frac{2}{(N-1)!}-\frac{1}{N!}\right)\\ & +\left(\frac{-1}{(N-1)!}+\frac{2}{N!}-\frac{1}{(N+1)!}\right)\\ & +\left(\frac{-1}{N!}+\frac{2}{(N+1)!}-\frac{1}{(N+2)!}\right)\end{aligned}$$

$$=-\frac{1}{2}+\frac{1}{(N+1)!}-\frac{1}{(N+2)!}$$

$$=-\frac{1}{2}+\frac{N+1}{(N+2)!}\ \text{(Shown)}$$

10



The diagram shows the region R bounded by the x-axis and the two curves $y = 2\cos x$ and $x^2 + y^2 = 1$.

(i) Find the exact area of the region R. [3]

(ii) Using integration by parts, show that

$$\int u^2 \sin u \, \mathrm{d}u = -u^2 \cos u + 2u \sin u + 2\cos u + C$$

where C is a real constant. [2]

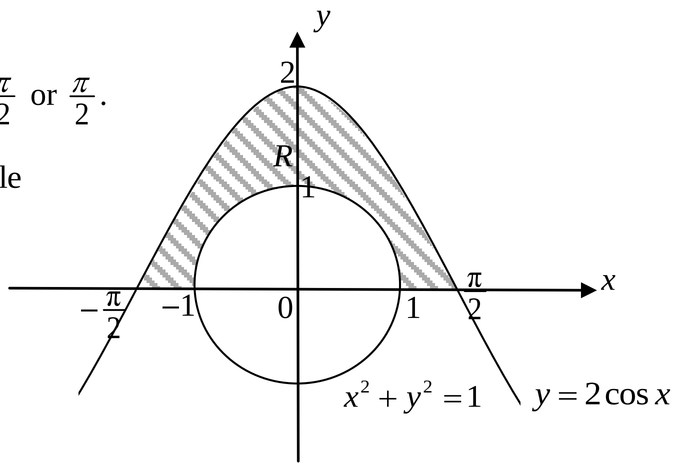
(iii) The region R is rotated π radians about the y-axis to form a solid of revolution S. Show that the volume of S can be expressed as

$$\pi \int_0^m \left(\cos^{-1} \frac{y}{2} \right)^2 \mathrm{d}y - k \,,$$

where m and k are exact values to be determined. [3]

Hence, by using the substitution $u = \cos^{-1} \dfrac{y}{2}$ and the result in part (ii), find the exact value of the volume of S. [4]

Solution:



(i) For the curve $y = 2\cos x$, when $y = 0$, $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$.

Shaded area $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos x \, dx - \text{Area of semi-circle}$

$= 2\int_0^{\frac{\pi}{2}} 2\cos x \, dx - \frac{1}{2}\pi(1)^2$

$= 2\left[2\sin x\right]_0^{\frac{\pi}{2}} - \frac{\pi}{2}$

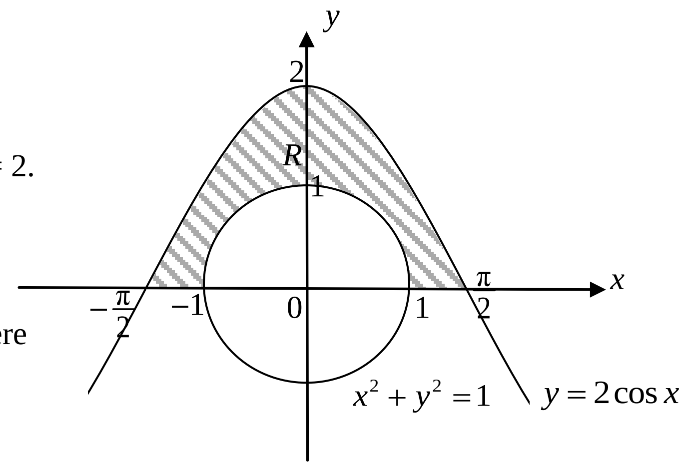
$= 4 - \frac{\pi}{2}$

(ii) $\int u^2 \sin u \, du = -u^2\cos u - \int 2u(-\cos u)\, du$

$= -u^2\cos u + \int 2u\cos u \, du$

$= -u^2\cos u + 2u\sin u - \int 2\sin u \, du$

$= -u^2\cos u + 2u\sin u + 2\cos u + C$



(iii) $y = 2\cos x \Rightarrow x = \cos^{-1}\frac{y}{2}$. When $x = 0$, $y = 2$.

Volume of hemisphere = $\frac{1}{2}\left(\frac{4}{3}\pi(1)^3\right) = \frac{2}{3}\pi$.

Volume of S = $\pi\int_0^2 x^2 dy - \text{Volume of hemisphere}$

$= \pi\int_0^2 \left(\cos^{-1}\frac{y}{2}\right)^2 dy - \frac{2}{3}\pi$

By substitution: $u = \cos^{-1}\frac{y}{2} \Rightarrow y = 2\cos u \Rightarrow \frac{dy}{du} = -2\sin u$

$y = 0 \Rightarrow u = \frac{\pi}{2}$

$y = 2 \Rightarrow u = 0$

The required volume of revolution $= \pi\int_0^2 \left(\cos^{-1}\frac{y}{2}\right)^2 dy - \frac{2}{3}\pi$

$= -2\pi\int_{\frac{\pi}{2}}^0 u^2 \sin u \, du - \frac{2}{3}\pi$

$= -2\pi\left[-u^2\cos u + 2u\sin u + 2\cos u\right]_{\pi/2}^0 - \frac{2}{3}\pi$

$= -2\pi[2 - \pi] - \frac{2}{3}\pi$

$= \pi\left[2\pi - \frac{14}{3}\right]$

11 **(a)** Show that

$$\int_0^{\frac{e}{2}} \frac{4x+e^2}{4x^2+e^2}\,\mathrm{d}x = m\ln 2 + ne\,,$$

where m and n are exact values to be determined. [6]

(b) In an experiment to study the spread of a soil disease, an area of 15 m^2 of soil was exposed to infection. In a simple model, it is assumed that the infected area grows at a rate which is proportional to the product of the infected area and the uninfected area. Initially, 5 m^2 was infected and the rate of growth of the infected area was 0.1 m^2 per hour. At time t hours after the start of the experiment, an area x m^2 is infected.

(i) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x(15-x)}{500}$. [2]

(ii) Solve the differential equation and express t in terms of x. [4]

(iii) Find the minimum time in hours needed for 95% of the soil area to become infected. [1]

Solution:

(a)

$$\int_0^{\frac{e}{2}} \frac{4x+e^2}{4x^2+e^2}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{e}{2}} \frac{8x}{4x^2+e^2}\,\mathrm{d}x + e^2\int_0^{\frac{e}{2}} \frac{1}{4x^2+e^2}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\ln\left(4x^2+e^2\right)\right]_0^{\frac{e}{2}} + \frac{e^2}{4}\int_0^{\frac{e}{2}} \frac{1}{x^2+\left(\frac{e}{2}\right)^2}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\ln\left(2e^2\right) - \ln\left(e^2\right)\right] + \frac{e^2}{4}\left[\frac{2}{e}\tan^{-1}\left(\frac{2x}{e}\right)\right]_0^{\frac{e}{2}}$$

$$= \frac{1}{2}\ln 2 + \frac{e}{2}\left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= \frac{1}{2}\ln 2 + \frac{\pi}{8}e$$

(b) **(i)** Let $\frac{dx}{dt} = kx(15-x)$.

Given that when $t = 0$, $x = 5$, $\frac{dx}{dt} = 0.1$. $\Rightarrow$ $0.1 = k(5)(15-5)$

$\Rightarrow$ $k = \frac{1}{500}$

Hence, $\frac{dx}{dt} = \frac{x(15-x)}{500}$.

(ii) $\frac{dx}{dt}=\frac{x(15-x)}{500} \quad \Rightarrow \quad 500\int \frac{1}{x(15-x)}\,dx=\int 1\,dt$

$$\Rightarrow \quad \frac{500}{15}\int\left(\frac{1}{x}+\frac{1}{15-x}\right)dx=t+c$$

$$\Rightarrow \quad \frac{100}{3}\left[\ln|x|-\ln|15-x|\ \right]=t+c$$

$$\Rightarrow \quad \frac{100}{3}\ln\left(\frac{x}{15-x}\right)=t+c \quad \text{as } 0\le x\le 15$$

When t = 0, x = 5 $\Rightarrow \quad c=\frac{100}{3}\ln(0.5)=-\frac{100}{3}\ln 2$

Hence, $t=\frac{100}{3}\ln\left(\frac{2x}{15-x}\right)$ is the solution of the DE.

(iii) When x = 0.95(15) = 14.25 m^2, $t=121.25$ hours. (2dp)

It required a minimum of 122 hours for 95% of the soil area to become infected.