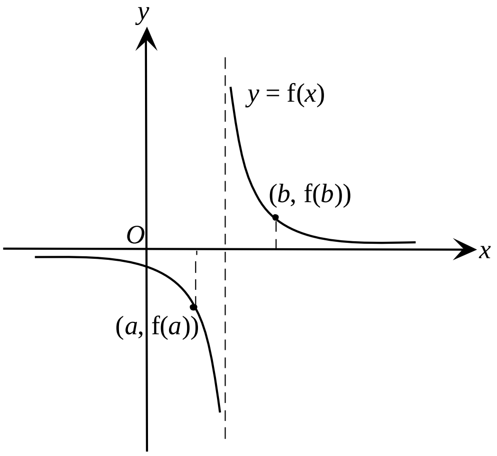
JPJC J2 Preliminary Examination 2024
Further Mathematics Paper 2 Solutions

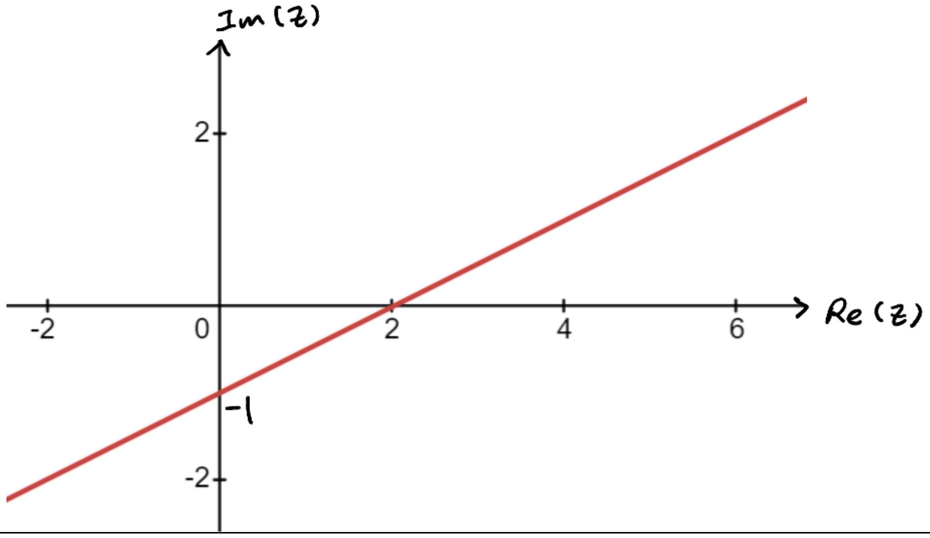
Section A: Pure Mathematics [50 marks]

<table>
<tr><th>Qn</th><th>Suggested Solutions</th><th></th></tr>
<tr><td>1(a)</td><td>
$\int_0^{\frac{\pi}{5}} \cos x\,\mathrm{d}x$
Let $\mathrm{f}(x)=\cos x$,
<table>
<tr><td>x</td><td>0</td><td>$\frac{\pi}{10}$</td><td>$\frac{\pi}{5}$</td></tr>
<tr><td>$\mathrm{f}(x)=\cos x$</td><td>1</td><td>$\cos\frac{\pi}{10}$</td><td>$\cos\frac{\pi}{5}$</td></tr>
</table>
Using Simpson's rule with 2 strips,
$\int_0^{\frac{\pi}{5}} \cos x\,\mathrm{d}x \approx \frac{1}{6}\left(\frac{\pi}{5}-0\right)\left[1+4\left(\cos\frac{\pi}{10}\right)+\cos\frac{\pi}{5}\right]$
$\sin 2\left(\frac{\pi}{10}\right) \approx \frac{\pi}{30}\left\{1+4\cos\frac{\pi}{10}+\cos 2\left(\frac{\pi}{10}\right)\right\}$
$2\sin\frac{\pi}{10}\cos\frac{\pi}{10} \approx \frac{\pi}{30}\left(1+4\cos\frac{\pi}{10}+2\cos^2\frac{\pi}{10}-1\right)$
$\sqrt{1-\cos^2\frac{\pi}{10}} \approx \frac{\pi}{30}\left(2+\cos\frac{\pi}{10}\right)$
$1-x^2 \approx \left(\frac{\pi}{30}\right)^2(2+x)^2, \quad \text{let } x=\cos\frac{\pi}{10}$
$1-x^2 \approx \left(\frac{\pi}{30}\right)^2\left(4+4x+x^2\right)$
$\left[\left(\frac{\pi}{30}\right)^2+1\right]x^2+\left(\frac{\pi}{15}\right)^2 x+\left[\left(\frac{\pi}{15}\right)^2-1\right]\approx 0$
Therefore $\cos\frac{\pi}{10}$ is approximately the root of
$\left[\left(\frac{\pi}{30}\right)^2+1\right]x^2+\left(\frac{\pi}{15}\right)^2 x+\left[\left(\frac{\pi}{15}\right)^2-1\right]=0.$
</td><td></td></tr>
<tr><td>(b)</td><td>
Solving $\left[\left(\frac{\pi}{30}\right)^2+1\right]x^2+\left(\frac{\pi}{15}\right)^2 x+\left[\left(\frac{\pi}{15}\right)^2-1\right]=0$ using GC,
$x \approx 0.951051199$ and $x \approx -0.9944402928$
Since $\frac{\pi}{10}$ is acute, $\cos\frac{\pi}{10}$ is positive.
$\therefore \cos\frac{\pi}{10} \approx 0.9511$ (4 dp)
</td><td></td></tr>
</table>

Qn	**Suggested Solutions**	
2(i)	C is a parabola, then $k=3$ C is an ellipse, then $k>3$ C is a hyperbola, then $0<k<3$	
(ii)	$r=\frac{4}{2+3\sin\theta}$ $2r+3r\sin\theta=4$ $2\sqrt{x^2+y^2}+3y=4$ $\sqrt{x^2+y^2}=2-\frac{3}{2}y$ $x^2+y^2=4-6y+\frac{9}{4}y^2$ $x^2-\frac{5}{4}y^2+6y=4$ $x^2-\frac{5}{4}\left(y-\frac{12}{5}\right)^2=-\frac{16}{5}$ $\frac{\left(y-\frac{12}{5}\right)^2}{\left(\frac{8}{5}\right)^2}-\frac{x^2}{\left(\frac{4}{\sqrt{5}}\right)^2}=1$	
(iii)	Eccentricity, $e=\frac{3}{2}$ and coordinates of the two foci are $(0,0)$ and $(0,4.8)$	

Qn	Suggested Solutions	
3(a)(i)	 The equation $\mathrm{f}(x)=0$ has no roots in the interval (a, b).	
(a)(ii)	Let $\mathrm{f}(x)=\operatorname{cosec}^2 x-3\ln x$ By using linear interpolation in the interval $[3, 4]$, $$x_1 = \frac{3\left\|\mathrm{f}(4)\right\|+4\left\|\mathrm{f}(3)\right\|}{\left\|\mathrm{f}(4)\right\|+\left\|\mathrm{f}(3)\right\|}$$ $= 3.\underline{\underline{95}}$ (3 sf) $\mathrm{f}(x)=\operatorname{cosec}^2 x-3\ln x=\frac{1}{\sin^2 x}-3\ln x$ is undefined at $x=\pi=3.14$ (3 sf) $\mathrm{f}(x)$ has a **discontinuity** at $x=\pi$ and $3<\pi<4$, hence the method is not suitable.	
(b)(i)	$x_1=\mathrm{G}(x_0)=\sin^{-1}\sqrt{\frac{1}{3\ln 2}}=0.76629\approx 0.766$ $x_2=\mathrm{G}(x_1)=\sin^{-1}\sqrt{\frac{1}{3\ln(0.76629)}}$ is undefined Since $\ln(0.76629)$ is negative.	
(b)(ii)	$x_1=\mathrm{G}(x_0)=\mathrm{e}^{\frac{1}{3}\operatorname{cosec}^2 2}=1.49653$ $x_2=\mathrm{G}(x_1)=1.39819$ $x_3=1.40982$ $x_4=1.40793\approx 1.408$ $x_5=1.40823\approx 1.408$ $\beta\approx 1.408$ The student can verify the correctness by evaluating $\mathrm{f}(1.4075)$ and $\mathrm{f}(1.4085)$ and check that $\mathrm{f}(1.4075)\mathrm{f}(1.4085)<0$.	

Qn	**Suggested Solutions**	
4(a)(i)	$\frac{1}{2}\left(z+z^{-1}\right)=\frac{1}{2}\left(\cos\theta+\mathrm{i}\sin\theta+\cos\theta-\mathrm{i}\sin\theta\right)$ $=\cos\theta$ (shown)	
(a)(ii)	$\cos^{n}\theta=\frac{1}{2^{n}}\left(z+z^{-1}\right)^{n}$ $=\frac{1}{2^{n}}\left[z^{n}+\binom{n}{1}z^{n-1}z^{-1}+\binom{n}{2}z^{n-2}z^{-2}+\ldots+\binom{n}{n-2}z^{2}z^{-n+2}+\binom{n}{n-1}z^{1}z^{-n+1}+z^{-n}\right]$ ---(1) Flip the equation, $\cos^{n}\theta=\frac{1}{2^{n}}\left[z^{-n}+\binom{n}{1}z^{-n+2}+\binom{n}{2}z^{-n+4}+\ldots+\binom{n}{n-2}z^{n-4}+\binom{n}{n-1}z^{n-2}+z^{n}\right]$ --- (2) Since $\binom{n}{k}=\binom{n}{n-k}$, where k is an integer $0\le k\le n$ (1)+(2), $2\cos^{n}\theta=\frac{1}{2^{n}}\left[\begin{aligned}&\left(z^{n}+z^{-n}\right)+\binom{n}{1}\left(z^{n-2}+z^{-n+2}\right)+\binom{n}{2}\left(z^{n-4}+z^{-n+4}\right)+\ldots\\&+\binom{n}{n-2}\left(z^{-n+4}+z^{n-4}\right)+\binom{n}{n-1}\left(z^{-n+2}+z^{n-2}\right)+\left(z^{n}+z^{-n}\right)\end{aligned}\right]$ $=\frac{2}{2^{n}}\left[\cos n\theta+\binom{n}{1}\cos(n-2)\theta+\binom{n}{2}\cos(n-4)\theta+\ldots+\cos(-n)\theta\right]$ $=\frac{2}{2^{n}}\sum_{k=0}^{n}\binom{n}{k}\cos(n-2k)\theta$ Therefore, $\cos^{n}\theta=\frac{1}{2^{n}}\sum_{k=0}^{n}\binom{n}{k}\cos(n-2k)\theta$	
(a)(iii)	When n = 3, $\cos^{3}\theta=\frac{1}{2^{3}}\sum_{k=0}^{3}\binom{3}{k}\cos(3-2k)\theta$ $=\frac{1}{8}\left[\cos 3\theta+3\cos\theta+3\cos(-\theta)+\cos(-3\theta)\right]$ $=\frac{1}{4}\left(\cos 3\theta+3\cos\theta\right)$	

	$\int_0^{\frac{\pi}{2}} \cos^3 \theta \, \mathrm{d}\theta = \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos 3\theta + 3\cos\theta \, \mathrm{d}\theta$ $= \frac{1}{4}\left[\frac{\sin 3\theta}{3} + 3\sin\theta\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{4}\left[-\frac{1}{3} + 3\right]$ $= \frac{2}{3}$	
(b)	$z = 2 + \lambda(2 + \mathrm{i})$ $x + y\mathrm{i} = (2 + 2\lambda) + \lambda\mathrm{i}$ $x = 2 + 2\lambda$ and $y = \lambda$ $x = 2 + 2y$ $y = \frac{x}{2} - 1$  	

Qn	Suggested Solutions	
5(i)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{b^2x}{a^2y}$ Equation of tangent at *P*, $y-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0)$ $a^2y_0y-a^2{y_0}^2=-b^2x_0x+b^2{x_0}^2$ $b^2x_0x+a^2y_0y=a^2{y_0}^2+b^2{x_0}^2$ $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=\frac{{y_0}^2}{b^2}+\frac{{x_0}^2}{a^2}$ $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$ (shown)	
(ii)	Equation of tangent at *P*, $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$ Intersect with directix $x=\frac{a}{e}$, $\frac{x_0}{a^2}\left(\frac{a}{e}\right)+\frac{y_0y}{b^2}=1$ $\frac{y_0y}{b^2}=1-\frac{x_0}{ae}$ $y=\frac{b^2\left(c-x_0\right)}{cy_0}$ since $e=\frac{c}{a}$ $\therefore T\left(\frac{a}{e},\frac{b^2\left(c-x_0\right)}{cy_0}\right)$ Gradient of $PF_2=\frac{y_0-0}{x_0-ae}$ $=\frac{y_0}{x_0-c}$ Gradient of $TF_2=\frac{\frac{b^2\left(c-x_0\right)}{cy_0}-0}{\frac{a}{e}-ae}$ $=\frac{b^2e\left(c-x_0\right)}{cy_0\left(a-ae^2\right)}$	

	Gradient of PF_2 × Gradient of TF_2 $= \frac{y_0}{x_0-c}\times\frac{b^2e\left(c-x_0\right)}{cy_0\left(a-ae^2\right)}$ $= -\frac{b^2e}{ac}\left(\frac{1}{1-e^2}\right)$ $= -\frac{b^2}{a^2}\left(\frac{1}{1-\frac{c^2}{a^2}}\right)$ since $e=\frac{c}{a}$ $= -b^2\left(\frac{1}{a^2-c^2}\right)$ $= -1$ $\because c^2=a^2-b^2$ Therefore, $\angle PF_2T = 90^\circ$	
(iii)	$\frac{PF_2}{PD_2}=e$ $\Rightarrow \frac{PF_2}{PT\cos\phi}=e$ $\Rightarrow \frac{PF_2}{PT}=e\cos\phi$ From part (ii), similarly, $\angle PF_1S$ is also 90° $\frac{PF_1}{PD_1}=e$ $\Rightarrow \frac{PF_1}{PS\cos\phi}=e$ $\Rightarrow \frac{PF_1}{PS}=e\cos\phi=\frac{PF_2}{PT}$	
(iv)	$\cos\left(\angle SPF_1\right)=\frac{PF_1}{PS}$ $\cos\left(\angle TPF_2\right)=\frac{PF_2}{PT}$ $\cos\left(\angle SPF_1\right)=\cos\left(\angle TPF_2\right)$ by (iii) $\therefore \angle SPF_1=\angle TPF_2$ (Since both angles are acute)	

Section B: Statistics [50 marks]

<table>
<tr><th>Qn</th><th>Suggested Solutions</th></tr>
<tr><td>6(a)</td><td>The sprint time of an athlete may not follow normal distribution and hence <i>t</i>-test is not appropriate.</td></tr>
<tr><td>(b)</td><td>Let <i>m</i> be the population median of difference between the timings (before – after)

H_0: $m = 0$
H_1: $m > 0$

<table>
<tr><td></td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td><td>F</td><td>G</td><td>H</td><td>I</td><td>J</td></tr>
<tr><td>before - after</td><td>6</td><td>4</td><td>10</td><td>11</td><td>0</td><td>−5</td><td>1</td><td>7</td><td>−2</td><td>3</td></tr>
<tr><td>Ranks</td><td>6</td><td>4</td><td>8</td><td>9</td><td></td><td>5</td><td>1</td><td>7</td><td>2</td><td>3</td></tr>
</table>

P = sum of positive rank = 6 + 4 + 8 + 9 + 1 + 7 + 3 = 38
Q = sum of negative rank = 5 + 2 = 7
Tcal = min(P, Q) = 7

At 5% level, we reject H_0 if $T \leq 8$

Since T cal ≤ 8, we reject H_0, and conclude that there is sufficient evidence at 5% level of significance that the new training is effective.</td></tr>
</table>

Qn	Suggested Solutions	
7(a)	**Assumptions:** (1) The delays occurring in a month is independent from the delays occuring in another month. (2) The average number of delays occurring in a month is a constant.	
(b)	$D \sim \text{Po}(\lambda)$ $\text{P}(D=0\ or\ 2\ or\ 4\ or\ 6) = \text{P}(D=0)+\text{P}(D=2)+\text{P}(D=4)+\text{P}(D=6)$ $= e^{-\lambda}+\frac{e^{-\lambda}\lambda^2}{2!}+\frac{e^{-\lambda}\lambda^4}{4!}+\frac{e^{-\lambda}\lambda^6}{6!}$ $= \frac{e^{-\lambda}}{2}\left(2+\lambda^2+\frac{\lambda^4}{12}+\frac{\lambda^6}{360}\right)$ $\text{P}(D=\text{even}) = \text{P}(D=0)+\text{P}(D=2)+\text{P}(D=4)+\text{P}(D=6)+\ldots$ $= \frac{e^{-\lambda}}{2}\left(2+\lambda^2+\frac{\lambda^4}{12}+\frac{\lambda^6}{360}+\ldots\right)$ $\frac{1}{2}\left(1+e^{-2\lambda}\right) = \frac{e^{-\lambda}}{2}\left(e^{\lambda}+e^{-\lambda}\right)$ $= \frac{e^{-\lambda}}{2}\left[\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\frac{\lambda^4}{4!}+\frac{\lambda^5}{5!}+\frac{\lambda^6}{6!}\ldots\right)+\left(1-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\frac{\lambda^4}{4!}-\frac{\lambda^5}{5!}+\frac{\lambda^6}{6!}\ldots\right)\right]$ $= \frac{e^{-\lambda}}{2}\left[2+2\left(\frac{\lambda^2}{2!}\right)+2\left(\frac{\lambda^4}{4!}\right)+2\left(\frac{\lambda^6}{6!}\right)\ldots\right]$ $= \frac{e^{-\lambda}}{2}\left(2+\lambda^2+\frac{\lambda^4}{12}+\frac{\lambda^6}{360}+\ldots\right)$ $\therefore \text{P}(D=\text{even}) = \frac{1}{2}\left(1+e^{-2\lambda}\right)$	

Qn	Suggested Solutions	
8(i)	Let T_H be the time spent in hours for customers during the holiday season. $\overline{t}_H = \frac{341}{100} = 3.41$, $s_H^2 = \frac{1}{99}\left[1563 - \frac{(341)^2}{100}\right] = \frac{40019}{9900}$ For 95% confidence interval, $z_{(0.025)}$ = 1.9600 95% CI for μ = $\left(3.41 - 1.9600\sqrt{\frac{\frac{40019}{9900}}{100}}, 3.41 + 1.9600\sqrt{\frac{\frac{40019}{9900}}{100}}\right)$ $= (3.02, 3.80)$	
(ii)	Let T_H and T_N be the time spent in hours for customers during the holiday and non-holiday seasons respectively. Let μ_H and μ_N be the population mean time spent in hours for customers during the holiday and non-holiday seasons respectively. $\mathrm{H}_0 : \mu_H - \mu_N = 0$ $\mathrm{H}_1 : \mu_H - \mu_N > 0$ $\overline{t}_N = \frac{257}{100} = 2.57$, $s_N^2 = \frac{1}{99}\left[1032 - \frac{(257)^2}{100}\right] = \frac{37151}{9900}$ Since $n = 100 \geq 50$ is large, by Central Limit Theorem, $\overline{T_H}$ and $\overline{T_N}$ is approximately normally distributed. Under H_0, test statistic $Z = \frac{\overline{T_H} - \overline{T_N} - 0}{\sqrt{\frac{s_H^2}{100} + \frac{s_N^2}{100}}} \sim \mathrm{N}(0,1)$ approximately $z = \frac{3.41 - 2.57}{\sqrt{\frac{\frac{40019}{9900}}{100} + \frac{\frac{37151}{9900}}{100}}} = 3.0086$ *p*-value = $\mathrm{P}(Z \geq 3.0086) \approx 0.0013121$ which is very small. H$_0$ is rejected as long as level of significance is more than 0.14%. Hence there is **very strong evidence** that the mean time spent in hours for customers during the holiday season is greater than non-holiday season.	

(iii)	As the sample sizes are large, by Central Limit Theorem, the distributions of the **sample means** $\overline{T}_H$ and $\overline{T}_N$ are approximately normal. Hence there are **no implications** for the validity of the test even if the time spent of the two groups of customers are not normally distributed.	
(iv)	If the statistician's advice had been followed the test procedure would be a paired-sample *z*-test. In this procedure, the manager should randomly select 200 customers during the non-holiday season and survey them regarding their time spent in the mall and then survey the same 200 customers on their time spent during the holiday season. This procedure will be more accurate because the pairing calculates the difference of the **same** customers. This will **eliminate any factors** that could **affect the time spent in the mall** which may vary from customer to customer.	Note: We do not use a paired - sample t-test here as the sample size is large. If paired sample t-test is used, we need to assume that the differences in the timings follow a normal distribution.

Qn	**Suggested Solutions**	

9 (i)

H_0: Types of movies are independent of whether snacks are bought for the movie

H_1: Types of movies are not independent of whether snacks are nought for the movie

Observed Frequency

		Snacks	No Snacks	**Total**
Type of movie	Action	214	196	**410**
	Comedy	545	445	**990**
	Family	206	194	**400**
	Horror	95	105	**200**
	Total	**1060**	**940**	**2000**

Under H_0, Expected Frequency

		Snacks	No Snacks	**Total**
Type of movie	Action	217.3	192.7	**410**
	Comedy	524.7	465.3	**990**
	Family	212	188	**400**
	Horror	106	94	**200**
	Total	**1040**	**960**	**2000**

Contribution to the test statistic

		Snacks	No Snacks	**Total**
Type of movie	Action	0.050115	0.056513	**410**
	Comedy	0.78538	0.88564	**990**
	Family	0.16981	0.19149	**400**
	Horror	1.1415	1.2872	**200**
	Total	**1040**	**960**	**2000**

Degree of freedom = (4 – 1)(2 – 1) = 3

Reject H_0 if $\chi^2_{\text{cal}} \geq 6.251$

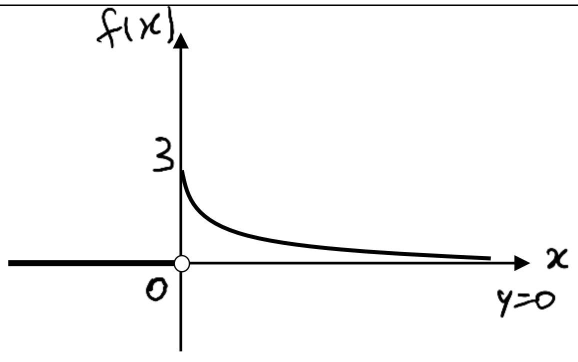
Using GC, *p*-value = 0.20633 > 0.1 (or $\chi^2_{\text{cal}} = 4.5677 <$ 6.251), we do not reject H_0. Hence there is insufficient evidence at 10% level of significance to conclude that the type of movie they saw is not independent from whether or not they bought snacks.

(ii)

When the sample size increases from 2000 to 4000, it means every cells will increase by 2 times. So

$$\chi^2_{\text{cal}} = \sum \frac{\left(O_i - E_i\right)^2}{E_i} = 9.1354 > 6.251,$$ we reject null hypothesis. So the conclusion of the test will be changed.

(iii)	Let *n* be the sample size required. We have to reject null hypothesis to support the manager's claim, i.e. $\chi^2_{\text{cal}} \geq 12.84$ $\chi^2_{\text{cal}} = \sum \frac{\left(O_i - E_i\right)^2}{E_i} \geq 12.84$ $\Rightarrow \left(\frac{4.5677}{2000}\right)n = 0.00228385n \geq 12.84$ $\Rightarrow n \geq 5622.0855$ Hence least n = 5623.	

Qn	Suggested Solutions	
10(i)	$\int_0^\infty \frac{k}{(x+1)^4}\,\mathrm{d}x = 1$ $-\frac{k}{3}\left[(x+1)^{-3}\right]_0^\infty = 1$ $-\frac{k}{3}(0-1) = 1$ $k = 3$ (shown) Let F(x) be the cdf for X $\mathrm{F}(x) = \mathrm{P}(X \le x)$ $= \int_0^x \frac{k}{(t+1)^4}\,\mathrm{d}t$ $= -\frac{k}{3}\left[(t+1)^{-3}\right]_0^x$ $= -\left(\frac{1}{(x+1)^3} - 1\right)$ $= 1 - \frac{1}{(x+1)^3}$ $\mathrm{F}(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 - \frac{1}{(x+1)^3} & \text{for} \quad x \ge 0. \end{cases}$ $\mathrm{P}(X < x) = \frac{7}{8}$ $\mathrm{F}(x) = \frac{7}{8}$ $1 - \frac{1}{(x+1)^3} = \frac{7}{8}$ $(x+1)^3 = 8$ $x + 1 = 2$ $x = 1$	
(ii)		

(iii)	$\mathrm{E}\left(X+1\right)=\int_0^\infty \frac{3(x+1)}{(x+1)^4}\,\mathrm{d}x$ $=\int_0^\infty \frac{3}{(x+1)^3}\,\mathrm{d}x$ $=-\frac{3}{2}\left[(x+1)^{-2}\right]_0^\infty$ $=\frac{3}{2}$ $\mathrm{E}\left(X+1\right)=\frac{3}{2}$ $\mathrm{E}\left(X\right)+1=\frac{3}{2}$ $\mathrm{E}\left(X\right)=\frac{1}{2}$	
(iv)	$\mathrm{E}\left(\left(X+1\right)^2\right)=\int_0^\infty \frac{3}{(x+1)^2}\,\mathrm{d}x$ $=-3\left[(x+1)^{-1}\right]_0^\infty$ $=3$ $\mathrm{Var}(X+1)=\mathrm{E}\left(\left(X+1\right)^2\right)-\left[\mathrm{E}\left(X+1\right)\right]^2$ $\mathrm{Var}(X)=\frac{3}{4}$	

The End