2024 ASRJC H3 Math Prelim Solutions

1 **(a)** $\frac{d^2}{dx}\sin x = -\sin x < 0$ for $x \in (0,\pi)$. $\frac{d^2}{dx}\ln x = -\frac{1}{x^2} < 0$ for $x \in (0,\infty)$.

(i) Given that A, B and C are angles of a triangle, then

$0 < A < \frac{\pi}{2}$, $0 < B < \frac{\pi}{2}$ and $0 < C < \frac{\pi}{2}$.

Applying Jensen inequality,

$$\frac{1}{3}\sum_{k=1}^{3}\sin(x_k) \le \sin\left(\frac{1}{3}\sum_{k=1}^{n}x_k\right)$$

$$\frac{1}{3}\left[\sin A + \sin B + \sin C\right] \le \sin\left(\frac{1}{3}(A+B+C)\right)$$

$$\frac{1}{3}\left[\sin A + \sin B + \sin C\right] \le \sin\left(\frac{1}{3}(\pi)\right)$$

$$\frac{1}{3}\left[\sin A + \sin B + \sin C\right] \le \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2} \text{ (Shown)}$$

(ii) Let $f(x) = \ln x$. Then $\frac{1}{n}\ \ln a_1 + \ln a_2 + \ldots + \ln a_n \ \le \ln\left[\frac{1}{n}\ a_1 + a_2 + \ldots + a_n\ \right]$

$$\ln(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \le \ln\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)$$

$$\therefore \quad \frac{a_1 + a_2 + \ldots + a_n}{n} \ge (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$$

(b) (i) $P(n) = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{9}{8}\cdot\ldots\ldots\cdot\frac{2^{n-1}+1}{2^{n-1}}$.

Using AM-GM inequality,

$$\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{9}{8}\cdot\ldots\ldots\cdot\frac{2^{n-1}+1}{2^{n-1}}\right)^{\frac{1}{n}} \le \frac{\frac{2}{1}+\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\ldots\ldots+\frac{2^{n-1}+1}{2^{n-1}}}{n}$$

$$\left(P(n)\right)^{\frac{1}{n}} \le \frac{\left(1+\frac{1}{1}\right)+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{4}\right)+\left(1+\frac{1}{8}\right)+\ldots\ldots+\left(1+\frac{1}{2^{n-1}}\right)}{n}$$

$$\left(P(n)\right)^{\frac{1}{n}} \le \frac{n+(1)+\left(\frac{1}{2}\right)+\left(\frac{1}{4}\right)+\left(\frac{1}{8}\right)+\ldots\ldots+\left(\frac{1}{2^{n-1}}\right)}{n}$$

$$(P(n))^{\frac{1}{n}} \le \frac{n + \dfrac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}}{n}$$

$$(P(n))^{\frac{1}{n}} \le \frac{n+2-\left(\frac{1}{2}\right)^{n-1}}{n}$$

$$(P(n))^{\frac{1}{n}} < \frac{n+2}{n}$$

$$P(n) < \left(1+\frac{2}{n}\right)^n$$

$$P(n) < \left(1+\frac{2}{n}\right)^n$$

$$= 1 + n\left(\frac{2}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{2}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{2}{n}\right)^3 + \ldots + \frac{n(n-1)(n-2)\ldots(2)(1)}{n!}\left(\frac{2}{n}\right)^n$$

(ii) $< 1 + 2 + \frac{1}{2!}(2)^2 + \frac{1}{3!}(2)^3 + \ldots + \frac{1}{n!}(2)^n$

$$< 1 + 2 + \frac{1}{2!}(2)^2 + \frac{1}{3!}(2)^3 + \ldots + \frac{1}{n!}(2)^n + \ldots$$

$$< e^2$$

2 $F_n - F_{n-1}$

$$= w_n^2 + w_{n-1}^2 - 4w_n w_{n-1} - w_{n-1}^2 - w_{n-2}^2 + 4w_{n-1}w_{n-2}$$

$$= w_n^2 - w_{n-2}^2 - 4w_n w_{n-1} + 4w_{n-1}w_{n-2}$$

$$= (w_n - w_{n-2})(w_n + w_{n-2}) - 4w_{n-1}(w_n - w_{n-2})$$

$$= (w_n - w_{n-2})(w_n + w_{n-2} - 4w_{n-1})$$

$F_n - F_{n-1} = (w_n - w_{n-2})(w_n + w_{n-2} - 4w_{n-1})$ ----------------------(1)

(a) Let w_n be u_n. Then $u_n + u_{n-2} - 4u_{n-1} = 0$.
So we have $F_n - F_{n-1} = 0$ for $n \ge 2$ by result above.
ie $F_n = F_{n-1}$ for $n \ge 2$
$F_1 = u_1^2 + u_0^2 - 4u_1u_0 = 2^2 + 1^2 - 4(2)(1) = -3$
Therefore $F_n = -3$ for all $n \ge 1$.
So we have $u_n^2 + u_{n-1}^2 = 4u_nu_{n-1} - 3$ for $n \ge 1$.

(b)(i) Let w_n be v_n.

$$v_1^2 + 1^2 = 4(v_1)(1) - 3$$
$$v_1^2 - 4v_1 + 4 = 0$$
$$\left(v_1 - 2\right)^2 = 0$$
$$v_1 = 2$$

$F_n = v_n^2 + v_{n-1}^2 - 4v_n v_{n-1} = -3$ for $n \geq 1$.

From (1), we have $v_n - v_{n-2} = 0$ or $v_n + v_{n-2} - 4v_{n-1} = 0$ for $n \geq 2.$.

(ii) Since 1,2,1,2,…. satisfies $v_n - v_{n-2} = 0$ for $n \geq 2.$, then from (1), F_n is constant. Also since $v_0 = 1$ and $v_1 = 2$, then $F_n = -3$. So the sequence satisfies (*).

(iii) Take the sequence 1,2,7,2,1,2,7,2,….. that satisfies $v_n - v_{n-2} = 0$ for odd $n \geq 2$ and $v_n + v_{n-2} - 4v_{n-1} = 0$ for even $n \geq 2$ with period 4.
Then by (1), F_n is constant and $F_n = -3$. So the sequence satisfies (*).

3 For all real values of *t*, $\left(t\mathrm{f}(x) + \mathrm{g}(x)\right)^2 \geq 0$

Thus we have

$$\int_a^b \left(t\mathrm{f}(x) + \mathrm{g}(x)\right)^2 \,\mathrm{d}x \geq 0$$
$$\Leftrightarrow \int_a^b t^2\left(\mathrm{f}(x)\right)^2 + 2t\mathrm{f}(x)\mathrm{g}(x) + \left(\mathrm{g}(x)\right)^2 \,\mathrm{d}x \geq 0$$
$$\Leftrightarrow t^2\int_a^b \left(\mathrm{f}(x)\right)^2 \,\mathrm{d}x + 2t\int_a^b \mathrm{f}(x)\mathrm{g}(x)\,\mathrm{d}x + \int_a^b \left(\mathrm{g}(x)\right)^2 \,\mathrm{d}x \geq 0$$

From above, we have

$$\int_a^b \left(t\mathrm{f}(x) + \mathrm{g}(x)\right)^2 \,\mathrm{d}x \geq 0$$
$$\Leftrightarrow \left(2\int_a^b \mathrm{f}(x)\mathrm{g}(x)\,\mathrm{d}x\right)^2 \leq 4\left(\int_a^b \left(\mathrm{f}(x)\right)^2 \,\mathrm{d}x\right)\left(\int_a^b \left(\mathrm{g}(x)\right)^2 \,\mathrm{d}x\right)$$
$$\Leftrightarrow \left(\int_a^b \mathrm{f}(x)\mathrm{g}(x)\,\mathrm{d}x\right)^2 \leq \left(\int_a^b \left(\mathrm{f}(x)\right)^2 \,\mathrm{d}x\right)\left(\int_a^b \left(\mathrm{g}(x)\right)^2 \,\mathrm{d}x\right)$$

Equality holds when $\int_a^b \left(t\mathrm{f}(x) + \mathrm{g}(x)\right)^2 \,\mathrm{d}x = 0$. Since f and g are continuous this means that we must have $t\mathrm{f}(x) + \mathrm{g}(x) = 0$ for all real *x*, i.e. f is a scalar multiple of g.

(i) Setting $f(x)=1$ and $g(x)=e^x$ in(*)

Since f is not a scalar multiple of g, we have

$$\left(\int_a^b e^x dx\right)^2 < \int_a^b 1\,dx \int_a^b e^{2x} dx$$

$$\left(e^b-e^a\right)^2 < (b-a)\frac{1}{2}\left(e^{2b}-e^{2a}\right)$$

$$\left(e^b-e^a\right)^2 < \frac{1}{2}(b-a)\left(e^b-e^a\right)\left(e^b+e^a\right)$$

$$e^b-e^a < \frac{1}{2}(b-a)\left(e^b+e^a\right)$$

Choosing $a=0$ and $b=t$ gives

$$e^t-1 < \frac{1}{2}t\left(e^t+1\right)$$

$$\frac{e^t-1}{e^t+1} < \frac{1}{2}t$$

(ii) Setting $f(x)=1$ and $g(x)=\sqrt{\sin x}$ and $a=0$ and $b=\frac{\pi}{2}$,

(*) becomes

$$\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}dx\right)^2 < \left(\int_0^{\frac{1}{2}\pi} 1dx\right)\left(\int_0^{\frac{1}{2}\pi} \sin xdx\right)$$

$$\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}dx\right)^2 < \frac{1}{2}\pi\left[-\cos x\right]_0^{\frac{1}{2}\pi} = \frac{\pi}{2}$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}dx < \sqrt{\frac{\pi}{2}}$$

Setting $f(x)=\cos x$ and $g(x)=(\sin x)^{\frac{1}{4}}$ and $a=0$ and $b=\frac{\pi}{2}$,

$$\left(\int_0^{\frac{1}{2}\pi} \cos x(\sin x)^{\frac{1}{4}}\, dx\right)^2 < \left(\int_0^{\frac{1}{2}\pi} \cos^2 x\, dx\right)\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}dx\right)$$

$$\left(\frac{4}{5}\left[(\sin x)^{\frac{5}{4}}\right]_0^{\frac{1}{2}\pi}\right)^2 < \left(\frac{1}{2}\int_0^{\frac{1}{2}\pi} 1+\cos 2x\, dx\right)\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}dx\right)$$

(*) becomes $\frac{16}{25} < \frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\frac{1}{2}\pi}\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}dx\right)$

$$\frac{16}{25} < \frac{1}{2}.\frac{\pi}{2}\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}dx\right)$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}dx > \frac{64}{25\pi}$$

Combining, $\frac{64}{25\pi} < \int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\, dx < \sqrt{\frac{\pi}{2}}.$

4 (i) $u=\dfrac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}(x)y \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+g(x)\dfrac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}'(x)y$

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\mathrm{f}(x)u=\mathrm{h}(x)$$
$$\Rightarrow\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+g(x)\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}'(x)y\right)+\mathrm{f}(x)u=\mathrm{h}(x)$$
$$\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+g(x)\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}'(x)y+\mathrm{f}(x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}(x)y\right)=\mathrm{h}(x)$$
$$\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\mathrm{g}(x)+\mathrm{f}(x)\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\mathrm{g}'(x)+\mathrm{f}(x)\mathrm{g}(x)\right)y=\mathrm{h}(x)$$

(ii) $\mathrm{g}(x)+\mathrm{f}(x)=2+\dfrac{2}{x}\Rightarrow f(x)=2+\dfrac{2}{x}-g(x)$

$$\mathrm{g}'(x)+\mathrm{f}(x)\mathrm{g}(x)=\frac{4}{x}$$
$$\Rightarrow \mathrm{g}'(x)+\left(2+\frac{2}{x}-g(x)\right)\mathrm{g}(x)=\frac{4}{x}$$

This is the first order differential equation satisfied by $\mathrm{g}(x)$.

If $\mathrm{f}(x)=kx^n$, then $\mathrm{g}(x)=2+\dfrac{2}{x}-kx^n$ and $\mathrm{g}'(x)=-\dfrac{2}{x^2}-knx^{n-1}$

Subst into the above first order de for $\mathrm{g}(x)$,

$$-\frac{2}{x^2}-knx^{n-1}+\left(2+\frac{2}{x}-kx^n\right)kx^n=\frac{4}{x}$$
$$-2-knx^{n+1}+2kx^{n+2}+2kx^{n+1}-k^2x^{2n+2}=4x$$

<u>Case 1:</u> $2n+2=n+2\Rightarrow n=0$. Rejected as there is a constant term -2 that cannot be eliminated

<u>Case 2</u>: $2n+2=n+1\Rightarrow n=-1$.

Comparing coefficients of x terms: $2k=4\Rightarrow k=2$

Checking constant terms: $LHS=-1+2+4-4=0=RHS$

So $\mathrm{f}(x)=\dfrac{2}{x}$, $\mathrm{g}(x)=2+\dfrac{2}{x}-\dfrac{2}{x}=2$

Using $\frac{\mathrm{d}u}{\mathrm{d}x}+\mathrm{f}(x)u=\mathrm{h}(x)$, we have

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\left(\frac{2}{x}\right)u=\frac{6}{x}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2}{x}\left(3-u\right)$$

$$-\ln\left|3-u\right|=2\ln\left|x\right|+C$$

$$3-u=\frac{A}{x^2}$$

$$u=3-\frac{A}{x^2}$$

Using $\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}(x)y=u$ and we are given $y=4$ and $\frac{dy}{dx}=-5$ at $x=1$

$-5+2\left(4\right)=u\Rightarrow u=3$ at $x=1$.

$\therefore A=0\Rightarrow u(x)=3$.

Using $\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}(x)y=u$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x}+2y=3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3-2y$$

$$-\frac{1}{2}\ln\left|3-2y\right|=x+D$$

$$\ln\left|3-2y\right|=-2x+E$$

$$3-2y=Be^{-2x}$$

When $x=1, y=4\Rightarrow B=-5e^2$

$$y=\frac{3}{2}+\frac{5}{2}e^{-2x+2}$$

5 $2r-1=2(2.3...p)-1$

Claim: All prime factors of $2r-1$ are greater than *p*.

Suppose there exists a prime factor p_1 such that $2 \le p_1 \le p$. Then $p_1 \mid (2r\text{-}1)$ and $p_1 \mid 2(2.3.....p)$ which implies $p_1 \mid 1$. (contradiction)

Now $2r-1=2(2.3...p)-1 \equiv -1(mod\,4) \equiv 3(mod\,4)$

ie $2r-1$ is of the form $4n+3$ for some integer *n*.

Since $2r-1$ is odd, the only possible prime factors are of the form $4k+1$ or $4k+3$ for integers *k*.

But $(4k_1+1)(4k_2+1)=4(4k_1k_2+k_1+k_2)+1$ ie of the form $4k+1$

Hence there must exists a prime factor of $2r-1$ of the form $4k+3$ ie congruent to 3 modulo 4

Therefore there exists a prime factor *q* of $2r-1$ such that $q>p$ and $q \equiv 3(\text{mod}\,4)$.

Suppose there is a finite number of primes of the form $4n+3$ and *p* is the largest prime of this form.

From result above, $2r-1$ has a prime factor *q* such that $q>p$ and $q \equiv 3(\text{mod}\,4)$ ie q is of the form $4k+3$ for some integer *k* (contradiction)

Hence there is an infinite number of primes of the form $4n+3$, where *n* is an integer.

$4n+3=3^{2k+1}$ where *k* is an integer

$4n=3(3^{2k}-1)=3(9^k-1)$

$$n=\frac{3(9^k-1)}{4}$$

Since $(9^k-1)=(9-1)(9^{k-1}+9^{k-2}+...+1)$ which is divisible by 4, therefore

$n=\dfrac{3(9^k-1)}{4}$ is always an integer for all non-negative integer k.

Therefore $n=\dfrac{3(9^k-1)}{4}$ where $k=0,1,2,\ldots$ give an infinite sequence of numbers for which the values of $4n+3$ are the odd powers of 3.

6 $(\Rightarrow)$ If y is a solution of the congruences $y \equiv a\left(\bmod p^{\alpha} q^{\beta}\right)$

then $y \equiv m p^{\alpha} q^{\beta}+a$ for some integer m

$y=m p^{\alpha} q^{\beta}+a\left(\bmod p^{\alpha}\right) \Rightarrow y \equiv a\left(\bmod p^{\alpha}\right)$

$y=m p^{\alpha} q^{\beta}+a\left(\bmod q^{\beta}\right) \Rightarrow y \equiv a\left(\bmod q^{\beta}\right)$

$(\Leftarrow)$ If y is a solution of both the congruences of

$y \equiv a\left(\bmod p^{\alpha}\right)$ and $y \equiv a\left(\bmod q^{\beta}\right)$

then

$y \equiv s p^{\alpha}+a$ and $y \equiv t q^{\beta}+a$ for some integers *s, t*.

$s p^{\alpha}+a=t q^{\beta}+a$

$\Rightarrow s p^{\alpha}=t q^{\beta}$

Since *p* and *q* are distinct prime numbers and hence coprime,
then

$q^{\beta} \nmid p^{\alpha} \Rightarrow q^{\beta} \mid s$

$\Rightarrow s=h q^{\beta}$ for some integer *h*

We have $y \equiv s p^{\alpha}+a \Rightarrow y=h q^{\beta} p^{\alpha}+a \Rightarrow y \equiv a\left(\bmod p^{\alpha} q^{\beta}\right)$ (shown)

$x^{3}+10 x+9 \equiv 0(\bmod 24) \Rightarrow x^{3}+10 x+9 \equiv 0\left(\bmod 2^{3} .3\right)$

ie it is equivalent to finding the solutions to

$x^{3}+10 x+9 \equiv 0(\bmod 3)$ and $x^{3}+10 x+9 \equiv 0\left(\bmod 2^{3}\right)$

$x^{3}+10 x+9 \equiv 0(\bmod 3)$

$x^{3}+x \equiv 0(\bmod 3)$

$x\left(x^{2}+1\right) \equiv 0(\bmod 3)$

Since 3 is prime, therefore $x \equiv 0(\bmod 3)$ or $x^{2}+1 \equiv 0(\bmod 3)$

Consider $x=3 k, 3 k+1,3 k+2$ where k is an integer

We have $x^{2}+1 \equiv 1(\bmod 3), x^{2}+1 \equiv 2(\bmod 3), x^{2}+1 \equiv 2(\bmod 3)$ respectively.

So there are no solution to $x^{2}+1 \equiv 0(\bmod 3)$

Then $x \equiv 0(\bmod 3) \Rightarrow x=3 n$ where n is an integer

$x^{3}+10 x+9 \equiv 0(\bmod 8)$

$x^{3}+2 x+1 \equiv 0(\bmod 8)$

For $x=3 n$, consider $n=2 r, 2 r+1$ where r is an integer

For $n=2 r$, LHS is odd and RHS is even ie no solution

For $n=2r+1$, $x=6r+3$ we have

$$(6r+3)^3+2(6r+3)+1\equiv 0(\text{mod}\,8)$$

$$(6r)^3+3(6r)^2 3+3(6r)+3^3+12r+6+1\equiv 0(\text{mod}\,8)$$

$$4r^2+6r+2\equiv 0(\text{mod}\,8)$$

$$2(2r^2+3r+1)\equiv 0(\text{mod}\,8)$$

$$2(2r+1)(r+1)\equiv 0(\text{mod}\,8)$$

$$(2r+1)(r+1)\equiv 0(\text{mod}\,4)$$

Since $2r+1$ is odd, we have $r+1\equiv 0(\text{mod}\,4)$ ie $r=4w-1$ where w is an integer

So the solution to $x^3+10x+9\equiv 0(\text{mod}\,24)$ is

$$x=6(4w-1)+3=24w-3, w\in\mathbb{Z}$$

7	**(i)** For 2 boxes each contains exactly 2 objects, all the other remaining $(r-4)$ identical boxes must each contains 1 object. Out of r distinct objects, choose 4 of them to be in the 2 boxes. This gives $\binom{r}{4}$ choices. These 4 objects (eg, A, B, C and D) are to be among 2 identical boxes equally and this gives only <u>3</u> ways of doing so, namely $\{\{A,B\},\{C,D\}\},\{\{A,C\},\{B,D\}\}$ and $\{\{A,D\},\{C,B\}\}$. Note that there is only 1 way to distribute the remaining $(r-4)$ objects into the remaining identical boxes. Hence, we have $3\binom{r}{4}$ ways. **(ii)** To distribute r distinct objects, where $r\geq 3$, into $(r-2)$ identical boxes, we can do via (i) or do so by distributing it such that exactly one of the boxes contains 3 objects. This gives $\binom{r}{3}$ ways to do so. Adding it with part (i), we get $S(r,r-2)=\binom{r}{3}+3\binom{r}{4}$ $=\dfrac{r(r-1)(r-2)}{3!}+\dfrac{3r(r-1)(r-2)(r-3)}{4!}$ $=\dfrac{4r(r-1)(r-2)+3r(r-1)(r-2)(r-3)}{4!}$

$$= \frac{r(r-1)(r-2)\left(4+3(r-3)\right)}{4!}$$

$$= \frac{r(r-1)(r-2)(3r-5)}{24}$$

(iii) Distributing 10 distinct cookies among 8 (identical) boxes

$$= S(10,8) = \frac{10(9)(8)(30-5)}{24} = 750$$

Distributing 10 distinct cookies among 8 (identical) boxes such that exactly 2 boxes each contains exactly 2 cookies $= 3\binom{10}{4} = 630$

Required probability $= \frac{630}{750} \times \frac{2}{8} = \frac{21}{100}$ or 0.21

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(i)(a) $\binom{m+k-1}{k-1}$ or $\binom{m+k-1}{m}$

(b) $\binom{m+k-1}{m} \times m! = \frac{(m+k-1)!}{(k-1)!}$

(ii) There is only 1 way to put *m* identical flags in 1 box

$T(m,1) = 1$

To arrange *m* identical flags in *m* boxes such that no box is empty,

$T(m,m) = m!$

(iii) To arrange *m* identical flags in *n* distinct boxes such that no box is empty,

<u>Suppose the 1st flag is alone in a box:</u>

There are *n* ways to choose which box this flag is in.

We arrange the remaining $(m-1)$ flags in the remaining $(n-1)$ boxes, giving us $T(m-1,n-1)$ ways.

So, total $n \times T(m-1,n-1)$ ways

	Suppose the 1st flag is not alone: We isolate this particular flag. We arrange the remaining $(m-1)$ flags in *n* boxes, giving us $T(m-1,n-1)$ ways. There are *n* choices where this 1st flag can be put in any of the already filled *n* boxes. So, total $n\times T(m-1,n)$ ways Adding the above cases up, we obtain $T(m,n)=n\times\left(T(m-1,n-1)+T(m-1,n)\right)$ **(iv)** Let A_i denote the set of distinct arrangements by distributing *m* distinct flags into *n* distinct boxes, such that at least *i* box(es) being empty. Where there is no restriction on the number of boxes, $\left\|A_0\right\|=n^m$ When at least 1 box is empty, $\left\|A_1\right\|=\binom{n}{1}(n-1)^m$ When at least 2 boxes are empty, $\left\|A_2\right\|=\binom{n}{2}(n-2)^m$ When at least *r* boxes are empty, $\left\|A_r\right\|=\binom{n}{r}(n-r)^m$ By principle of inclusion and exclusion, $T(m,n)=\sum_{r=0}^{n}(-1)^r\left\|A_r\right\|=\sum_{r=0}^{n}(-1)^r\binom{n}{r}(n-r)^m$