NJC 2023 H2 Mathematics Promotional Exam

1 The equations of three planes p_1, p_2 and p_3 are

$$\mathbf{r}\cdot\begin{pmatrix}3\\2\\-5\end{pmatrix}=3,\quad \mathbf{r}\cdot\begin{pmatrix}-5\\3\\2\end{pmatrix}=-5 \text{ and } \mathbf{r}\cdot\begin{pmatrix}17\\5\\a\end{pmatrix}=b$$

respectively, where a and b are constants.

If $a=-20.9$ and $b=16.6$, find the coordinates of the point at which these planes meet. [2]

The planes p_1 and p_2 intersect in a line l.

(i) Find a vector equation of l. [2]

(ii) Given that all three planes meet in the line l, find the values of a and b. [3]

2 A curve D has equation $\dfrac{(x+2)^2}{2^2}-(y-1)^2=1$.

(i) Another curve E has equation $x^2-y^2=1$. Describe a sequence of transformations that transforms the graph of E onto D. [3]

(ii) Find the equations of the asymptotes of D. [2]

(iii) Sketch D, stating the equations of the asymptotes and the coordinates of the vertices. [3]

(iv) Given an ellipse with equation $\left(\dfrac{x+2}{c}\right)^2+(y-1)^2=1$, state the possible values of c such that the ellipse does not intersect D. [1]

3 It is given that $\mathrm{f}(x)=x\mathrm{e}^{-\frac{x}{k}}$, where k is a positive constant.

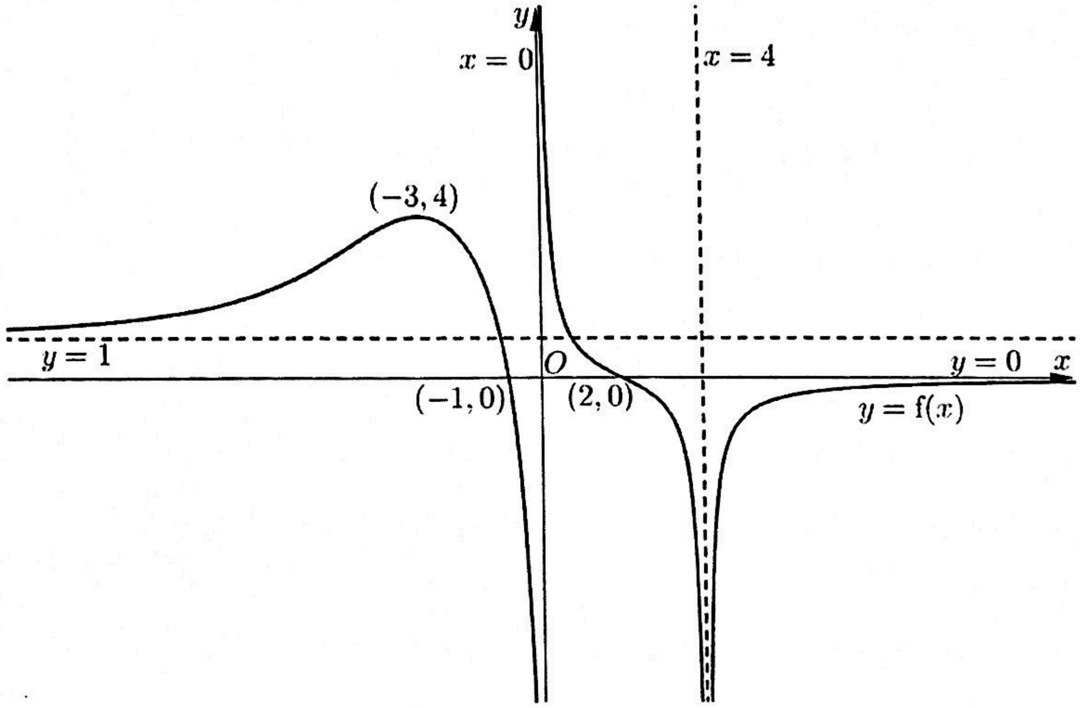
(i) Find the range of values of x for which the curve with equation $y=\mathrm{f}(x)$ is concave upwards. [4]

Assume for the remainder of this question that $k=2$.

(ii) R is the region bounded by the curve, the x-axis, the line $x=4$ and the line $x=n$, where $n>4$. The solid obtained when rotating R about the x-axis through 2π radians has a volume of V units^3. Find the exact value that V converges to as n approaches infinity. [5]

[You may use the result that $x^n\mathrm{e}^{-x}$ approaches 0 as x approaches infinity for all real values of n.]

4 **(a)** The diagram below shows the curve $y = f(x)$. It has a maximum point at $(-3, 4)$ and crosses the x-axis at $(-1, 0)$ and $(2, 0)$. The lines $x = 0$, $x = 4$, $y = 0$ and $y = 1$ are asymptotes of the curve.



On separate diagrams, sketch the following graphs. Label clearly, where they can be determined, the equations of the asymptote(s), the coordinates of the stationary point(s) and the point(s) where the curves cross the x- and y-axes.

(i) $y = f\left(2 - |x|\right)$, [3]

(ii) $y = \dfrac{1}{f(x)}$, [4]

(iii) $y = f'(x)$, [3]

(b) Write a possible non-constant function g such that $\dfrac{1}{g(x)} = g(a - x)$ for all $x \in (0, a)$, where a is some constant to be stated. [1]

5 **(a)** **(i)** Find the exact shortest distance from the origin O to the plane p with equation $2x + 3y + 6z = 10$. [2]

(ii) Given that point A lies on plane p and $OA = 10$, find the acute angle between plane p and line OA. [2]

(b) Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $|\mathbf{a} \times \mathbf{b}| = \sqrt{3}$, $|\mathbf{a} - \mathbf{b}| = 1$ and the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is $30°$, find the possible values of $|\mathbf{a}|$ and $|\mathbf{b}|$. [5]

(c) Two fixed points C and D have position vectors $\mathbf{c}$ and $\mathbf{d}$ respectively. By considering the result $\mathbf{c} \times \mathbf{c} = \mathbf{0}$, give a geometrical interpretation of the vector equation $\mathbf{r} \times (\mathbf{d} - \mathbf{c}) = \mathbf{c} \times \mathbf{d}$. [3]

6 A curve C has parametric equations

$$x = e^{t}\sin t, \quad y = e^{-t}\cos t, \quad \text{for } 0 \le t \le \frac{3\pi}{4}.$$

(i) Sketch C, labelling the exact coordinates of the end-point(s). [2]

(ii) In the same diagram as part (i), shade the region of points (X, Y) such that there are exactly two distinct tangents to C that pass through (X, Y). [2]

(iii) Show that the tangent to C at the point $\left(e^{p}\sin p, e^{-p}\cos p\right)$ can be expressed as

$$xe^{-p} + ye^{p} = \cos p + \sin p.$$ [3]

(iv) Find the exact area of the region bounded by C, the y-axis and the line $y = -\frac{1}{\sqrt{2}e^{\frac{3\pi}{4}}}$. [5]

7 A function f is defined by $f : x \mapsto \frac{2x+3}{x-2}$, $x \in \mathbb{R}$, $x \ne 2$.

(i) Find the domain of f^{-1}. [2]

A function h is said to be self-inverse if h and h^{-1} are the same function.

(ii) Show that f is self-inverse. [2]

(iii) State the value of $f^{2023}(6)$. [1]

Another function g has range $(k, k+10]$, where k is a constant.

(iv) Find the range of values of k such that the composite function fg exists. [3]

Assume for the remainder of this question that fg exists and $k \ne 2$.

(v) Find the range of fg in terms of k. [3]

(vi) Given that $fg(x) = \frac{2x+2}{1-6x}$, find $g(x)$ in terms of x. [3]

8 **(a)** Use the substitution $x = 2\tan\theta$ to find $\int \frac{3x+5}{\sqrt{4+x^2}}\,dx$. [5]

(b) **(i)** Given that $\frac{8x^3+7x^2-12x}{(x^2+2x+3)(4x^2-9)} = \frac{2x+1}{x^2+2x+3} + \frac{Cx+D}{4x^2-9}$, find the values of the constants C and D. [2]

(ii) Find $\int \frac{8x^3+7x^2-12x}{(x^2+2x+3)(4x^2-9)}\,dx$. [5]

9 **(i)** A function f is such that $f'(k)=0$ and $f''(k)<0$ for some $k \in [p, q]$. Justify with the aid of a sketch whether this must mean that $f(k)$ is the largest possible value of $f(x)$ for any x in the interval $[p, q]$. [2]

A particle is travelling along a straight line from an origin O such that its displacement $s(t)$ metres away from O at time t seconds is given by

$$s(t) = t^3 - 6t^2 + 9t,\ 0 \le t \le 5.$$

(ii) State the largest possible value of $s(t)$ for $0 \le t \le 5$. [1]

The *instantaneous speed* $V(t)$ ms^{-1} of the particle at time t seconds is given by $V(t) = |s'(t)|$.

(iii) Find $s'(t)$ in terms of t and hence sketch the graph of $V(t)$ against t for $0 \le t \le 5$. [3]

The total distance $D(T)$ metres travelled by the particle at time T seconds is the total area of the region bounded between the graph of $V(t)$ and the t-axis from $t=0$ to $t=T$.

(iv) Show that for $1 < T \le 3$, $D(T) = 8 - T^3 + 6T^2 - 9T$. Find $D(T)$ in a similar form for $3 < T \le 5$. [4]

The *average speed* $V_{ave}(t)$ ms^{-1} of the particle over the first t seconds is given by $V_{ave}(t) = \frac{D(t)}{t}$.

(v) Express $V_{ave}(t)$ as a function of t by filling in the blanks below. [2]

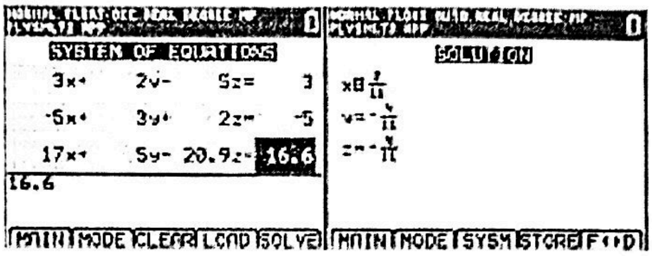
$$V_{ave}(t) = \begin{cases} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ , & 0 \le t \le 1 \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ , & \_\_ < t \le \_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ , & \_\_ < t \le \_\_ \end{cases}$$

(vi) Hence find the times at which the average speed of the particle is 3 ms^{-1}. [2]

Question 1 (Systems of Linear Equations, Vectors II)

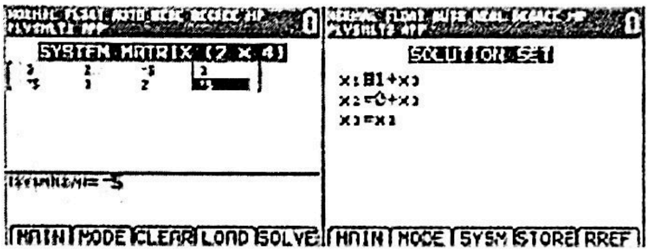
Rewriting the equations into cartesian form,

$$\begin{aligned} 3x + 2y - 5z &= 3 \\ -5x + 3y + 2z &= -5 \\ 17x + 5y - 20.9z &= 16.6 \end{aligned}$$



Thus, the three planes meet at $\left(\frac{7}{11}, -\frac{4}{11}, -\frac{4}{11}\right)$.

(i)



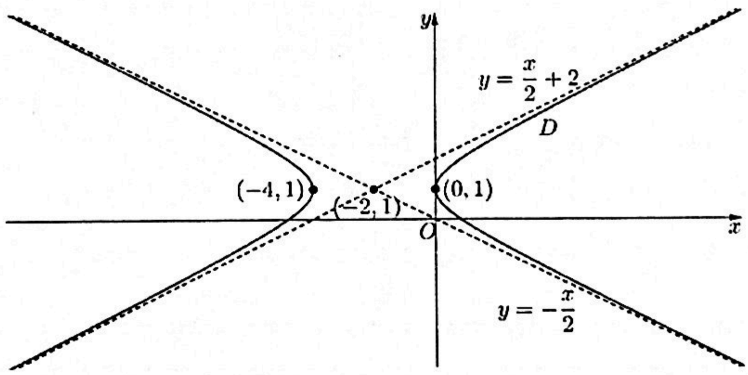
The two planes meet at the line l: $\mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\lambda \in \mathbb{R}$.

(ii)

$$\begin{pmatrix}17\\5\\a\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \Rightarrow 17 + 5 + a = 0 \Rightarrow a = -22$$

$$\begin{pmatrix}17\\5\\-22\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = b \Rightarrow b = 17$$

Question 2 (Conics and Linear Transformations)

(i)	$D: \dfrac{(x+2)^2}{2^2} - (y-1)^2 = 1 \Rightarrow \left(\dfrac{x}{2}+1\right)^2 - (y-1)^2 = 1$ Step 1: Translate in the positive y-direction by 1 unit. Step 2: Translate in the negative x-direction by 1 unit. Step 3: Scale parallel to the x-axis by factor 2. **Alternatively,** Step 1: Translate in the positive y-direction by 1 unit. Step 3: Scale parallel to the x-axis by factor 2. Step 2: Translate in the negative x-direction by 2 units.
(ii)	$\dfrac{(x+2)^2}{2^2} - (y-1)^2 = 1$ Asymptotes: $(y-1) = \pm\dfrac{1}{2}(x+2)$ $y = \dfrac{1}{2}x + 2$ and $y = -\dfrac{1}{2}x$
(iii)	
(iv)	All possible values of c: $-2 < c < 2,\ c \neq 0$ Or $-2 < c < 0$ or $0 < c < 2$

Question 3 (Concavity, Integration by Parts, Volume of Revolution)

(i)

For the curve to be concave upwards, the gradient must be increasing, i.e. $f''(x) > 0$.

$$f(x) = xe^{-\frac{x}{k}}$$

$$\Rightarrow f'(x) = x\left(-\frac{1}{k}e^{-\frac{x}{k}}\right) + e^{-\frac{x}{k}} = e^{-\frac{x}{k}}\left(1-\frac{x}{k}\right)$$

$$\Rightarrow f''(x) = e^{-\frac{x}{k}}\left(-\frac{1}{k}\right) + \left(1-\frac{x}{k}\right)\left(-\frac{1}{k}e^{-\frac{x}{k}}\right)$$

$$= -\frac{e^{-\frac{x}{k}}}{k}\left(1+1-\frac{x}{k}\right)$$

$$= -\frac{e^{-\frac{x}{k}}}{k}\left(\frac{2k-x}{k}\right)$$

Next we have $-\frac{e^{-\frac{x}{k}}}{k}\left(\frac{2k-x}{k}\right) > 0 \Rightarrow x-2k > 0 \Rightarrow x > 2k$.

(ii)

Volume of solid obtained

$$= \pi\int_{2k}^{\infty} y^2 \, dx$$

$$= \pi\int_{4}^{\infty} x^2 e^{-\frac{2x}{2}} \, dx$$

$$= -\pi\int_{4}^{\infty} x^2\left(-e^{-x}\right) dx$$

$$= -\pi\left\{\left[x^2e^{-x}\right]_4^{\infty} - \int_4^{\infty} 2xe^{-x} \, dx\right\}$$

$$= -\pi\left[0-16e^{-4}\right] - 2\pi\int_4^{\infty} x\left(-e^{-x}\right) dx$$

$$= 16\pi e^{-4} - 2\pi\left\{\left[xe^{-x}\right]_4^{\infty} - \int_4^{\infty} e^{-x} \, dx\right\}$$

$$= 16\pi e^{-4} - 2\pi\left[0-4e^{-4}\right] - 2\pi\int_4^{\infty} -e^{-x} \, dx$$

$$= 16\pi e^{-4} + 8\pi e^{-4} - 2\pi\left[e^{-x}\right]_4^{\infty}$$

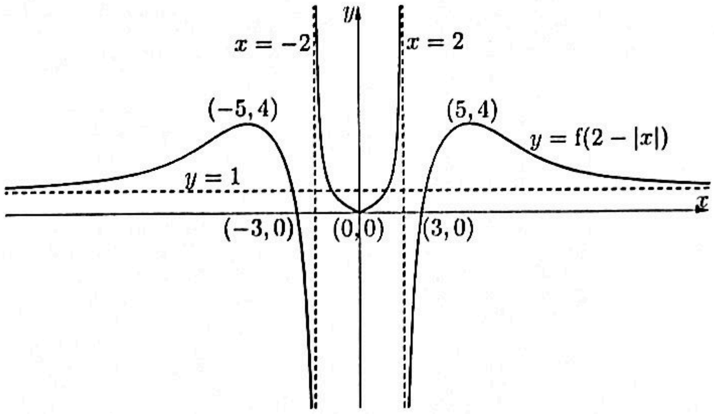
$$= 24\pi e^{-4} - 2\pi\left[0-e^{-4}\right]$$

$$= 24\pi e^{-4} + 2\pi e^{-4}$$

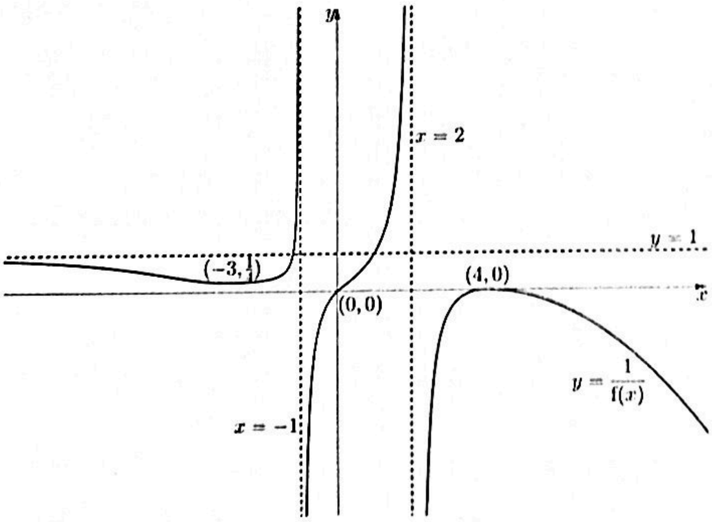
$$= 26\pi e^{-4}$$

Question 4 (Transformations of Graphs, Graph of Gradient Function)

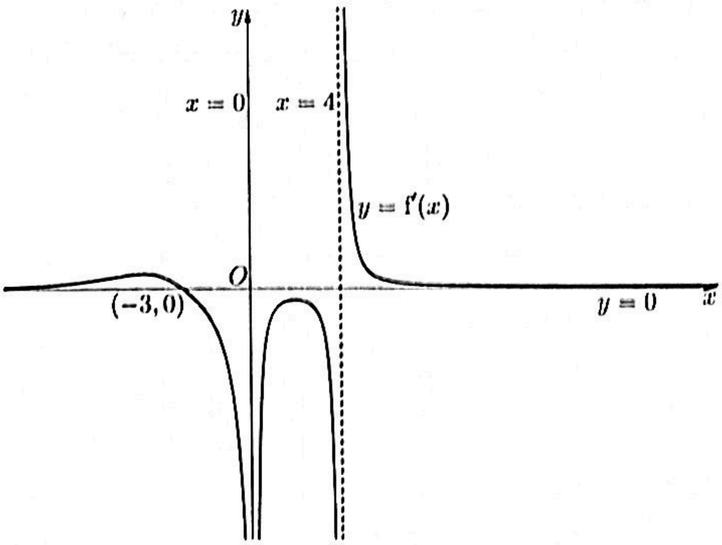
(a)(i)



(a)(ii)



(a)(iii)



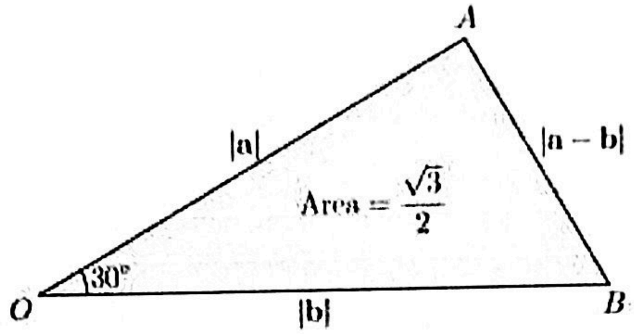
(b)

$g(x) = \tan x$ or $\cot x$, $a = \frac{\pi}{2}$ or $g(x) = A^{b-x}$, where A is a positive constant such that $A \neq 1$, $a = \frac{b}{2}$

Question 5 (Vectors I and II)

(a)(i)	Shortest distance $= \left\| \dfrac{D}{\|\mathbf{n}\|} \right\|$ $= \left\| \dfrac{10}{\sqrt{2^2+3^2+6^2}} \right\|$ $= \dfrac{10}{7}$ units
(a)(ii)	Angle between line OA and plane p $= \sin^{-1}\left(\dfrac{10/7}{10}\right)$ $= 0.143$ rads (to 3 s.f.) or $8.2°$ (to 1 d.p.)
(b)	$\|\mathbf{a}\times\mathbf{b}\| = \sqrt{3} \Rightarrow \|\mathbf{a}\|\|\mathbf{b}\|\sin 30° = \sqrt{3} \Rightarrow \|\mathbf{b}\| = \dfrac{2\sqrt{3}}{\|\mathbf{a}\|}$ (1) $\|\mathbf{a}-\mathbf{b}\| = 1 \Rightarrow \|\mathbf{a}-\mathbf{b}\|^2 = 1$ $\Rightarrow (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = 1$ $\Rightarrow \mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) - \mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) = 1$ $\Rightarrow \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} = 1$ $\Rightarrow \|\mathbf{a}\|^2 - 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2 = 1$ $\Rightarrow \|\mathbf{a}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos 30° + \|\mathbf{b}\|^2 = 1$ $\Rightarrow \|\mathbf{a}\|^2 - \sqrt{3}\|\mathbf{a}\|\|\mathbf{b}\| + \|\mathbf{b}\|^2 = 1$ (2) Substituting equation (1) into equation (2), $\|\mathbf{a}\|^2 - \sqrt{3}\|\mathbf{a}\|\left(\dfrac{2\sqrt{3}}{\|\mathbf{a}\|}\right) + \left(\dfrac{2\sqrt{3}}{\|\mathbf{a}\|}\right)^2 = 1$ $\|\mathbf{a}\|^2 - 7 + \dfrac{12}{\|\mathbf{a}\|^2} = 1$ $\|\mathbf{a}\|^4 - 7\|\mathbf{a}\|^2 + 12 = 0$ $\left(\|\mathbf{a}\|^2 - 3\right)\left(\|\mathbf{a}\|^2 - 4\right) = 0$ $\|\mathbf{a}\|^2 = 3$ or $\|\mathbf{a}\|^2 = 4$ $\|\mathbf{a}\| = \sqrt{3}$ or $\|\mathbf{a}\| = 2$ Substituting into (1), we have $\|\mathbf{a}\| = \sqrt{3},\ \|\mathbf{b}\| = 2$ or $\|\mathbf{a}\| = 2,\ \|\mathbf{b}\| = \sqrt{3}$

(b) Alternatively, let A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Then $AB = |\mathbf{a}-\mathbf{b}| = 1$ and area of triangle $OAB = \frac{1}{2}|\mathbf{a}\times\mathbf{b}| = \frac{\sqrt{3}}{2}$



Area of triangle $OAB = \frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin 30° = \frac{1}{4}|\mathbf{a}||\mathbf{b}|$.

Thus, $\frac{1}{4}|\mathbf{a}||\mathbf{b}| = \frac{\sqrt{3}}{2} \Rightarrow |\mathbf{a}||\mathbf{b}| = 2\sqrt{3} \Rightarrow |\mathbf{b}| = \frac{2\sqrt{3}}{|\mathbf{a}|}$ (1)

By using Cosine Rule,

$$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos 30°$$

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\times\frac{\sqrt{3}}{2} = 1$$

$$|\mathbf{a}|^2 - \sqrt{3}|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 = 1 \qquad (2)$$

Substituting equation (1) into equation (2),

$$|\mathbf{a}|^2 - \sqrt{3}|\mathbf{a}|\left(\frac{2\sqrt{3}}{|\mathbf{a}|}\right) + \left(\frac{2\sqrt{3}}{|\mathbf{a}|}\right)^2 = 1$$

$$|\mathbf{a}|^2 - 7 + \frac{12}{|\mathbf{a}|^2} = 1$$

$$|\mathbf{a}|^4 - 7|\mathbf{a}|^2 + 12 = 0$$

$$\left(|\mathbf{a}|^2 - 3\right)\left(|\mathbf{a}|^2 - 4\right) = 0$$

$$|\mathbf{a}|^2 = 3 \text{ or } |\mathbf{a}|^2 = 4$$

$$|\mathbf{a}| = \sqrt{3} \text{ or } |\mathbf{a}| = 2$$

Substituting into (1), we have $|\mathbf{a}| = \sqrt{3},\ |\mathbf{b}| = 2$ or $|\mathbf{a}| = 2,\ |\mathbf{b}| = \sqrt{3}$

(c) $\mathbf{r}\times(\mathbf{d}-\mathbf{c})=\mathbf{c}\times\mathbf{d}$

$\mathbf{r}\times(\mathbf{d}-\mathbf{c})=\mathbf{c}\times\mathbf{d}-\mathbf{c}\times\mathbf{c}$

$\mathbf{r}\times(\mathbf{d}-\mathbf{c})=\mathbf{c}\times(\mathbf{d}-\mathbf{c})$

$\mathbf{r}\times(\mathbf{d}-\mathbf{c})-\mathbf{c}\times(\mathbf{d}-\mathbf{c})=\mathbf{0}$

$(\mathbf{r}-\mathbf{c})\times(\mathbf{d}-\mathbf{c})=\mathbf{0}$

$\mathbf{r}-\mathbf{c}$ is parallel to $\mathbf{d}-\mathbf{c}$.

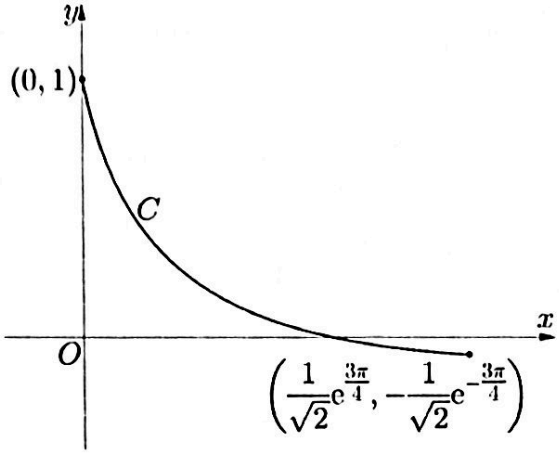
$\mathbf{r}-\mathbf{c}=\lambda(\mathbf{d}-\mathbf{c})$

$\mathbf{r}=\mathbf{c}+\lambda(\mathbf{d}-\mathbf{c})$.

Thus, the equation represents the line *CD* and **r** represents the position vector of a variable point on this line.

Question 6 (Parametric Curves, Tangents & Normals, Parametric Area)

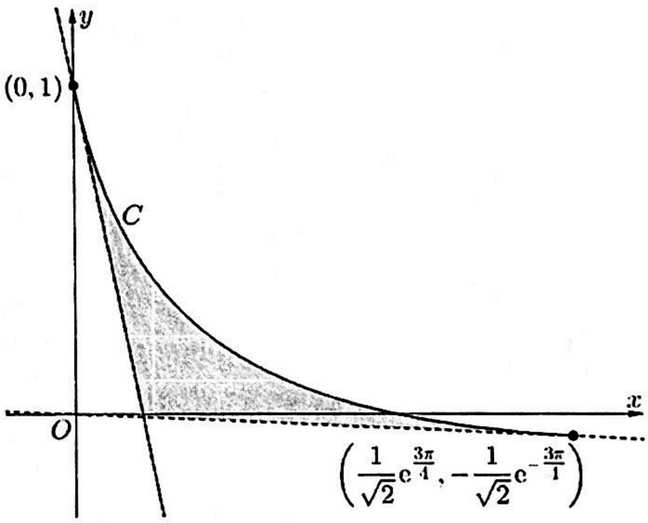
(i)



When $t=0,\ x=e^0\sin 0=0,\ y=e^0\cos 0=1$.

When $t=\frac{3\pi}{4},\ x=e^{\frac{3\pi}{4}}\sin\frac{3\pi}{4}=\frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}, y=e^{-\frac{3\pi}{4}}\cos\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}$.

(iii)



(ii)

$x = e^t \sin t \Rightarrow \frac{dx}{dt} = e^t \cos t + e^t \sin t$

$y = e^{-t} \cos t \Rightarrow \frac{dy}{dt} = e^{-t}\left(-\sin t\right) + \left(-e^{-t}\right)\cos t$

Therefore $\frac{dy}{dx} = \frac{e^{-t}\left(-\sin t\right) + \left(-e^{-t}\right)\cos t}{e^t \cos t + e^t \sin t} = \frac{-e^{-t}}{e^t} = -e^{-2t}$

At point $\left(e^p \sin p, e^{-p} \cos p\right)$, the equation of the tangent to C is

$\frac{y - e^{-p}\cos p}{x - e^p \sin p} = -e^{-2p}$

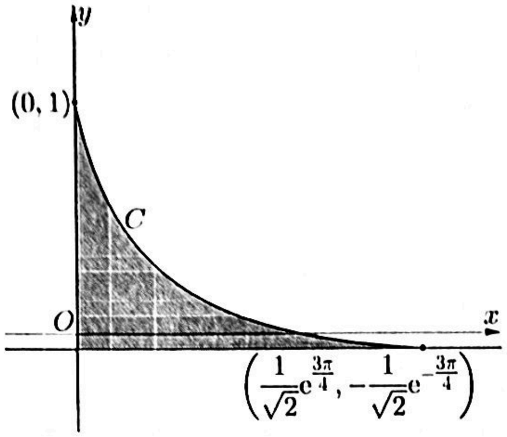
$\Rightarrow y - e^{-p}\cos p = -e^{-2p}\left(x - e^p \sin p\right)$

$\Rightarrow y = -e^{-2p}x + e^{-p}\cos p + e^{-p}\sin p$

$\Rightarrow ye^p = -e^{-p}x + \cos p + \sin p$

$\Rightarrow xe^{-p} + ye^p = \cos p + \sin p$ (shown)

(iv)



Area of region

$= \int_{\frac{1}{\sqrt{2}e^{\frac{3\pi}{4}}}}^{1} x \, dy$

$= \int_{\frac{3\pi}{4}}^{0} e^{t} \sin t \left(-e^{-t} \cos t - e^{-t} \sin t\right) dt$

$= \int_{0}^{\frac{3\pi}{4}} \sin t \cos t + \sin^2 t \, dt$

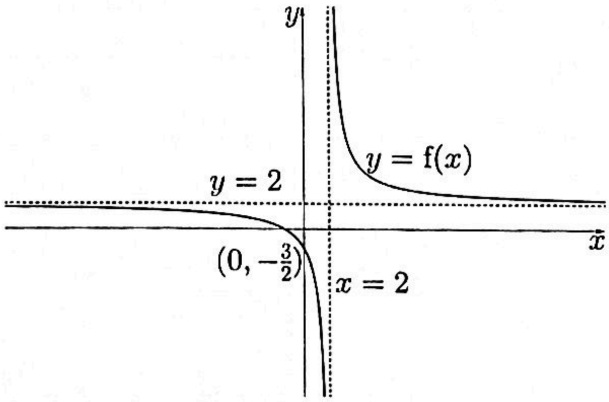
$= \int_{0}^{\frac{3\pi}{4}} \frac{1}{2} \sin 2t + \frac{1}{2}(1 - \cos 2t) \, dt$

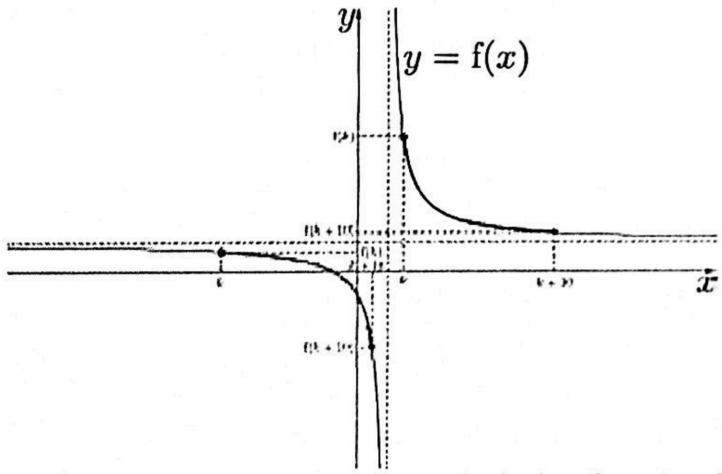
$= \frac{1}{2}\left[-\frac{1}{2}\cos 2t + t - \frac{1}{2}\sin 2t\right]_{0}^{\frac{3\pi}{4}}$

$= \frac{1}{2}\left[\left(0 + \frac{3\pi}{4} + \frac{1}{2}\right) - \left(-\frac{1}{2}\right)\right]$

$= \frac{3\pi}{8} + \frac{1}{2}$

Question 7 (Functions)

(i)	 $D_{f^{-1}} = R_f = (-\infty, 2) \cup (2, \infty)$
(ii)	Let $y = \dfrac{2x+3}{x-2}$. $y(x-2) = 2x+3$ $xy - 2y = 2x+3$ $xy - 2x = 2y+3$ $x(y-2) = 2y+3$ $x = \dfrac{2y+3}{y-2}$ $f^{-1}(x) = \dfrac{2x+3}{x-2} = f(x)$. Furthermore, $D_{f^{-1}} = (-\infty, 2) \cup (2, \infty) = D_f$. $\therefore f = f^{-1}$. Thus, f is self-inverse (shown).
(iii)	$f^{2023}(6) = f(6) \quad (\because f = f^{-1})$ $= \dfrac{2(6)+3}{6-2}$ $= \dfrac{15}{4}$.

(iv)	For fg to exist, $R_g \subseteq D_f$ i.e. $(k, k+10] \subseteq (-\infty, 2) \cup (2, \infty)$ $(k, k+10] \subseteq (-\infty, 2)$ or $(k, k+10] \subseteq (2, \infty)$ $k+10 < 2$ or $k \geq 2$ $k < -8$ or $k \geq 2$.
(v)	 We see from the graph of $y = f(x)$ that regardless of whether $k < -8$ or $k \geq 2$, restricting the domain of f to the range of g i.e., $(k, k+10]$ will yield the range $[f(k+10), f(k))$ as f is a <u>decreasing</u> on $(-\infty, 2)$ as well as on $(2, \infty)$. Thus, the range of fg is $\left[\dfrac{2k+23}{k+8}, \dfrac{2k+3}{k-2}\right)$.

(vi)	Method 1: Applying f^{-1} to both sides.

$$fg(x) = \frac{2x+2}{1-6x}$$

$$f^{-1}fg(x) = f^{-1}\left(\frac{2x+2}{1-6x}\right)$$

$$g(x) = f\left(\frac{2x+2}{1-6x}\right) \quad (\because f = f^{-1})$$

$$= \frac{2\left(\frac{2x+2}{1-6x}\right)+3}{\left(\frac{2x+2}{1-6x}\right)-2}$$

$$= \frac{4x+4+3-18x}{2x+2-2+12x}$$

$$= \frac{7-14x}{14x}$$

$$= \frac{1-2x}{2x} \quad \left(\text{or } \frac{1}{2x}-1\right).$$

Method 2: Using $g(x)$ as an input for f.

$$fg(x) = \frac{2x+2}{1-6x}$$

$$\frac{2g(x)+3}{g(x)-2} = \frac{2x+2}{1-6x}$$

Let $u = g(x)$. Then,

$$\frac{2u+3}{u-2} = \frac{2x+2}{1-6x}$$

$$(2u+3)(1-6x) = (u-2)(2x+2)$$

$$2u+3-12ux-18x = 2ux-4x+2u-4$$

$$7-14x = 14ux$$

$$u = \frac{7-14x}{14}$$

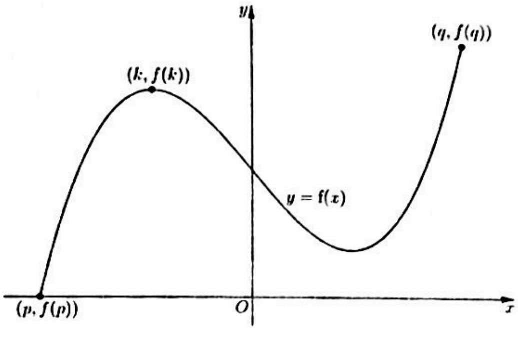
$$g(x) = \frac{1-2x}{2x} \quad \left(\text{or } \frac{1}{2x}-1\right).$$

Question 8 (Integration Techniques)

(a)	$x = 2\tan\theta \Rightarrow \dfrac{dx}{d\theta} = 2\sec^2\theta$ $\displaystyle\int \frac{3x+5}{\sqrt{4+x^2}}\,dx$ $\displaystyle= \int \frac{6\tan\theta+5}{\sqrt{4+4\tan^2\theta}}(2\sec^2\theta)\,d\theta$ $\displaystyle= \int \frac{6\tan\theta+5}{2\sec\theta}(2\sec^2\theta)\,d\theta$ $\displaystyle= \int 6\tan\theta\sec\theta + 5\sec\theta\,d\theta$ $= 6\sec\theta + 5\ln\lvert\tan\theta+\sec\theta\rvert + c$ $\displaystyle= 3\sqrt{4+x^2} + 5\ln\left\lvert\frac{x+\sqrt{4+x^2}}{2}\right\rvert + c$
(b) (i)	$\displaystyle\int \frac{8x^3+7x^2-12x}{(x^2+2x+3)(4x^2-9)}\,dx$ $\displaystyle= \int \frac{2x+1}{x^2+2x+3} + \frac{3}{4x^2-9}\,dx$
(ii)	$\displaystyle\int \frac{2x+1}{x^2+2x+3}\,dx$ $\displaystyle= \int \frac{2x+2}{x^2+2x+3} - \frac{1}{x^2+2x+3}\,dx$ $\displaystyle= \int \frac{2x+2}{x^2+2x+3}\,dx - \int \frac{1}{(x+1)^2+\sqrt{2}^2}\,dx$ $\displaystyle= \ln\lvert x^2+2x+3\rvert - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$ $\displaystyle\int \frac{3}{4x^2-9}\,dx = \frac{3}{4}\int \frac{1}{x^2-(\frac{3}{2})^2}\,dx$ $\displaystyle= \frac{3}{4}\left(\frac{1}{2(\frac{3}{2})}\right)\ln\left\lvert\frac{x-\frac{3}{2}}{x+\frac{3}{2}}\right\rvert + D$ $\displaystyle= \frac{1}{4}\ln\left\lvert\frac{2x-3}{2x+3}\right\rvert + D$ Thus, $\displaystyle\int \frac{8x^3+7x^2-12x}{(x^2+2x+3)(4x^2-9)}\,dx$ $\displaystyle= \ln\lvert x^2+2x+3\rvert - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{4}\ln\left\lvert\frac{2x-3}{2x+3}\right\rvert + c$

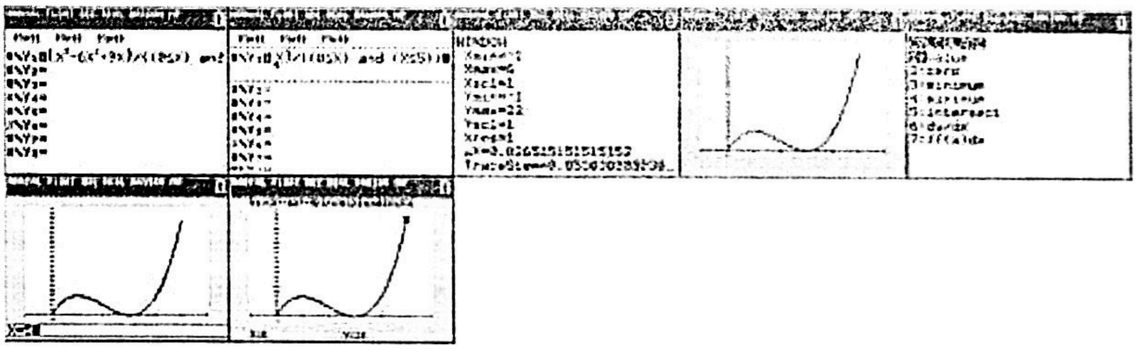
Question 9 (Contextual Problem)

(i)



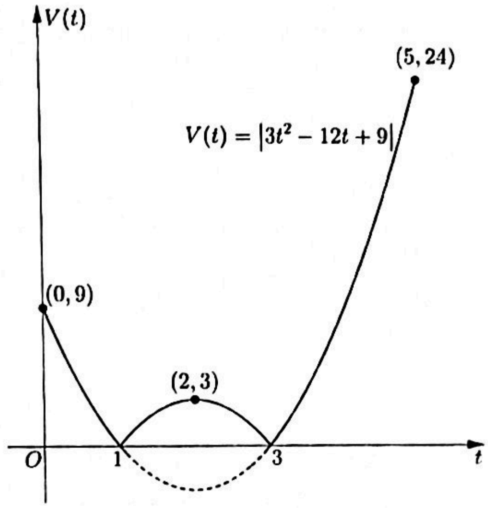
$f'(k)=0$ and $f''(k)<0$ implies that the point $(k,\ f(k))$ is a maximum turning point of the curve. However, as seen from the sketch above, the maximum value of $f(x)$ can be attained at one of the endpoints of $y=f(x)$ instead. Thus the condition $f'(k)=0$ and $f''(k)<0$ is not sufficient to prove that $f(k)$ is the largest possible value of $f(x)$ for any x in the interval $[p, q]$.

(ii)



From the sketch, we see that the largest possible value is 20.

(iii) $s'(t) = 3t^2 - 12t + 9$



(iv) For $1 < T \le 3$,

$$D(T) = \int_0^1 3t^2 - 12t + 9 \, dt - \int_1^T 3t^2 - 12t + 9 \, dt$$

$$= \left[t^3 - 6t^2 + 9t\right]_0^1 - \left[t^3 - 6t^2 + 9t\right]_1^T$$

$$= 1^3 - 6(1)^2 + 9(1) - \left(T^3 - 6T^2 + 9T\right)$$

$$+ \left[1^3 - 6(1)^2 + 9(1)\right]$$

$$= 8 - \left(T^3 - 6T^2 + 9T\right)$$

$$= 8 - T^3 + 6T^2 - 9T$$

For $3 < T \le 5$,

$$D(T) = \int_0^1 3t^2 - 12t + 9 \, dt - \int_1^3 3t^2 - 12t + 9 \, dt$$

$$+ \int_3^T 3t^2 - 12t + 9 \, dt$$

$$= \left[t^3 - 6t^2 + 9t\right]_0^1 - \left[t^3 - 6t^2 + 9t\right]_1^3$$

$$+ \left[t^3 - 6t^2 + 9t\right]_3^T$$

$$= 1^3 - 6(1)^2 + 9(1) - \left[3^3 - 6(3)^2 + 9(3)\right]$$

$$+ \left[1^3 - 6(1)^2 + 9(1)\right]$$

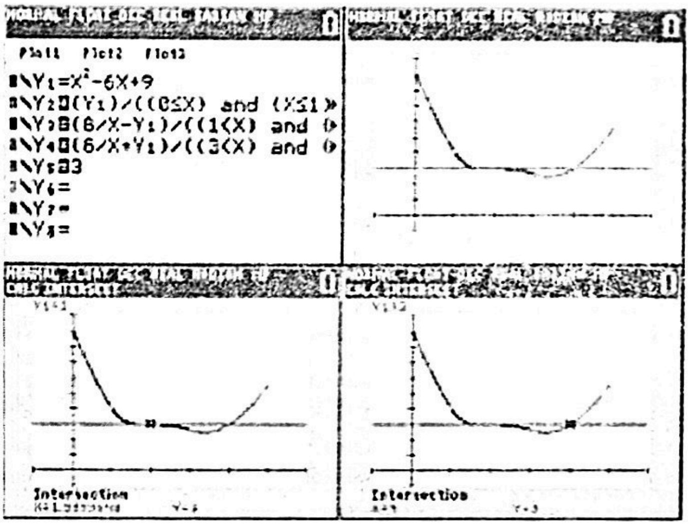
$$+ \left(T^3 - 6T^2 + 9T\right) - \left[3^3 - 6(3)^2 + 9(3)\right]$$

$$= 8 + T^3 - 6T^2 + 9T$$

(v)

$$V_{\text{ave}}(t) = \frac{D(t)}{t} = \begin{cases} t^2 - 6t + 9, & 0 \le t \le 1 \\ -t^2 + 6t - 9 + \dfrac{8}{t}, & 1 \le t \le 3 \\ t^2 - 6t + 9 + \dfrac{8}{t}, & 3 \le t \le 5 \end{cases}$$

(vi)



From the GC, $V_{\text{ave}}(t) = 3$ when $t = 2$ or $t = 4$.