2012 MJC H2 MATH (9740) JC 2 PRELIMINARY EXAM PAPER 1 – SOLUTIONS

Qn	Solution
1	**Inequalities**

$$x-4 \ge \frac{4-6x}{x^2-1}\ ,\ x \ne \pm 1$$

$$\frac{(x-4)\left(x^2-1\right)-(4-6x)}{x^2-1} \ge 0$$

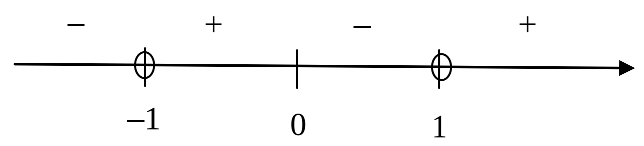
$$\frac{\left(x^3-4x^2-x+4\right)-4+6x}{x^2-1} \ge 0$$

$$\frac{x^3-4x^2+5x}{x^2-1} \ge 0$$

$$\frac{x\left(x^2-4x+5\right)}{x^2-1} \ge 0$$

since $\left(x^2-4x+5\right)=(x-2)^2+1>0$ for all real values of x,

$$\frac{x}{(x-1)(x+1)} \ge 0$$



$\therefore -1 < x \le 0$ or $x > 1$

Qn	Solution
2	**SLE**

Let $\mathrm{f}(x)=ax^3+bx^2+cx+d$.

Method 1:

$\mathrm{f}(0)=0 \quad \Rightarrow \quad d=0$

$\mathrm{f}(1)=a+b+c=3 \qquad \cdots (1)$

$\mathrm{f}'(1)=3a+2b+c=0 \quad \cdots (2)$

Since the stationary point is a point of inflexion, $\mathrm{f}''(1)=6a+2b=0 \ \cdots (3)$

Using GC to solve, a = 3, b = – 9, c = 9

$\therefore \mathrm{f}(x)=3x^3-9x^2+9x$

Method 2:

$\mathrm{f}(0)=0 \quad \Rightarrow \quad d=0$

$\mathrm{f}(1)=a+b+c=3 \qquad \cdots (1)$

$\mathrm{f}'(1)=3a+2b+c=0 \quad \cdots (2)$

$\mathrm{f}'(x)=3ax^2+2bx+c$

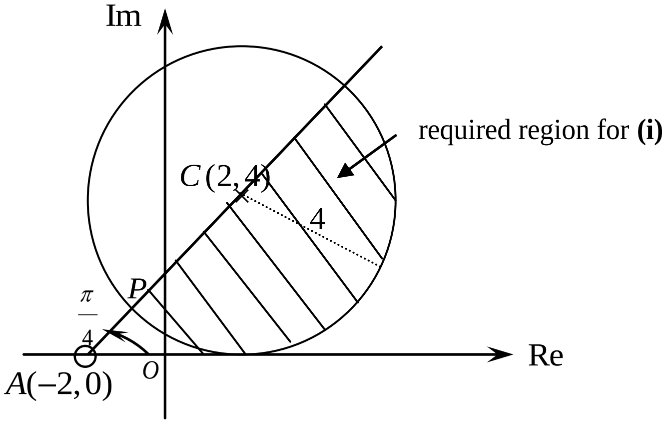
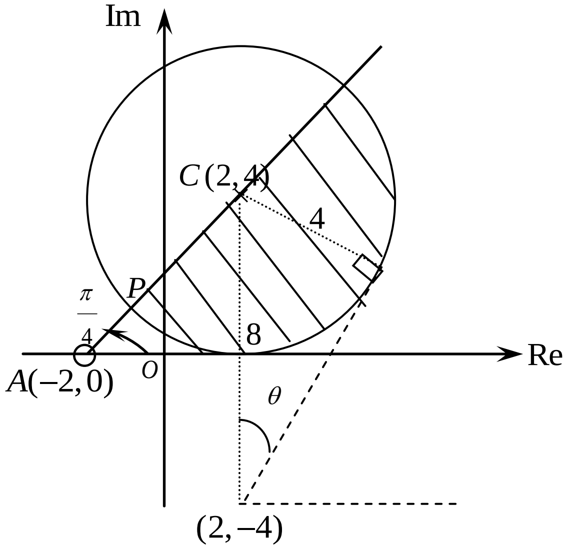
Since there is only one stationary point, $\mathrm{f}'(x)=0$ has only one solution.

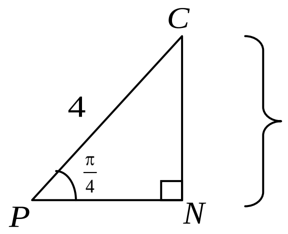
Hence, Discriminant = 0

$$(2b)^2-4(3a)(c)=0$$

	$4b^2 - 12ac = 0 \quad \cdots (3)$ Solving equations (1), (2) & (3), we have $a = 3, b = -9, c = 9$ $\therefore \mathrm{f}(x) = 3x^3 - 9x^2 + 9x$
	Method 3: $\mathrm{f}(0) = 0 \;\Rightarrow\; d = 0$ $\mathrm{f}(1) = a + b + c = 3 \;\cdots\; (1)$ $\because x = 1$ is the only stationary point $\therefore \mathrm{f}'(x) = k(x-1)^2 = 3ax^2 + 2bx + c$ $\therefore k = 3a = -b = c \quad \cdots (2)$ Solving (1) and (2), $a = 3, b = -9, c = 9$ $\therefore \mathrm{f}(x) = 3x^3 - 9x^2 + 9x$

Qn	Solution
3	**Differentiation (Implicit) + Techniques of Integration**
(a)	$-x^2 + xy + \ln y = 2$ $-2x + x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y + \dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(x + \dfrac{1}{y}\right) = 2x - y$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x - y}{x + \frac{1}{y}} = \dfrac{2xy - y^2}{xy + 1}$
(bi)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(2^{2x}\right) = 2^{2x+1}\ln 2$
(bii)	$\int 2^{2x} \ln 2^x \mathrm{d}x$ $= \dfrac{1}{2}\int (x)\left(2^{2x+1}\ln 2\right)\mathrm{d}x$ $= \dfrac{1}{2}\left[2^{2x}x - \int 2^{2x}\mathrm{d}x\right]$ $= \dfrac{1}{2}\left[2^{2x}x - 2^{2x}\dfrac{1}{2\ln 2}\right] + C$ $= 2^{2x-1}\left(x - \dfrac{1}{2\ln 2}\right) + C$

Qn	Solution
4	**Complex 3 (include intersection of loci)**
(i)	
(ii)	 $\sin\theta = \frac{4}{8} = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{6}$ smallest value of $\arg(z-2+4\text{i}) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$
(iii)	Method 1: $AP = \sqrt{4^2+4^2} - 4 = 4\left(\sqrt{2}-1\right)$ $w = \left(-2+4\left(\sqrt{2}-1\right)\cos\frac{\pi}{4}\right) + \left(4\left(\sqrt{2}-1\right)\sin\frac{\pi}{4}\right)\text{i}$ $= \left(-2+4\left(\sqrt{2}-1\right)\left(\frac{1}{\sqrt{2}}\right)\right) + 4\left(\sqrt{2}-1\right)\left(\frac{1}{\sqrt{2}}\right)\text{i}$ $= 2\left(1-\sqrt{2}\right) + 2\left(2-\sqrt{2}\right)\text{i}$ Method 2: Equation of circle: $(x-2)^2 + (y-4)^2 = 16$ -------(1) Equation of half line: $y - 0 = \tan\left(\frac{\pi}{4}\right)(x+2) \Rightarrow y = x+2, x > -2$ ----(2) Sub (2) into (1):

	$(x-2)^2+(x-2)^2=16$ $(x-2)^2=8$ $(x-2)=\pm 2\sqrt{2}$ $x=2-2\sqrt{2}$ or $x=2+2\sqrt{2}\,(rej.\because x<0)$ $y=4-2\sqrt{2}$ $\therefore w=2\left(1-\sqrt{2}\right)+2\left(2-\sqrt{2}\right)\mathrm{i}$ Method 3: $PN=4\cos\left(\frac{\pi}{4}\right)=2\sqrt{2}$ $CN=4\sin\left(\frac{\pi}{4}\right)=2\sqrt{2}$  $x=2-2\sqrt{2}$ $y=4-2\sqrt{2}$ $\therefore w=2\left(1-\sqrt{2}\right)+2\left(2-\sqrt{2}\right)\mathrm{i}$

Qn	**Solution**
5	**Vectors 1 & 2 (Ratio Thm, application of dot & cross product)**
(i)	Using ratio theorem, $\overrightarrow{OB}=\frac{4\overrightarrow{OM}+\overrightarrow{OA}}{5}$ $\overrightarrow{OM}=\frac{5\overrightarrow{OB}-\overrightarrow{OA}}{4}$ $=\frac{1}{4}\left(5\begin{pmatrix}-1\\3\\2\end{pmatrix}-\begin{pmatrix}1\\-1\\2\end{pmatrix}\right)$ $=\frac{1}{2}\begin{pmatrix}-3\\8\\4\end{pmatrix}$
(ii)	$\frac{\left\lvert\overrightarrow{OA}\times\overrightarrow{OB}\right\rvert}{\left\lvert\overrightarrow{OB}\right\rvert}$ represents the shortest/perpendicular distance from point *A* to the line *OB.*
(iii)	$\overrightarrow{OA}\times\overrightarrow{OB}=\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}-1\\3\\2\end{pmatrix}$ $=\begin{pmatrix}-8\\-4\\2\end{pmatrix}$ $=2\begin{pmatrix}-4\\-2\\1\end{pmatrix}$

$\therefore$ Normal to the plane containing O, A and B is $\begin{pmatrix}-4\\-2\\1\end{pmatrix}$.

Method 1:

$$\text{shortest distance } = \frac{\left|\begin{pmatrix}1\\3\\8\end{pmatrix}\cdot\begin{pmatrix}-4\\-2\\1\end{pmatrix}\right|}{\sqrt{21}}$$

$$= \frac{2}{\sqrt{21}} \quad \frac{\sqrt{21}}{\sqrt{21}}$$

$$= \frac{2\sqrt{21}}{21}\text{units}$$

Method 2:

Let N be the foot of perpendicular from C to the plane.

Equation of line CN is $\mathbf{r} = \begin{pmatrix}1\\3\\8\end{pmatrix} + \lambda\begin{pmatrix}-4\\-2\\1\end{pmatrix}, \lambda \in \mathbb{R}$

Equation of plane: $\mathbf{r}\cdot\begin{pmatrix}-4\\-2\\1\end{pmatrix} = 0$

Since N lies on the plane, $\begin{pmatrix}1-4\lambda\\3-2\lambda\\8+\lambda\end{pmatrix}\cdot\begin{pmatrix}-4\\-2\\1\end{pmatrix} = 0$

$$\Rightarrow \lambda = \frac{2}{21}$$

$$\overrightarrow{ON} = \begin{pmatrix}1-4\left(\frac{2}{21}\right)\\3-2\left(\frac{2}{21}\right)\\8+\left(\frac{2}{21}\right)\end{pmatrix} = \begin{pmatrix}\frac{13}{21}\\\frac{59}{21}\\\frac{170}{21}\end{pmatrix}$$

So $\overrightarrow{CN} = \overrightarrow{ON} - \overrightarrow{OC} = \begin{pmatrix}\frac{13}{21}\\\frac{59}{21}\\\frac{170}{21}\end{pmatrix} - \begin{pmatrix}1\\3\\8\end{pmatrix} = \begin{pmatrix}-\frac{8}{21}\\-\frac{4}{21}\\\frac{2}{21}\end{pmatrix}$

$$CN = \sqrt{\left(-\frac{8}{21}\right)^2 + \left(-\frac{4}{21}\right)^2 + \left(\frac{2}{21}\right)^2} = \sqrt{\frac{4}{21}}$$

$$= \frac{2\sqrt{21}}{21} \text{ units}$$

Qn	Solution
6	**Maclaurin + Binomial Series (include concept of approximation)**
	$\mathrm{f}(x)=\ln\left(\mathrm{e}x+2\right)$ $\qquad$ $\mathrm{f}(0)=\ln 2$ $\mathrm{f}'(x)=\dfrac{\mathrm{e}}{\mathrm{e}x+2}$ $\qquad$ $\mathrm{f}'(0)=\dfrac{\mathrm{e}}{2}$ $\mathrm{f}''(x)=-\dfrac{\mathrm{e}^2}{\left(\mathrm{e}x+2\right)^2}$ $\qquad$ $\mathrm{f}''(0)=-\dfrac{\mathrm{e}^2}{4}$ $\mathrm{f}^{(3)}(0)=\dfrac{2\mathrm{e}^3}{\left(\mathrm{e}x+2\right)^3}$ $\qquad$ $\mathrm{f}^{(3)}(0)=\dfrac{\mathrm{e}^3}{4}$
(i)	$\mathrm{f}\left(x\right)=\mathrm{f}\left(0\right)+x\,\mathrm{f}'\left(0\right)+\dfrac{x^2}{2!}\mathrm{f}''\left(0\right)+\dfrac{x^3}{3!}\mathrm{f}^{(3)}\left(0\right)+\ldots$ $\mathrm{f}\left(x\right)=\ln 2+x\left(\dfrac{\mathrm{e}}{2}\right)+\dfrac{x^2}{2!}\left(-\dfrac{\mathrm{e}^2}{4}\right)+\dfrac{x^3}{3!}\left(\dfrac{\mathrm{e}^3}{4}\right)+\ldots$ $\mathrm{f}\left(x\right)=\ln 2+\dfrac{\mathrm{e}x}{2}-\dfrac{\mathrm{e}^2x^2}{8}+\dfrac{\mathrm{e}^3x^3}{24}+\ldots$
(ii)	$\ln\left(\mathrm{e}x+2\right)=\ln 2+\dfrac{\mathrm{e}x}{2}-\dfrac{\mathrm{e}^2x^2}{8}+\dfrac{\mathrm{e}^3x^3}{24}+\ldots$ Differentiate both sides wrt x: $\dfrac{\mathrm{e}}{\mathrm{e}x+2}=\dfrac{\mathrm{e}}{2}-\dfrac{\mathrm{e}^2x}{4}+\dfrac{\mathrm{e}^3x^2}{8}+\ldots$ $\dfrac{2}{\mathrm{e}x+2}=1-\dfrac{\mathrm{e}x}{2}+\dfrac{\mathrm{e}^2x^2}{4}+\ldots$
(iii)	$\dfrac{2}{\mathrm{e}x+2}=2\left(\mathrm{e}x+2\right)^{-1}$ $=2\left(2\right)^{-1}\left(1+\dfrac{\mathrm{e}x}{2}\right)^{-1}$ $=1+\left(-1\right)\left(\dfrac{\mathrm{e}x}{2}\right)+\dfrac{\left(-1\right)\left(-2\right)}{2!}\left(\dfrac{\mathrm{e}x}{2}\right)^2+\ldots$ $=1-\dfrac{\mathrm{e}x}{2}+\dfrac{\mathrm{e}^2x^2}{4}+\ldots$

Qn	Solution
7	**MI + Summation**
(i)	$a_2=\dfrac{1}{2}+\dfrac{1}{2\left(1\right)\left(2\right)}=\dfrac{3}{4}=1-\dfrac{1}{4}=1-\dfrac{1}{2\left(2\right)}$ $a_3=\dfrac{3}{4}+\dfrac{1}{2\left(2\right)\left(3\right)}=\dfrac{5}{6}=1-\dfrac{1}{6}=1-\dfrac{1}{2\left(3\right)}$ $a_4=\dfrac{5}{6}+\dfrac{1}{2\left(3\right)\left(4\right)}=\dfrac{7}{8}=1-\dfrac{1}{8}=1-\dfrac{1}{2\left(4\right)}$ $\therefore a_n=1-\dfrac{1}{2n},c=2$

(ii)	Let P_n be the statement $a_n = 1-\frac{1}{2n}$ for all $n \in \mathbb{Z}^+$. When $n=1$, $$\text{LHS} = a_1 = \frac{1}{2}$$ $$\text{RHS} = 1-\frac{1}{2(1)} = \frac{1}{2}$$ $$\therefore \mathrm{P}_1 \text{ is true}$$ Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $a_k = 1-\frac{1}{2k}$. To prove P_{k+1} is true. i.e. $a_{k+1} = 1-\frac{1}{2(k+1)}$ LHS of $\mathrm{P}_{k+1}: a_{k+1} = a_k + \frac{1}{2k(k+1)}$ $= 1-\frac{1}{2k}+\frac{1}{2k(k+1)}$ (from assumption) $= 1-\left[\frac{k+1}{2k(k+1)}-\frac{1}{2k(k+1)}\right]$ $= 1-\frac{k}{2k(k+1)}$ $= 1-\frac{1}{2(k+1)} = \text{RHS}$ $$\therefore \mathrm{P}_k \text{ is true} \Rightarrow \mathrm{P}_{k+1} \text{ is true}$$ Since P_1 is true and P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.
(iii)	$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+...+\frac{1}{N^2+N}$ $= \frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\frac{1}{(3)(4)}+...+\frac{1}{N(N+1)} = \sum_{n=1}^{N}\frac{1}{n(n+1)}$ $\sum_{n=1}^{N}\frac{1}{n(n+1)} = 2\sum_{n=1}^{N}\frac{1}{2n(n+1)}$ $= 2\sum_{n=1}^{N}(a_{n+1}-a_n)$ $= 2(a_2-a_1$ $+a_3-a_2$ $+a_4-a_3$ $+ \vdots$ $+a_N-a_{N-1}$ $+a_{N+1}-a_N)$ $= 2(a_{N+1}-a_1)$ $= 2\left(1-\frac{1}{2(N+1)}-\frac{1}{2}\right)$

	$= 1 - \frac{1}{(N+1)}$

Qn	Solution
8	**Application of Differentiation (Max/Min, Rate of Change)**
(i)	Method 1: Area of triangle PQR = $\frac{1}{2}(QR)(PQ)$ since $\angle PQR = 90^\circ$ (angle inscribed in a semicircle is a right angle). Let $\angle QPR = \theta$. Then $QR = 2r\sin\theta$ and $PQ = 2r\cos\theta$ Area of $PQR = \frac{1}{2}(2r\sin\theta)(2r\cos\theta)$ $= 2r^2\sin\theta\cos\theta = r^2\sin 2\theta$. For maximum area of triangle PQR, $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(r^2\sin 2\theta\right) = 0$ $2r^2\cos 2\theta = 0$ $\Rightarrow \cos 2\theta = 0$ $2\theta = \frac{\pi}{2}$ $\therefore \theta = \frac{\pi}{4}$ $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(2r^2\cos 2\theta\right) = -4r^2\sin 2\theta = -4r^2 < 0$ when $\theta = \frac{\pi}{4}$ $\Rightarrow$Area is maximum. $PQ = 2r\cos\frac{\pi}{4} = \sqrt{2}r$ $QR = 2r\sin\frac{\pi}{4} = \sqrt{2}r$ Method 2: Let $\angle POQ = \theta$. Area of PQR = Area of POQ + Area of ROQ $= \frac{1}{2}r^2\sin\theta + \frac{1}{2}r^2\sin(\pi-\theta)$ note: $\sin(\pi-\theta) = \sin\theta$ $= r^2\sin\theta$. For maximum area of triangle PQR, $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(r^2\sin\theta\right) = 0$ $r^2\cos\theta = 0$ $\Rightarrow \cos\theta = 0$ $\theta = \frac{\pi}{2}$ $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(r^2\cos\theta\right) = -r^2\sin\theta = -r^2 < 0$ when $\theta = \frac{\pi}{4}$ $\Rightarrow$Area is maximum. $PQ = \sqrt{r^2 + r^2} = \sqrt{2}r$ $QR = \sqrt{r^2 + r^2} = \sqrt{2}r$

Method 3:
Let the length of PQ be x units.
Let A be the area of triangle PQR

$$A = \frac{1}{2}(PQ)(QR) = \frac{1}{2}x\sqrt{4r^2 - x^2}$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{2}x\left(\frac{1}{2}\right)\left(4r^2 - x^2\right)^{-\frac{1}{2}}(-2x) + \frac{1}{2}\left(4r^2 - x^2\right)^{\frac{1}{2}}$$

$$= -\frac{1}{2}x^2\left(4r^2 - x^2\right)^{-\frac{1}{2}} + \frac{1}{2}\left(4r^2 - x^2\right)^{\frac{1}{2}}$$

$$= \left(4r^2 - x^2\right)^{-\frac{1}{2}}\left[-\frac{1}{2}x^2 + \frac{1}{2}\left(4r^2 - x^2\right)\right]$$

$$= \left(4r^2 - x^2\right)^{-\frac{1}{2}}(-x^2 + 2r^2)$$

For maximum area of triangle PQR,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 0$$

$$\left(4r^2 - x^2\right)^{-\frac{1}{2}}(-x^2 + 2r^2) = 0$$

$\left(4r^2 - x^2\right)^{-\frac{1}{2}} = 0$ (No solution) or $-x^2 + 2r^2 = 0$

$x = \pm\sqrt{2}r$

$\therefore x = \sqrt{2}r$

Using 1st derivative test,

x	$\left(\sqrt{2}r\right)^-$	$\left(\sqrt{2}r\right)$	$\left(\sqrt{2}r\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}x}$	$/$	$-$	$\backslash$

Area is maximised when $x = \sqrt{2}r$

$\therefore PQ = \sqrt{2}r$

$QR = \sqrt{4r^2 - 2r^2} = \sqrt{2}r$

(ii) For $r = 2$, $QR = 4\sin\theta$

$$\frac{\mathrm{d}(QR)}{\mathrm{d}\theta} = 4\cos\theta$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}(QR)} \cdot \frac{\mathrm{d}(QR)}{\mathrm{d}t}$$

$$= \frac{1}{4\cos\theta} \cdot \frac{1}{5} = \frac{1}{20\cos\theta}$$

When $\theta = \frac{\pi}{3}$,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{20\left(\frac{1}{2}\right)}$$

$= \frac{1}{10}$ radians per second.

Qn	Solution
9	**Arithmetic and Geometric Progressions**
(i)	$S_{15} = \dfrac{2400\left[\left(\frac{11}{10}\right)^{15} - 1\right]}{\left(\frac{11}{10}\right) - 1}$ $= 76253.96$ $= 76254$ m (to the nearest metre)
(ii)	$(4000) + (n-1)800 \geq 42000$ $n \geq 48.5$ least $n = 49$ Or By GC, When $n = 48, (4000) + (n-1)800 = 41600 < 42000$ When $n = 49, (4000) + (n-1)800 = 42400 > 42000$ $\therefore$ least $n = 49$
(iii)	Total distance covered by runner B in 3 days $= \dfrac{3}{2}\left[2(4000) + (2)(800)\right]$ $= 14400$ $14400 = \dfrac{2400\left[r^3 - 1\right]}{r - 1}$ $6(r-1) = r^3 - 1$ $r^3 - 6r + 5 = 0$ By GC, $r = -2.7913$ (rej. $\because$ distance covered cannot be negative) or $r = 1$ (rej. $r \neq 1$) or $r = 1.7913$ $x\% = (1.7913 - 1) \times 100\%$ $= 79.13\%$ $\approx 79.1\%$ $x = 79.1$

Qn	Solution
10	**Differential Equations**
(a)	$y = xz ----(1)$ $\frac{dy}{dx} = x\frac{dz}{dx} + z ----(2)$ Sub (1) and (2) into D.E.: $\left(e^x+1\right)\left(x\frac{dz}{dx}+\frac{y}{x}-\frac{y}{x}\right)=\frac{x^2}{xz}\left(e^x-1\right)$ $\left(e^x+1\right)\left(\frac{dz}{dx}\right)=\frac{\left(e^x-1\right)}{z}$ $z\frac{dz}{dx}=\frac{e^x-1}{e^x+1}$ $\int z\ dz=\int\frac{e^x-1}{e^x+1}\ dx$ Method 1: $\int z\ dz=\int\frac{\left(e^{\frac{x}{2}}\right)\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)}{\left(e^{\frac{x}{2}}\right)\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)}\ dx$ $\int z\ dz=\int\frac{\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)}{\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)}\ dx$ $\frac{z^2}{2}=2\ln\left\|e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right\|+C,$ where C is an arbitrary constant. $\left(\frac{y}{x}\right)^2=4\ln\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)+D,$ where $D=2C$ $y^2=4x^2\ln\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)+Dx^2$ Method 2: $\int z\ dz=\int\left(\frac{e^x}{e^x+1}-\frac{1}{e^x+1}\frac{e^{-x}}{e^{-x}}\right)dx$ $\frac{z^2}{2}=\ln\left\|e^x+1\right\|+C'-\int\left(\frac{1}{e^x+1}\frac{e^{-x}}{e^{-x}}\right)dx$ $\frac{z^2}{2}=\ln\left\|e^x+1\right\|+C'-\int\left(\frac{e^{-x}}{1+e^{-x}}\right)dx$ $\frac{z^2}{2}=\ln\left\|e^x+1\right\|+C'+\int\left(\frac{-e^{-x}}{1+e^{-x}}\right)dx$

$$\frac{z^2}{2} = \ln\left|e^x + 1\right| + \ln\left|1 + e^{-x}\right| + C \quad \text{, where } C \text{ is an arbitrary constant}$$

$$z^2 = 2\ln\left(e^x + 1\right) + 2\ln\left(1 + e^{-x}\right) + D,\ D = 2C$$

$$\left(\frac{y}{x}\right)^2 = 2\ln\left(e^x + 1\right) + 2\ln\left(1 + e^{-x}\right) + D$$

$$y^2 = 2x^2\ln\left(e^x + 1\right) + 2x^2\ln\left(1 + e^{-x}\right) + Dx^2$$

Method 3:

$$\int z \ \mathrm{d}z = \int\left(1 - \frac{2}{e^x + 1}\right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + C' - 2\int\left(\frac{1}{e^x + 1}\right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + C' - 2\int\left(\frac{1}{e^x + 1}\frac{e^{-x}}{e^{-x}}\right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + C' - 2\int\left(\frac{e^{-x}}{1 + e^{-x}}\right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + C' + 2\int\left(\frac{-e^{-x}}{1 + e^{-x}}\right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + 2\ln\left|1 + e^{-x}\right| + C \quad \text{, where } C \text{ is an arbitrary constant}$$

$$z^2 = 2x + 4\ln\left(1 + e^{-x}\right) + D,\ D = 2C$$

$$\left(\frac{y}{x}\right)^2 = 2x + 4\ln\left(1 + e^{-x}\right) + D$$

$$y^2 = 2x^3 + 4x^2\ln\left(1 + e^{-x}\right) + Dx^2$$

Method 4:

$$\int z \ \mathrm{d}z = \int \frac{e^x - 1}{e^x + 1} \mathrm{d}x \qquad \text{Let } u = e^x$$

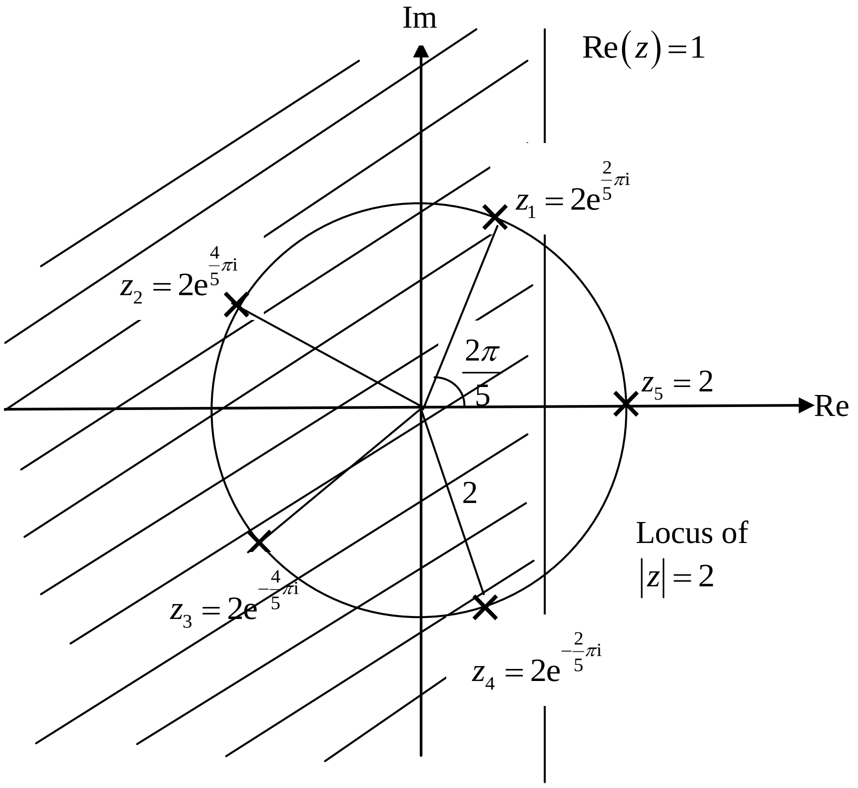
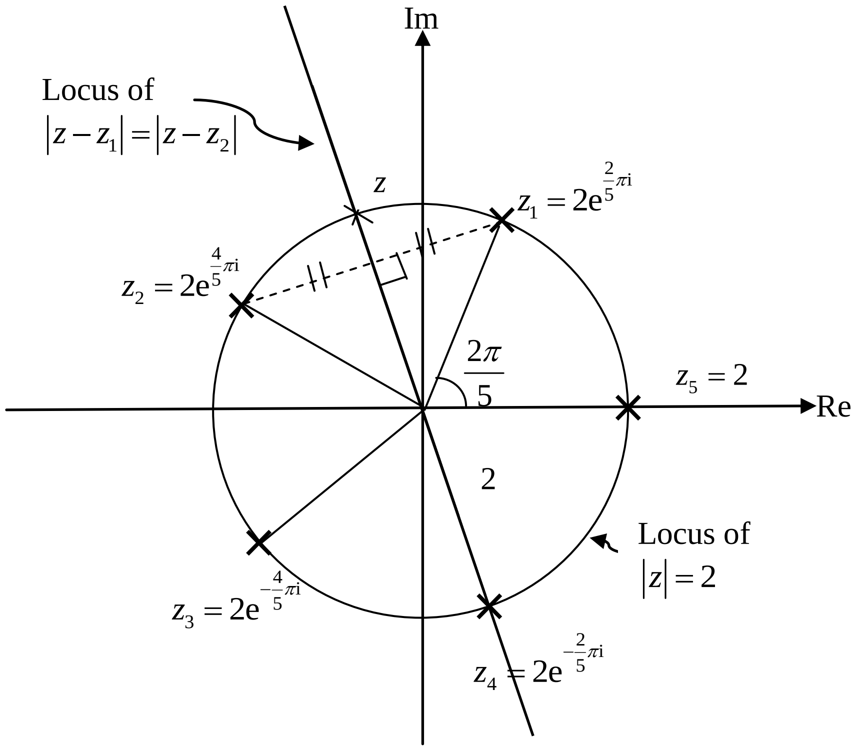
$$\int z \ \mathrm{d}z = \int\left(\frac{u - 1}{u + 1}\frac{1}{u}\right) \mathrm{d}u \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = e^x$$

$$\int z \ \mathrm{d}z = \int\left(\frac{u - 1}{u(u + 1)}\right) \mathrm{d}u$$

$$\int z \ \mathrm{d}z = \int\left(\frac{2}{u + 1} - \frac{1}{u}\right) \mathrm{d}x$$

$$\frac{z^2}{2} = 2\ln\left|u + 1\right| - \ln\left|u\right| + C \quad \text{, where } C \text{ is an arbitrary constant}$$

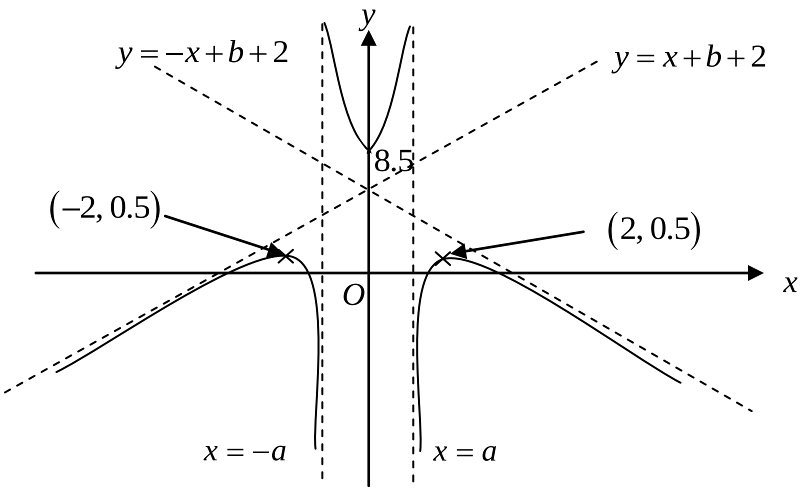
	$\frac{z^2}{2} = 2\ln\left(e^x + 1\right) - \ln\left(e^x\right) + C$ $z^2 = 4\ln\left(e^x + 1\right) - 2\ln\left(e^x\right) + D,\ D = 2C$ $\left(\frac{y}{x}\right)^2 = 4\ln\left(e^x + 1\right) - 2\ln\left(e^x\right) + D$ $y^2 = 4x^2\ln\left(e^x + 1\right) - 2x^2\ln\left(e^x\right) + Dx^2$
(b)	$\frac{dx}{dt} = \frac{A}{9-x} - \frac{x}{20}$ When $x = 4$, $\frac{dx}{dt} = 0,$ $0 = \frac{A}{5} - \frac{4}{20}$ $A = 1$
	$\frac{dx}{dt} = -\frac{x}{20} + \frac{1}{9-x}$ $\frac{dx}{dt} = \frac{-x(9-x)+20}{20(9-x)}$ $\frac{dx}{dt} = \frac{x^2 - 9x + 20}{20(9-x)}$ $\frac{dx}{dt} = \frac{(x-4)(x-5)}{20(9-x)}$ (shown)
	$\frac{dx}{dt} = \frac{(x-4)(x-5)}{20(9-x)}$ $\int \frac{9-x}{(x-4)(x-5)}\,dx = \int \frac{1}{20}\,dt$ $\int \frac{-5}{x-4} + \frac{4}{x-5}\,dx = \frac{t}{20} + C'$ $-5\ln\lvert x-4\rvert + 4\ln\lvert x-5\rvert = \frac{t}{20} + C'$ $t = -100\ln\lvert x-4\rvert + 80\ln\lvert x-5\rvert + C$ where $C = 20C'$ When $t = 0,\ x = 0,$ $0 = -100\ln 4 + 80\ln 5 + C$ $C = 100\ln 4 - 80\ln 5$ $t = -100\ln\lvert x-4\rvert + 80\ln\lvert x-5\rvert + 100\ln 4 - 80\ln 5$ $t = -100\ln\left\lvert\frac{x-4}{4}\right\rvert + 80\ln\left\lvert\frac{x-5}{5}\right\rvert$ When $x = 2$, $t = 28.449$ $t = 28.4$ months (3 s.f.)

Qn	Solution
11	**Complex no. 1-3 (Roots + Loci)**
(i)	$z^5 = 32$ $\quad = 32e^{i(2k\pi)}$ $z = 32^{\frac{1}{5}} e^{\frac{i2k\pi}{5}}$, where $k = -2, -1, 0, 1, 2$ $\therefore z = 2e^{-\frac{4}{5}\pi i}, 2e^{-\frac{2}{5}\pi i}, 2, 2e^{\frac{2}{5}\pi i}, 2e^{\frac{4}{5}\pi i}$
(iii)	 There are 4 points within the region given.
(iii)	

	Method 1: Since the argument of $z=\frac{1}{2}\left(\frac{2\pi}{5}\right)+\frac{2\pi}{5}=\frac{3\pi}{5}$ or $-\frac{2\pi}{5}$, $z=2\mathrm{e}^{\frac{3}{5}\pi \mathrm{i}}$ or $z=2\mathrm{e}^{-\frac{2}{5}\pi \mathrm{i}}$ $=-0.62+1.90i$ $\quad$ $=0.62-1.90i$ Method 2: Equation of circle centred 0+0i and radius 2: $x^2+y^2=2^2$ Gradient of the perpendicular bisector = $\tan\frac{3\pi}{5}=-3.077683$ Equation of perpendicular bisector that passes through the Origin: $y=-3.077683x$ $x^2+\left(-3.077683x\right)^2=4$ $x^2=0.381966$ $x=-0.618034$ or $x=0.618034$ $y=1.9021127$ or $y=-1.9021127$ $\therefore z=-0.62+1.90\mathrm{i}$ or $z=0.62-1.90\mathrm{i}$
(iv)	Method 1: $\left(w-2\right)^4+2\left(w-2\right)^3+4\left(w-2\right)^2+8\left(w-2\right)+16=0$ $\frac{16\left[1-\left(\frac{w-2}{2}\right)^5\right]}{1-\frac{w-2}{2}}=0$ $1-\left(\frac{w-2}{2}\right)^5=0, w\neq 4$ $\left(w-2\right)^5=2^5=32$ $\therefore$ replace z by $w-2$ in previous answer (excluding $z=2$ since $w\neq 4$) $w-2=2\mathrm{e}^{-\frac{4}{5}\pi\mathrm{i}},2\mathrm{e}^{-\frac{2}{5}\pi\mathrm{i}},2\mathrm{e}^{\frac{2}{5}\pi\mathrm{i}},2\mathrm{e}^{\frac{4}{5}\pi\mathrm{i}}$ $w=2+2\mathrm{e}^{-\frac{4}{5}\pi\mathrm{i}},2+2\mathrm{e}^{-\frac{2}{5}\pi\mathrm{i}},2+2\mathrm{e}^{\frac{2}{5}\pi\mathrm{i}},2+2\mathrm{e}^{\frac{4}{5}\pi\mathrm{i}}$ Method 2: $z^5-32=\left(z-2\right)\left(z^4+2z^3+4z^2+8z+16\right)$ For $\left(w-2\right)^4+2\left(w-2\right)^3+4\left(w-2\right)^2+8\left(w-2\right)+16=0$ $\therefore$ replace z by w - 2 in previous answer (excluding $z=2$), $w-2=2\mathrm{e}^{-\frac{4}{5}\pi\mathrm{i}},2\mathrm{e}^{-\frac{2}{5}\pi\mathrm{i}},2\mathrm{e}^{\frac{2}{5}\pi\mathrm{i}},2\mathrm{e}^{\frac{4}{5}\pi\mathrm{i}}$ $w=2+2\mathrm{e}^{-\frac{4}{5}\pi\mathrm{i}},2+2\mathrm{e}^{-\frac{2}{5}\pi\mathrm{i}},2+2\mathrm{e}^{\frac{2}{5}\pi\mathrm{i}},2+2\mathrm{e}^{\frac{4}{5}\pi\mathrm{i}}$ ---

	$\Rightarrow w = 2\left(1+\mathrm{e}^{\frac{p\pi \mathrm{i}}{5}}\right)$,where $p = -4, -2, 2, 4$ $w = 2\mathrm{e}^{\frac{p\pi \mathrm{i}}{10}}\left(\mathrm{e}^{-\frac{p\pi \mathrm{i}}{10}} + \mathrm{e}^{\frac{p\pi \mathrm{i}}{10}}\right)$ $= 2\mathrm{e}^{\frac{p\pi \mathrm{i}}{10}}\left[2\cos\left(\frac{p\pi}{10}\right)\right]$ $= 4\cos\left(\frac{p\pi}{10}\right)\mathrm{e}^{\frac{p\pi \mathrm{i}}{10}}$,where $p = -4, -2, 2, 4$

Qn	Solution
12	**Curve Sketching + Transformations + Integration**
a(i)	Method 1 1. Translation of 1 unit in the direction of x–axis. 2. Scaling parallel to the x–axis by factor $\frac{1}{2}$. 3. Scaling parallel to the y-axis by factor $\frac{1}{2}$. Method 2 1. Scaling parallel to the y-axis by factor $\frac{1}{2}$. 2. Scaling parallel to the x-axis by factor $\frac{1}{2}$ 3. Translation of $\frac{1}{2}$ unit in the direction of x–axis.
(ii)	Area $= \int_{-1}^{-0.5}\left\lvert\frac{2x+1}{x^2+4}\right\rvert \mathrm{d}x$ $= \int_{-1}^{-0.5} -\left(\frac{2x+1}{x^2+4}\right)\mathrm{d}x$ $= \int_{-1}^{-0.5} -\left(\frac{2x}{x^2+4} + \frac{1}{x^2+4}\right)\mathrm{d}x$ $= -\left[\ln\lvert x^2+4\rvert + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{-0.5}$ $= -\left[\left[\ln\left(\frac{1}{4}+4\right) + \frac{1}{2}\tan^{-1}\left(\frac{-0.5}{2}\right)\right] - \left[\ln(1+4) + \frac{1}{2}\tan^{-1}\left(\frac{-1}{2}\right)\right]\right]$ $= -\ln\left(\frac{17}{4}\right) + \frac{1}{2}\tan^{-1}\left(\frac{1}{4}\right) + \ln(5) - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$ $= \ln(5) - \ln\left(\frac{17}{4}\right) + \frac{1}{2}\tan^{-1}\left(\frac{1}{4}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$ $= \ln\left(\frac{20}{17}\right) + \frac{1}{2}\left(\tan^{-1}\left(\frac{1}{4}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right)$

(b)	**(i)**  (ii) $y^2 = \dfrac{\mathrm{d}y}{\mathrm{d}x}$