		Class	Reg No.
Name:	**Teachers'**		
Date:		4	

Chapter 12: Differentiation
Reference Book: Additional Maths 360 Volume B, Marshall Cavendish
You will learn how to,

- Relate the derivative of a function to the gradient of tangent.
- Differentiate functions using the basic rules of differentiation,
  - ➔ Constant multiple rule: $\frac{\mathrm{d}}{\mathrm{d}x}\left[a\mathrm{f}(x)\right]=a\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]$, where a is a constant
  - ➔ Power rule: $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$, n is a rational number
  - ➔ Sum and difference rule: $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\pm\mathrm{g}(x)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]\pm\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{g}(x)\right]$
  - ➔ Chain rule: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(u^n\right)=nu^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}$ to differentiate functions of the form $y = u^n$ where $u = f(x)$
  - ➔ Product rule: $\frac{\mathrm{d}}{\mathrm{d}x}(uv)=u\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}x}$
  - ➔ Quotient rule: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)=\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Introduction

The word "calculus" comes from Latin (*calculus*) and means a small stone used for counting. ***Calculus*** is the mathematical study of change and has two major branches; ***Differential calculus***, which concerns itself with the rate of change and the gradient of curves, as well as ***integral calculus***, which is regarding the accumulation of quantities and area under curves.

The First Derivative of *y*

The first ***derivative*** of y is denoted by $\mathrm{f}'(x)$ or , pronounced as "dee y dee x".

It is the ***gradient function*** of y, that is, it is the function that provides the value of the instantaneous gradient of y with respect to x at any required point.

➔ Hence, the **gradient (at a particular point) of a curve** is defined as the derivative of a function or gradient function.

It is the result of differentiating y with respect to x.

The process of finding $\frac{\mathrm{d}y}{\mathrm{d}x}$ is called ***differentiation***.

Notation

The notation $\frac{\mathrm{d}y}{\mathrm{d}x}$, is also known as the first derivative of y with respect to the variable x.

If f is a function of x, i.e. f(x), the first derivative of f with respect to x can be denoted by
→ If $y = \mathrm{f}(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)$.

Sometimes, you may be asked to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right)$. This represents the first derivative of x^n with respect to x. $\frac{\mathrm{d}}{\mathrm{d}x}(\ )$ is the differential operator. You simply need to differentiate the expression in the brackets with respect to the variable x. Similarly, $\frac{\mathrm{d}}{\mathrm{d}t}(\ )$ is the differential operator with respect to the variable t. t is commonly used to represent time.

The Power Rule

If $y = x^n$, where n is a rational constant, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$$

*We ***“bring down” the power of the variable,*** followed by ***reducing the power by one.***

The Constant Multiple Rule

If $\mathrm{f}(x)$ is a function and a is a constant, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[a\mathrm{f}(x)\right] = a\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]$$

The Constant Rule

Given a general function $y = k$, where k is a constant, we can use Constant Multiple Rule and the Power Rule to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(k)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(kx^0\right)$$

$$= k\frac{\mathrm{d}}{\mathrm{d}x}\left(x^0\right)$$ **Constant Multiple Rule**

$$= k\left(0 \times x^{0-1}\right)$$ **Power Rule**

$$= 0$$

Class Practice 1:
Differentiate the following functions with respect to x.

	y	$\frac{dy}{dx}$		y	$\frac{dy}{dx}$
a	x^2	$(2)x^1 = 2x$	**b**	x^5	$(5)x^4 = 5x^4$
c	$3x^2$	$3(2)x^1 = 6x$	**d**	$5x^4$	$5(4)x^3 = 20\,x^3$
e	πx^{10}	$10\pi x^9$	**f**	$-12x^{-\frac{7}{2}}$	$42x^{-\frac{9}{2}}$
g	$\sqrt[6]{x^7}$	$\frac{7}{6}\sqrt[6]{x}$	**h**	$17x^{\frac{7}{4}}$	$\frac{119}{4}x^{\frac{3}{4}}$
i	$23x^{-9}$	$-207x^{-10}$	**j**	$\frac{27}{\sqrt[3]{x}}$	$\frac{-9}{\sqrt[3]{x^4}}$

The Sum Rule and Difference Rule

If $y = f(x) \pm g(x)$,

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Example 1

Differentiate $x^2 + 4x$ with respect to x.

$$\frac{d}{dx}\left[x^2 + 4x\right] = 2x + 4$$

Example 2

Differentiate $-2x^3 + 4\sqrt{x} - 5$ with respect to x.

$$\frac{d}{dx}\left[-2x^3 + 4\sqrt{x} - 5\right]$$

$$= \frac{d}{dx}\left[-2x^3 + 4(x)^{\frac{1}{2}} - 5\right]$$

$$= -6x^2 + 2x^{-\frac{1}{2}} \quad \text{OR} \quad -6x^2 + \frac{2}{\sqrt{x}}$$

Class Practice 2:

1 Differentiate the following functions with respect to x.

(a) $\frac{\pi^2x^2}{3}-\frac{2x^4}{5}$ (b) $\frac{3x^8}{20}-\frac{\pi x^9}{11}$

(a) $\frac{d}{dx}\left(\frac{\pi^2x^2}{3}-\frac{2x^4}{5}\right)$ $=\frac{2\pi^2x}{3}-\frac{8x^3}{5}$	(b) $\frac{d}{dx}\left(\frac{3x^8}{20}-\frac{\pi x^9}{11}\right)$ $=\frac{24x^7}{20}-\frac{9\pi x^8}{11}$ $=\frac{6x^7}{5}-\frac{9\pi x^8}{11}$

2 Differentiate the following functions with respect to x.

(a) $\frac{6}{x^3}-\frac{1}{x}+3$ (b) $3x+2\sqrt{x}-3$

(a) $\frac{d}{dx}\left(\frac{6}{x^3}-\frac{1}{x}+3\right)$ $=\frac{d}{dx}\left(6x^{-3}-x^{-1}+3\right)$ $=-18x^{-4}+x^{-2}$ $=-\frac{18}{x^4}+\frac{1}{x^2}$	(b) $\frac{d}{dx}\left(3x+2\sqrt{x}-3\right)$ $=3+2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$ $=3+x^{-\frac{1}{2}}$ $=3+\frac{1}{\sqrt{x}}$

3 Differentiate $\frac{6x^2-\sqrt{x}+2}{2x}$, $x\neq 0$, with respect to x.

$$\frac{d}{dx}\left(\frac{6x^2-\sqrt{x}+2}{2x}\right)$$

$$=\frac{d}{dx}\left(3x-\frac{1}{2}x^{-\frac{1}{2}}+x^{-1}\right)$$

$$=3+\frac{1}{4}x^{-\frac{3}{2}}-x^{-2}$$

$$=3+\frac{1}{4\sqrt{x^3}}-\frac{1}{x^2}$$

4 Differentiate the following functions with respect to x.

(a) $g(x)=\left(1+\sqrt{x}\right)\left(1-\sqrt{x}\right)$ (b) $t=3x^2\left(2-\sqrt{x}\right)$

(a) $g'(x)=\frac{d}{dx}\left[\left(1+\sqrt{x}\right)\left(1-\sqrt{x}\right)\right]$ $=\frac{d}{dx}\left[1^2-\left(\sqrt{x}\right)^2\right]$ $=\frac{d}{dx}\left[1-x\right]$ $=-1$	(b) $\frac{d}{dx}(t)=\frac{d}{dx}\left[3x^2\left(2-\sqrt{x}\right)\right]$ $\frac{dt}{dx}=\frac{d}{dx}\left[6x^2-3x^{\frac{5}{2}}\right]$ $=12x-\frac{15}{2}x^{\frac{3}{2}}$ $=12x-\frac{15\sqrt{x^3}}{2}$

5 Calculate the gradient of each curve at the given point

(a) $y=5x^2-4x+2$, at $(1,3)$ (b) $y=\frac{(x-1)(2x+3)}{x}$, at $x=-2$

(a) $y=5x^2-4x+2$ $\frac{dy}{dx}=\frac{d}{dx}\left(5x^2-4x+2\right)$ $=10x-4$ At $(1,3)$, $\frac{dy}{dx}=10(1)-4$ $=6$	(b) $y=\frac{(x-1)(2x+3)}{x}$ $=\frac{2x^2+x-3}{x}$ $=2x+1-\frac{3}{x}$ $\frac{dy}{dx}=\frac{d}{dx}\left(2x+1-\frac{3}{x}\right)$ $=2+\frac{3}{x^2}$ At $x=-2$, $\frac{dy}{dx}=2+\frac{3}{(-2)^2}$ $=2\frac{3}{4}$

6 Given that the equation of a curve is $y = \frac{10}{x} - x$, find the coordinates of the points on the curve at which the gradient is $-\frac{7}{2}$.

$$y = \frac{10}{x} - x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{10}{x} - x\right) \\ &= \frac{d}{dx}\left(10x^{-1} - x\right) \\ &= -10x^{-2} - 1\end{aligned}$$

When the gradient is $-\frac{7}{2}$, $\frac{dy}{dx} = -\frac{7}{2}$

$$\begin{aligned}-10x^{-2} - 1 &= -\frac{7}{2} \\ 10x^{-2} &= \frac{5}{2} \\ x^2 &= 4 \\ x &= 2 \text{ or } -2\end{aligned}$$

When $x = 2$, $y = \frac{10}{2} - 2 = 3$

When $x = -2$, $y = \frac{10}{-2} - (-2) = -3$

$\therefore$ the coordinates of the points are (2, 3) and (–2, –3).

7 A curve has the equation $y = x^3 + px + q$ where p and q are constants. The gradient of the curve at the point (3, 16) is 20.

(i) Find the value of p and of q.

(ii) Find the coordinates of the other point on the curve where the gradient is 20.

(i) $y = x^3 + px + q$

Since the point (3, 16) lies on the curve,

$$16 = (3)^3 + p(3) + q$$

$$3p + q = -11 \qquad -(1)$$

$$\frac{dy}{dx} = 3x^2 + p$$

Substituting $x = 3$ and $\frac{dy}{dx} = 20$,

$$20 = 3(3)^2 + p$$

$$p = -7$$

Substituting $p = -7$ into (1),

$$3(-7) + q = -11$$

$$q = 10$$

(ii) Given: $\frac{dy}{dx} = 20$

$$3x^2 - 7 = 20$$

$$3x^2 = 27$$

$$x^2 = 9$$

$$x = 3 \text{ or } -3$$

When $x = -3$, $y = (-3)^3 - 7(-3) + 10$

$$= 4$$

The coordinates of the other point where the gradient is 20 are $(-3, 4)$.

The Chain Rule

How would you differentiate $\left(3x^2+2\right)^2$ w.r.t. x?

How would you differentiate $\left(3x^2+2\right)^5$ w.r.t. x?

To expand first then differentiate is rather tedious. Is there a shorter way?

To obtain the above solution quickly, we apply the **Chain Rule**.

If $y=\mathrm{f}(u)$ and $u=\mathrm{g}(x)$, $\dfrac{\mathrm{d}y}{\mathrm{d}u}$ and $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ **exist**, then the derivative of the function $y=\mathrm{f}\left[\mathrm{g}\left(x\right)\right]$ exists and is given by,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$$

Note: The Chain Rule looks as if the du's are cancelled, when in fact, $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ is a single **notation** that **cannot be taken as a fraction**.

To differentiate $y=\left(3x^2+2\right)^5$, let $u=3x^2+2$.
$\therefore y=u^5$

Note, y is a function of u and u is a function of x.

$$\frac{\mathrm{d}y}{\mathrm{d}u}=5u^4, \qquad \frac{\mathrm{d}u}{\mathrm{d}x}=6x$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}\\&=5u^4\left(6x\right)\\&=30x\left(3x^2+2\right)^4\end{aligned}$$

Example 3

Differentiate $\sqrt{5x^3+4}$ with respect to *x*.

$$\frac{d}{dx}\left[\sqrt{5x^3+4}\right]$$
$$=\frac{d}{dx}\left[\left(5x^3+4\right)^{\frac{1}{2}}\right]$$
$$=\frac{15}{2}x^2\left(5x^3+4\right)^{-\frac{1}{2}}$$
$$=\frac{15x^2}{2\sqrt{5x^3+4}}$$

Example 4

Differentiate $\left(x^2+\frac{2}{x}\right)^8$ with respect to *x*.

$$\frac{d}{dx}\left[\left(x^2+\frac{2}{x}\right)^8\right]$$
$$=8\left(x^2+\frac{2}{x}\right)^7\left(2x-\frac{2}{x^2}\right)$$

Class Practice 3:

1 Use Chain Rule to differentiate $(1-6x)^9$ with respect to *x*.

$$\frac{d}{dx}(1-6x)^9$$
$$=9(1-6x)^8(-6)$$
$$=-54(1-6x)^8$$

2 Differentiate $\sqrt{x^2-2x+3}$ with respect to *x*.

$$\frac{d}{dx}\sqrt{x^2-2x+3}$$
$$=\frac{d}{dx}\left(x^2-2x+3\right)^{\frac{1}{2}}$$
$$=\frac{1}{2}\left(x^2-2x+3\right)^{-\frac{1}{2}}(2x-2)$$
$$=\frac{x-1}{\sqrt{x^2-2x+3}}$$

3 Use Chain Rule to find the derivative of $\dfrac{1}{2\left(x^3-1\right)^6}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2\left(x^3-1\right)^6}\right]$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(x^3-1\right)^{-6}\right]$$
$$=\frac{1}{2}(-6)\left(x^3-1\right)^{-7}\left(3x^2\right)$$
$$=-\frac{9x^2}{\left(x^3-1\right)^7}$$

4 Find the derivative of $h=\dfrac{1}{\left(2-3t+5t^2\right)^4}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(h)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\left(2-3t+5t^2\right)^4}\right]$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left(2-3t+5t^2\right)^{-4}$$
$$=-4\left(2-3t+5t^2\right)^{-5}(-3+10t)$$
$$=\frac{4(3-10t)}{\left(2-3t+5t^2\right)^5}$$

5 Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and the gradient of the curve at the point with the given value of y.

$$y=\frac{1}{(2x-5)^3},\quad y=\frac{1}{8}$$

Given: $y=\dfrac{1}{(2x-5)^3}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{(2x-5)^3}\right]$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}(2x-5)^{-3}$$
$$=-3(2x-5)^{-4}(2)$$
$$=\frac{-6}{(2x-5)^4}$$

Substituting $y=\frac{1}{8}$ into the equation of curve,

$$\frac{1}{8}=\frac{1}{(2x-5)^3}$$

$$(2x-5)^3=8$$

$$2x-5=2$$

$$x=\frac{7}{2}$$

When $x=\frac{7}{2}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-6}{\left[2\left(\frac{7}{2}\right)-5\right]^4}$$

$$=-\frac{3}{8}$$

The gradient of the curve at $y=\frac{1}{8}$ is $-\frac{3}{8}$.

6 (i) Given that $\frac{2x-1}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}$, find the value of A and B.

(ii) Hence find the derivative of $\frac{2x-1}{(x-1)^2}$.

(i)	(ii)
Given: $\frac{2x-1}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}$ $\Rightarrow \frac{2x-1}{(x-1)^2}=\frac{A(x-1)+B}{(x-1)^2}$ $\Rightarrow 2x-1=A(x-1)+B$ Comparing x terms: $A=2$ When $x=1$, $B=1$ Hence $A=2$ and $B=1$.	From (i), $\frac{d}{dx}\left[\frac{2x-1}{(x-1)^2}\right]=\frac{d}{dx}\left[\frac{2}{x-1}+\frac{1}{(x-1)^2}\right]$ $=\frac{d}{dx}\left[\frac{2}{x-1}\right]+\frac{d}{dx}\left[\frac{1}{(x-1)^2}\right]$ $=-2(x-1)^{-2}-2(x-1)^{-3}$ $=-2(x-1)^{-3}\left[(x-1)+1\right]$ $=-\frac{2x}{(x-1)^3}$

7 Calculate the coordinates of the point on the curve $y=(2-5x)^3+1$ at which the gradient is 0.

Given: $y=(2-5x)^3+1$

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left[(2-5x)^3+1\right]\\&=3(2-5x)^2(-5)\\&=-15(2-5x)^2\end{aligned}$$

When the gradient is 0,

$$\begin{aligned}-15(2-5x)^2&=0\\2-5x&=0\\x&=\frac{2}{5}\end{aligned}$$

When the $x=\frac{2}{5}$, $y=\left[2-5\left(\frac{2}{5}\right)\right]^3+1$

$$=1$$

The coordinates of the point at which the gradient is 0 are $\left(\frac{2}{5},\ 1\right)$.

8 Calculate the coordinates of the point on the curve $y=\sqrt{x^2-4x+1}$ at which the gradient is 2.

Given: $y=\sqrt{x^2-4x+1}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2-4x+1}\\&=\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-4x+1\right)^{\frac{1}{2}}\\&=\frac{1}{2}\left(x^2-4x+1\right)^{-\frac{1}{2}}(2x-4)\\&=(x-2)\left(x^2-4x+1\right)^{-\frac{1}{2}}\end{aligned}$$

When the gradient is 2, $\frac{\mathrm{d}y}{\mathrm{d}x}=2$

$$\begin{aligned}\frac{x-2}{\sqrt{x^2-4x+1}}&=2\\(x-2)^2&=\left[2\sqrt{x^2-4x+1}\right]^2\\x^2-4x+4&=4\left(x^2-4x+1\right)\\x^2-4x+4&=4x^2-16x+4\\3x^2-12x&=0\\3x(x-4)&=0\\x=0 \text{ or } x&=4\end{aligned}$$

When $x=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=(0-2)\left[0^2-4(0)+1\right]^{-\frac{1}{2}}$
$=-2$

$x=0$ is rejected.

$\Rightarrow x=4$ and $y=\sqrt{(4)^2-4(4)+1}$
$=1$

The coordinates of the point on the curve at which the gradient is 2 are (4, 1).

The Product Rule

Product Rule states that if $y=u(x)v(x)$, then

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

Example 5:

Differentiate $x(3x+2)^2$ with respect to x.

$$\frac{d}{dx}\left[x(3x+2)^2\right]$$
$$=x\frac{d}{dx}(3x+2)^2+(3x+2)^2\frac{dx}{dx}$$
$$=x\left[2(3x+2)(3)\right]+(3x+2)^2$$
$$=(3x+2)\left[6x+(3x+2)\right]$$
$$=(3x+2)(9x+2)$$

Example 6:

Differentiate $x^2\sqrt{1+2x^2}$ with respect to x.

$$\frac{d}{dx}\left[x^2\sqrt{1+2x^2}\right]$$
$$=x^2\frac{d}{dx}\left[(1+2x^2)^{\frac{1}{2}}\right]+(1+2x^2)^{\frac{1}{2}}\frac{d}{dx}(x^2)$$
$$=x^2\left(\frac{1}{2}\right)(1+2x^2)^{-\frac{1}{2}}(4x)+(1+2x^2)^{\frac{1}{2}}(2x)$$
$$=2x(1+2x^2)^{-\frac{1}{2}}\left[x^2+(1+2x^2)\right]$$
$$=\frac{2x(3x^2+1)}{\sqrt{1+2x^2}}$$

Class Practice 4:

1 Differentiate $(2x-1)\sqrt{5x^2-1}$ with respect to x.

$$\frac{d}{dx}(2x-1)\sqrt{5x^2-1}$$
$$=(2x-1)\frac{d}{dx}(5x^2-1)^{\frac{1}{2}}+(5x^2-1)^{\frac{1}{2}}\frac{d}{dx}(2x-1)$$
$$=(2x-1)\left(\frac{1}{2}\right)(5x^2-1)^{-\frac{1}{2}}(10x)+2(5x^2-1)^{\frac{1}{2}}$$
$$=(5x^2-1)^{-\frac{1}{2}}\left[5x(2x-1)+2(5x^2-1)\right]$$
$$=\frac{20x^2-5x-2}{\sqrt{5x^2-1}}$$

2 Express $x^2(x-1)\sqrt{5+6x}$ as a product of two factors.
Then differentiate it with respect to x.

$$x^2(x-1)\sqrt{5+6x}=(x^3-x^2)\sqrt{5+6x}$$

$$\frac{d}{dx}\left[x^2(x-1)\sqrt{5+6x}\right]$$
$$=\frac{d}{dx}\left[(x^3-x^2)\sqrt{5+6x}\right]$$
$$=(x^3-x^2)\frac{d}{dx}(5+6x)^{\frac{1}{2}}+(5+6x)^{\frac{1}{2}}\frac{d}{dx}(x^3-x^2)$$
$$=(x^3-x^2)\left(\frac{1}{2}\right)(5+6x)^{-\frac{1}{2}}(6)+(5+6x)^{\frac{1}{2}}(3x^2-2x)$$
$$=x(5+6x)^{-\frac{1}{2}}\left[3(x^2-x)+(5+6x)(3x-2)\right]$$
$$=\frac{x(3x^2-3x+18x^2+3x-10)}{\sqrt{5+6x}}$$
$$=\frac{x(21x^2-10)}{\sqrt{5+6x}}$$

3 Given that $y = x\sqrt{5-x^2}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5-2x^2}{\sqrt{5-x^2}}$.

Given: $y = x\sqrt{5-x^2}$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[x\sqrt{5-x^2}\right] \\
&= x\frac{\mathrm{d}}{\mathrm{d}x}\left(5-x^2\right)^{\frac{1}{2}} + \left(5-x^2\right)^{\frac{1}{2}}\frac{\mathrm{d}x}{\mathrm{d}x} \\
&= x\left(\frac{1}{2}\right)\left(5-x^2\right)^{-\frac{1}{2}}(-2x) + \left(5-x^2\right)^{\frac{1}{2}} \\
&= \left(5-x^2\right)^{-\frac{1}{2}}\left[-x^2 + \left(5-x^2\right)\right] \\
&= \frac{5-2x^2}{\sqrt{5-x^2}}
\end{aligned}$$

4 The equation of a curve is $y = \sqrt{x}\,(x-3)^4$. Find

(i) $\frac{\mathrm{d}y}{\mathrm{d}x}$,

(ii) the x-coordinate(s) of the point(s) where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

(i) Given: $y = \sqrt{x}\,(x-3)^4$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\frac{1}{2}}(x-3)^4\right] \\
&= x^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(x-3)^4 + (x-3)^4\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}}\right) \\
&= x^{\frac{1}{2}}(4)(x-3)^3 + (x-3)^4\left(\frac{1}{2}\right)\left(x^{-\frac{1}{2}}\right) \\
&= \frac{(x-3)^3\left[8x + (x-3)\right]}{2\sqrt{x}} \\
&= \frac{3(x-3)^3\left[3x-1\right]}{2\sqrt{x}}
\end{aligned}$$

(ii) When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, then

$$\frac{3(x-3)^3\left[3x-1\right]}{2\sqrt{x}} = 0$$

$$\Rightarrow (x-3)^3(3x-1) = 0$$

$$x = 3 \text{ or } x = \frac{1}{3}$$

5 The equation of a curve is $y = x\sqrt{9-x^2}$. Find the gradient of the curve at each of the points where it meets the line $y = x$.

Given: $y = x\sqrt{9-x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[x\sqrt{9-x^2}\right] \\ &= x\frac{d}{dx}\left(9-x^2\right)^{\frac{1}{2}} + \left(9-x^2\right)^{\frac{1}{2}}\frac{dx}{dx} \\ &= x\left(\frac{1}{2}\right)\left(9-x^2\right)^{-\frac{1}{2}}(-2x) + \left(9-x^2\right)^{\frac{1}{2}} \\ &= \left(9-x^2\right)^{-\frac{1}{2}}\left[-x^2 + \left(9-x^2\right)\right] \\ &= \frac{9-2x^2}{\sqrt{9-x^2}}\end{aligned}$$

Substituting $y = x$ into $y = x\sqrt{9-x^2}$,

$$\begin{aligned}x\sqrt{9-x^2} &= x \\ x\sqrt{9-x^2} - x &= 0 \\ x\left(\sqrt{9-x^2} - 1\right) &= 0\end{aligned}$$

$$x = 0 \quad \text{or} \quad \sqrt{9-x^2} - 1 = 0$$

$$\begin{aligned}\sqrt{9-x^2} &= 1 \\ 9 - x^2 &= 1 \\ x^2 &= 8 \\ x &= \pm 2\sqrt{2}\end{aligned}$$

When $x = 0$, $\frac{dy}{dx} = \frac{9-2(0)^2}{\sqrt{9-0^2}}$

$= 3$

When $x = 2\sqrt{2}$, $\frac{dy}{dx} = \frac{9-2(8)}{\sqrt{9-(8)}}$

$= -7$

When $x = -2\sqrt{2}$, $\frac{dy}{dx} = \frac{9-2(8)}{\sqrt{9-(8)}}$

$= -7$

The gradients of the curve at the points where it meets the line $y = x$ are 3 and –7.

6 Find the coordinates of the point on the curve $y=(3x-1)(x-2)$ at which the tangent is parallel to the line $5x-y=0$.

Given: $y=(3x-1)(x-2)$

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}(3x-1)(x-2)\\&=(3x-1)\frac{d}{dx}(x-2)+(x-2)\frac{d}{dx}(3x-1)\\&=3x-1+3(x-2)\\&=6x-7\end{aligned}$$

The gradient of the line 5x – y = 0 is 5; so the gradient of the tangent that is parallel to this line is 5.

When $\frac{dy}{dx}=5$, $6x-7=5$

$$6x=12$$
$$x=2$$

Substituting $x=2$ into the equation of curve,

$$\begin{aligned}y&=[3(2)-1](2-2)\\&=0\end{aligned}$$

The coordinates of the point on the curve are (2, 0).

Quotient Rule

To differentiate the function $y=\frac{x}{(x+2)^3}$, you could write it as $y=x(x+2)^{-3}$ and then apply the product rule. Therefore, you can also apply the quotient rule directly.

Quotient Rule states that if $y=\frac{u(x)}{v(x)}$, where $v\neq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)=\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Notes: (i) $y=\frac{u(x)}{v(x)}$ can also be written as $y=u(x)v(x)^{-1}$.

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)\neq\frac{\frac{\mathrm{d}u}{\mathrm{d}x}}{\frac{\mathrm{d}v}{\mathrm{d}x}}$.

__**Example 7:**__

Differentiate the following with respect to x:

(i) $y=\frac{3x^2}{1-4x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{3x^2}{1-4x}\right]\\&=\frac{(1-4x)\frac{\mathrm{d}}{\mathrm{d}x}(3x^2)-3x^2\frac{\mathrm{d}}{\mathrm{d}x}(1-4x)}{(1-4x)^2}\\&=\frac{(1-4x)(6x)-3x^2(-4)}{(1-4x)^2}\\&=\frac{6x-24x^2+12x^2}{(1-4x)^2}\\&=\frac{6x(1-2x)}{(1-4x)^2}\end{aligned}$$

(ii) $y = \dfrac{x^2}{(1+x)^2}$

$$\frac{dy}{dx} = \frac{(1+x)^2 \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(1+x)^2}{(1+x)^4}$$

$$= \frac{(1+x)^2(2x) - x^2(2)(1+x)}{(1+x)^4}$$

$$= \frac{2x(1+x)^2 - 2x^2(1+x)}{(1+x)^4}$$

$$= \frac{2x(1+x)[(1+x)-x]}{(1+x)^4}$$

$$= \frac{2x}{(1+x)^3}$$

Class Practice 5:

1 Use Quotient Rule to differentiate $y = \dfrac{x^2+2}{3-x^3}$.

Given: $y = \dfrac{x^2+2}{3-x^3}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2+2}{3-x^3}\right]$$

$$= \frac{(3-x^3)\frac{d}{dx}(x^2+2) - (x^2+2)\frac{d}{dx}(3-x^3)}{(3-x^3)^2}$$

$$= \frac{(3-x^3)(2x) - (x^2+2)(-3x^2)}{(3-x^3)^2}$$

$$= \frac{x[6-2x^3+3x^3+6x]}{(3-x^3)^2}$$

$$= \frac{x[x^3+6x+6]}{(3-x^3)^2}$$

2 Differentiate the following functions with respect to x and simplify your answer.

(a) $\dfrac{\sqrt{x}}{1-x}$ (b) $\dfrac{5x^2}{\sqrt{3x^2-1}}$

(a)
$$\frac{d}{dx}\left[\frac{\sqrt{x}}{1-x}\right]$$
$$=\frac{(1-x)\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-\left(x^{\frac{1}{2}}\right)\frac{d}{dx}(1-x)}{(1-x)^2}$$
$$=\frac{(1-x)\left(\frac{1}{2}\right)\left(x^{-\frac{1}{2}}\right)-\left(x^{\frac{1}{2}}\right)(-1)}{(1-x)^2}$$
$$=\frac{\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\left[(1-x)-2(x)(-1)\right]}{(1-x)^2}$$
$$=\frac{\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\left[1-x+2x\right]}{(1-x)^2}$$
$$=\frac{x+1}{2\sqrt{x}(1-x)^2}$$

(b)
$$\frac{d}{dx}\left[\frac{5x^2}{\sqrt{3x^2-1}}\right]$$
$$=\frac{(3x^2-1)^{\frac{1}{2}}\frac{d}{dx}(5x^2)-(5x^2)\frac{d}{dx}(3x^2-1)^{\frac{1}{2}}}{\left[(3x^2-1)^{\frac{1}{2}}\right]^2}$$
$$=\frac{(3x^2-1)^{\frac{1}{2}}(10x)-(5x^2)\left(\frac{1}{2}\right)(3x^2-1)^{-\frac{1}{2}}(6x)}{\left[(3x^2-1)^{\frac{1}{2}}\right]^2}$$
$$=\frac{5x(3x^2-1)^{-\frac{1}{2}}\left[2(3x^2-1)-3(x^2)\right]}{3x^2-1}$$
$$=\frac{5x(3x^2-2)}{(3x^2-1)\sqrt{3x^2-1}}$$

3 The equation of a curve is $y=\frac{\sqrt{2x-1}}{x}$. Calculate the gradient of the tangents to the curve at the point where $x=1$.

Given: $y=\frac{\sqrt{2x-1}}{x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sqrt{2x-1}}{x}\right]\\&=\frac{x\frac{\mathrm{d}}{\mathrm{d}x}(2x-1)^{\frac{1}{2}}-(2x-1)^{\frac{1}{2}}\frac{\mathrm{d}x}{\mathrm{d}x}}{x^2}\\&=\frac{x\left(\frac{1}{2}\right)(2x-1)^{-\frac{1}{2}}(2)-(2x-1)^{\frac{1}{2}}(1)}{x^2}\\&=\frac{(2x-1)^{-\frac{1}{2}}\left[x-(2x-1)\right]}{x^2}\\&=\frac{1-x}{x^2\sqrt{2x-1}}\end{aligned}$$

When $x=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-(1)}{(1)^2\sqrt{2(1)-1}}$

$$=0$$

The gradient of the tangent to the curve at the point where x = 1 is 0.

4 (i) Differentiate $\left(1-x^2\right)\sqrt{x+3}$ with respect to x.

(ii) Use the Quotient Rule to find the rate of change of $y=\dfrac{\left(1-x^2\right)\sqrt{x+3}}{x^3}$ with respect to x when $x=1$.

(i)	(ii)
$\frac{d}{dx}\left[\left(1-x^2\right)\sqrt{x+3}\right]$ $=\left(1-x^2\right)\frac{d}{dx}(x+3)^{\frac{1}{2}}+(x+3)^{\frac{1}{2}}\frac{d}{dx}\left(1-x^2\right)$ $=\left(1-x^2\right)\left(\frac{1}{2}\right)(x+3)^{-\frac{1}{2}}(1)+(x+3)^{\frac{1}{2}}(-2x)$ $=\left(\frac{1}{2}\right)(x+3)^{-\frac{1}{2}}\left[1-x^2-4x(x+3)\right]$ $=\frac{1-12x-5x^2}{2\sqrt{x+3}}$	$y=\frac{\left(1-x^2\right)\sqrt{x+3}}{x^3}$ $\frac{dy}{dx}$ $=\frac{x^3\frac{d}{dx}\left[\left(1-x^2\right)\sqrt{x+3}\right]-\left[\left(1-x^2\right)\sqrt{x+3}\right]\frac{d}{dx}\left(x^3\right)}{\left(x^3\right)^2}$ $=\frac{x^3\left[\frac{1-12x-5x^2}{2\sqrt{x+3}}\right]-\left[\left(1-x^2\right)\sqrt{x+3}\right]\left(3x^2\right)}{x^6}$ $=\frac{x\left(1-12x-5x^2\right)-6\left(1-x^2\right)(x+3)}{2x^4\sqrt{x+3}}$ $=\frac{x-12x^2-5x^3+6x^3+18x^2-6x-18}{2x^4\sqrt{x+3}}$ $=\frac{x^3+6x^2-5x-18}{2x^4\sqrt{x+3}}$ At $x=1$, $\frac{dy}{dx}=\frac{(1)^3+6(1)^2-5(1)-18}{2(1)^4\sqrt{(1)+3}}$ $=-4$ The rate of change of y when x = 1 is –4.

5 The equation of a curve is $y=\left(\frac{5x+3}{10x-6}\right)^4$. Calculate the gradients of the tangents to the curve at the points where $y=16$.

Given: $y=\left(\frac{5x+3}{10x-6}\right)^4$

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(\frac{5x+3}{10x-6}\right)^4\\&=4\left(\frac{5x+3}{10x-6}\right)^3\times\frac{d}{dx}\left(\frac{5x+3}{10x-6}\right)\\&=4\left(\frac{5x+3}{10x-6}\right)^3\left[\frac{(10x-6)\frac{d}{dx}(5x+3)-(5x+3)\frac{d}{dx}(10x-6)}{(10x-6)^2}\right]\\&=4\left(\frac{5x+3}{10x-6}\right)^3\left[\frac{5(10x-6)-10(5x+3)}{(10x-6)^2}\right]\\&=\frac{-240(5x+3)^3}{(10x-6)^5}\end{aligned}$$

When $y=16$, $\left(\frac{5x+3}{10x-6}\right)^4=16$

$\Rightarrow \frac{5x+3}{10x-6}=\sqrt[4]{16}$ or $\frac{5x+3}{10x-6}=-\sqrt[4]{16}$

$\frac{5x+3}{10x-6}=2$ $\qquad$ $\frac{5x+3}{10x-6}=-2$

$5x+3=2(10x-6)$ $\qquad$ $5x+3=-2(10x-6)$

$15x=15$ $\qquad$ $25x=9$

$x=1$ $\qquad$ $x=\frac{9}{25}$

When $x=1$, $\frac{dy}{dx}=\frac{-240[5(1)+3]^3}{[10(1)-6]^5}$

$=-120$

When $x = \frac{9}{25}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-240\left[5\left(\frac{9}{25}\right)+3\right]^3}{\left[10\left(\frac{9}{25}\right)-6\right]^5}$

$= 333\frac{1}{3}$

The gradients of the tangents to the curve at the points where $y = 16$ are –120 and $333\frac{1}{3}$.

6 A curve had the equation $y=\sqrt{\frac{x-a}{b-x}}$, where $a<x<b$, and a and b are constants. Show that the gradient of the curve at $x=\frac{a+b}{2}$ is $\frac{2}{b-a}$.

Given: $y=\sqrt{\frac{x-a}{b-x}}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\sqrt{\frac{x-a}{b-x}} \\
&= \frac{d}{dx}\left[\frac{x-a}{b-x}\right]^{\frac{1}{2}} \\
&= \frac{1}{2}\left[\frac{x-a}{b-x}\right]^{-\frac{1}{2}} \times \frac{d}{dx}\left(\frac{x-a}{b-x}\right) \\
&= \frac{(b-x)^{\frac{1}{2}}}{2(x-a)^{\frac{1}{2}}} \times \frac{(b-x)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(b-x)}{(b-x)^2} \\
&= \frac{(b-x)^{\frac{1}{2}}}{2(x-a)^{\frac{1}{2}}} \times \frac{(b-x)(1)-(x-a)(-1)}{(b-x)^2} \\
&= \frac{b-a}{2(x-a)^{\frac{1}{2}}(b-x)^{\frac{3}{2}}}
\end{aligned}$$

When $x=\frac{a+b}{2}$,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{b-a}{2\left(\frac{a+b}{2}-a\right)^{\frac{1}{2}}\left(b-\frac{a+b}{2}\right)^{\frac{3}{2}}} \\
&= \frac{b-a}{2\left(\frac{a+b-2a}{2}\right)^{\frac{1}{2}}\left(\frac{2b-(a+b)}{2}\right)^{\frac{3}{2}}} \\
&= \frac{b-a}{2\left(\frac{b-a}{2}\right)^{\frac{1}{2}}\left(\frac{b-a}{2}\right)^{\frac{3}{2}}} \\
&= \frac{b-a}{2\left(\frac{b-a}{2}\right)^{2}} \\
&= \frac{2}{b-a} \quad \text{[Shown]}
\end{aligned}$$

**All questions are taken from Additional Mathematics 360 textbook, 2nd Edition (Volume B).
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